Classification of C^*-algebras, flow equivalence of shift spaces, and graph and Leavitt path algebras

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Content

Outline

Move **(C)**

"Cuntz splice" on a vertex supporting two cycles

We have seen and used that when $E^\dagger$ arises from E by **(C)**, then

$$C^*(E^\dagger) \sim_{\mathrm{ME}} C^*(E)$$

provided E was (gauge) simple. But the following is open:

Question

Does $C^*(E^\dagger) \sim_{\mathrm{ME}} C^*(E)$ hold true for any graph E?

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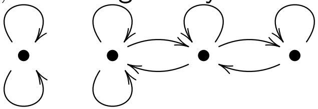
Affirmative answers when

- E is essential and finite (Rørdam)
- E_0 is finite (E/Restorff/Ruiz/Sørensen)
- $C^*(E)$ has at most one non-trivial ideal (E/Tomforde)
- $C^*(E)$ is purely infinite and has a finite number of ideals (Bentmann/Meyer)

The situation is graver for the Leavitt path algebra case. Returning to the two graphs $E, F = E^\dagger$ given by



we must ask

Question

Is $L_{\mathsf{k}}(E) \sim_{\mathrm{ME}} L_{\mathsf{k}}(E^\dagger)$?

Outline

Simple graphs

Let $\mathcal{G}_s[n]$ denote the set of simple graphs with n vertices.

n	$\lvert\mathcal{G}_s[n]\rvert$	$\lvert\mathcal{G}_s[n]/\sim_{C^*}\rvert$	$\lvert\mathcal{G}_s[n]/\sim_{\mathrm{LPA}}\rvert$
1	2	2	2
2	10	8	8
3	104	35	35
4	3044	206	{206,207,208,209}

	C^*-Morita equivalent	LPA-Morita equivalent
(S)	$\surd$	$\surd$
(O)	$\surd$	$\surd$
(I)	$\surd$	$\surd$
(R)	$\surd$	$\surd$
(C)	$(\surd)$	?

Definition

$E \sim_m F$ when there is a finite sequence of moves of type

(S),**(R)**,**(O)**,**(I)**

and their inverses, leading from E to F.

Definition

$E \sim_M F$ when there is a finite sequence of moves of type

(S),**(R)**,**(O)**,**(I)**,**(C)**

and their inverses, leading from E to F.

Key questions

Geometric classification

1. Which equivalence relation $\sim_{C^*}$ is induced on $\mathcal{G}$ by

$$C^*(E) \sim_{\mathrm{ME}} C^*(F)?$$

2. Which equivalence relation $\sim_{\mathrm{LPA}}$ is induced on $\mathcal{G}$ by

$$L_{\mathbb{C}}(E) \sim_{\mathrm{ME}} L_{\mathbb{C}}(F)?$$

Could the answer be $\sim_M$? It is finer as seen above, and Restorff proved that $\sim_{C^*} = \sim_M$ for finite essential graphs with condition (K).

Lemma (Basic move)

When $A \geq 0$ *with* $a_{ij} > 0$ *we have that* $\mathsf{X}_A \sim_{\mathrm{FE}} \mathsf{X}_{A^{(ij)}}$ *where*

$$A^{(ij)} = \begin{bmatrix} a_{11} & \dots & a_{1j} & \dots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{i1} + a_{j1} & \dots & a_{ij} + a_{jj} - 1 & \dots & a_{in} + a_{jn} \\ \vdots & & \vdots & & \vdots \\ a_{n1} & \dots & a_{nj} & \dots & a_{nn} \end{bmatrix}$$

Lemma (Row addition)

When A is the adjacency matrix of E with $a_{ij} + a_{jj} > 0$ we have that $E \sim_m E'$ where

$$A^{(ij)} = \begin{bmatrix} a_{11} & \dots & a_{1j} & \dots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{i1} + a_{j1} & \dots & a_{ij} + a_{jj} - 1 & \dots & a_{in} + a_{jn} \\ \vdots & & \vdots & & \vdots \\ a_{n1} & \dots & a_{nj} & \dots & a_{nn} \end{bmatrix}$$

is the adjacency matrix for E', provided that

- *there is a path in E from v_i to v_j*
- *v_j is regular*

Lemma (Column addition)

When A is the adjacency matrix of E with $a_{ji} + a_{jj} > 0$ we have that $E \sim_m E'$ where

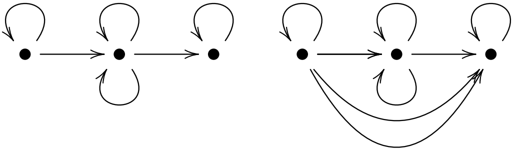
$$A^{(ij)} = \begin{bmatrix} a_{11} & \dots & a_{1i} + a_{1j} & \dots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{j1} & \dots & a_{ji} + a_{jj} - \delta^\bullet(j) & \dots & a_{jn} \\ \vdots & & \vdots & & \vdots \\ a_{n1} & \dots & a_{ni} + a_{nj} & \dots & a_{nn} \end{bmatrix}$$

is the adjacency matrix for E', provided that

- *there is a path in E from v_j to v_i*
- $\begin{bmatrix} a_{j1} & \dots & a_{ji} + a_{jj} - \delta^\bullet(j) & \dots & a_{jn} \end{bmatrix}$ *is not zero*

where $\delta^\bullet(j) = 1$ precisely when v is regular.

Let E and F be the graphs:



Theorem (E-Ruiz-Sørensen)

$E \not\sim_M F$, *yet*

$$C^*(E) \sim_{\mathrm{ME}} C^*(F)$$

Outline

Definition

A C^*-algebra $\mathfrak{A}$ has *real rank zero* if the invertible elements in $\mathfrak{A}_{\mathsf{sa}}$ are dense.

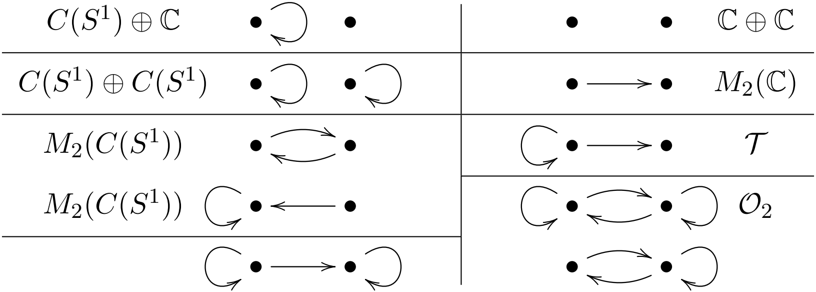
Proposition

The following are equivalent

1. $C^*(E)$ has real rank zero
2. If $v \in E_0$ supports a simple cycle, is supports another

The condition on the graph is called *condition (K)*.

$n = 2$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$M_2(\mathbb{C})$
$M_2(C(S^1))$
$\mathcal{T}$
$M_2(C(S^1))$
$\mathcal{O}_2$

Definition

- $V \subseteq E^0$ is **hereditary** when $s(e) \in V \implies r(e) \in V$
- $V \subseteq E^0$ is **saturated** when for every regular v
  $r(s^{-1}(v)) \subseteq V \implies v \in V$
- $v \in V$ is **breaking** for a hereditary and saturated set V when

$$|s^{-1}(v) \cap V| = \infty$$

  and

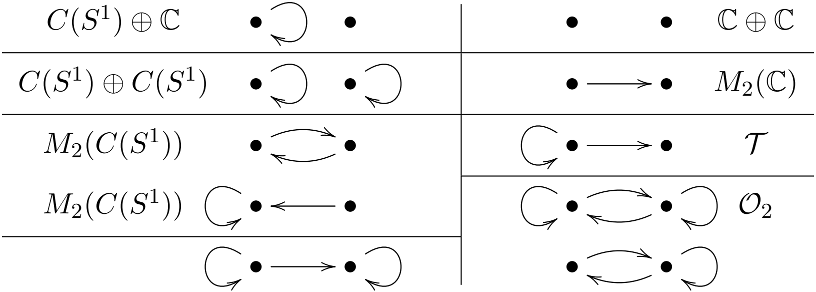
$$0 < |s^{-1}(v) \backslash V| < \infty$$

Theorem

When E has no breaking vertices, there is a $1-1$ correspondence between the gauge invariant ideals of $C^*(E)$ and hereditary and saturated subsets of E^0.

- Drinen-Tomforde singularization allows us to replace any E with E' so that $C^*(E) \sim_{\mathrm{ME}} C^*(E')$ and E' has no breaking vertices.
- When E has only finitely many vertices there is another procedure to obtain this, having also E' with finitely many vertices and $E \sim_m E'$.
- This tells us that when E^0 is finite, condition (K) implies that there are only finitely many ideals in $C^*(E)$.
- When $\mathfrak{I}$ is given by V we have $C^*(E)/\mathfrak{I} \simeq C^*(E \backslash V)$ and $\mathfrak{I} \sim_{\mathrm{ME}} C^*(V)$

$n = 2$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Outline

Filtered K-theory

Definition

Let $\mathfrak{A}$ be a C^*-algebra with only finitely many ideals. The collection of all sequences

$$\begin{array}{ccccc} K_0(\mathfrak{J}/\mathfrak{I}) & \longrightarrow & K_0(\mathfrak{K}/\mathfrak{I}) & \longrightarrow & K_0(\mathfrak{K}/\mathfrak{J}) \\ \uparrow & & & & \downarrow \\ K_1(\mathfrak{K}/\mathfrak{J}) & \longleftarrow & K_1(\mathfrak{K}/\mathfrak{I}) & \longleftarrow & K_1(\mathfrak{J}/\mathfrak{I}) \end{array}$$

with $\mathfrak{I} \triangleleft \mathfrak{J} \triangleleft \mathfrak{K} \triangleleft \mathfrak{A}$ is called the *filtered K-theory* of $\mathfrak{A}$ and denoted $FK(\mathfrak{A})$. Equipping all K_0-groups with order we arrive at the *ordered, filtered K-theory* $FK^+(\mathfrak{A})$.

There are similar definitions of $FK^\gamma(-)$, $FK^{\gamma,+}(-)$ where one only considers the gauge invariant ideals.

Definition

The **reduced** ordered, filtered K-theory $FK^{+,red}(\mathfrak{A})$ consists of

$$\begin{array}{ccccc} K_0(\mathfrak{I}) & \longrightarrow & K_0(\mathfrak{I}_0) & \longrightarrow & K_0(\mathfrak{I}_0/\mathfrak{I}) \\ \uparrow & & & & \\ K_1(\mathfrak{I}_0/\mathfrak{I}) & & & & \end{array}$$

with $\mathfrak{I}_0$ a smallest ideal properly containing a prime ideal $\mathfrak{I}$, along with

$$K_0(\mathfrak{I}_i) \to K_0(\mathfrak{I})$$

whenever $\mathfrak{I}, \mathfrak{I}_i$ are prime with $\mathfrak{I} = \bigcup_{i=1}^{n} \mathfrak{I}_i$.

There is a similar definition of $FK^{\gamma,+,red}(-)$ where one only considers the gauge invariant ideals.

Definition

The **tempered ideal space** of $\mathfrak{A}$ with finitely many ideals is the gauge invariant primitive ideal space $\mathrm{Prim}^\gamma(\mathfrak{A})$ equipped with a map

$$\tau : \mathrm{Prim}^\gamma(\mathfrak{A}) \to \mathbb{Z}$$

given by

$$\tau(\mathfrak{I}) = \begin{cases} -2 & \mathfrak{I}_0/\mathfrak{I} \text{ is not simple} \\ -1 & \mathfrak{I}_0/\mathfrak{I} \text{ is } AF \\ \operatorname{rank} K_0(\mathfrak{I}_0/\mathfrak{I}) - \operatorname{rank} K_1(\mathfrak{I}_0/\mathfrak{I}) & \text{otherwise} \end{cases}$$

when $\mathfrak{I}_0$ is the smallest ideal of $\mathfrak{A}$ containing $\mathfrak{I}$ properly.

Theorem (E-Ruiz-Sørensen)

Let E and F be finite graphs with heredity of negative temperatures. Then the following are equivalent

(i) $C^*(E) \sim_{\mathrm{ME}} C^*(F)$

(ii) $E \sim_M F$

(iii) $\tau_E = \tau_F$ *and* $FK^{\gamma,+,red}(C^*(E)) \simeq FK^{\gamma,+,red}(C^*(F))$

(iv) $FK^{\gamma,+}(C^*(E)) \simeq FK^{\gamma,+}(C^*(F))$

The example given above shows that the condition is necessary. It remains possible that (i)$\Longleftrightarrow$ (iv).

Theorem (E-Restorff-Ruiz-Sørensen)

Let $C^*(E)$ and $C^*(F)$ be unital graph algebras with real rank zero. Then the following are equivalent

(i) $C^*(E) \sim_{\mathrm{ME}} C^*(F)$

(ii) $E \sim_M F$

(iii) $\tau_E = \tau_F$ *and* $FK^{+,red}(C^*(E)) \simeq FK^{+,red}(C^*(F))$

(iv) $FK^+(C^*(E)) \simeq FK^+(C^*(F))$

Outline

Question

Suppose $\mathcal{C}[X]$ is a family of C^*-algebras with real rank zero and primitive ideal space X, so that it is known that K_* i(with order) s a complete invariant for all simple subquotients of $\mathfrak{A} \in \mathcal{C}$.
When can we conclude that $FK^+(-)$ is a complete invariant for the $\mathfrak{A}$'s themselves?

Working conjecture (E-Restorff-Ruiz)

FK^+ is a complete invariant for all graph C^*-algebras with finitely many ideals.

Status quo

$FK^+(-)$ is known to be a complete invariant for graph C^*-algebras over X when

- $|X| = 2$ (E-Tomforde)
- $|X| = 3$ and all K-groups are finitely generated (E/Restorff/Ruiz)
- $|X| = 4$ and the graph C^*-algebra is purely infinite (Arklint/Bentmann/Katsura,Arklint/Restorff/Ruiz)