Classification of C^*-algebras, flow equivalence of shift spaces, and graph and Leavitt path algebras

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Lecture 3
May 13, 2015

Content

Theorem (Boyle/Huang, Boyle)

Let X_A and X_B be reducible edge shifts with isomorphic colored partial order given by their irreducible components. Then $\mathsf{X}_A \sim_{\mathrm{FE}} \mathsf{X}_B$ in a way preserving the given isomorphism precisely when there exist block SL matrices U, V such that

$$U(I - A')V = I - B'$$

where $\mathsf{X}_{A'} \sim_{\mathrm{FE}} \mathsf{X}_A$ and $\mathsf{X}_{B'} \sim_{\mathrm{FE}} \mathsf{X}_B$ are prepared on the form

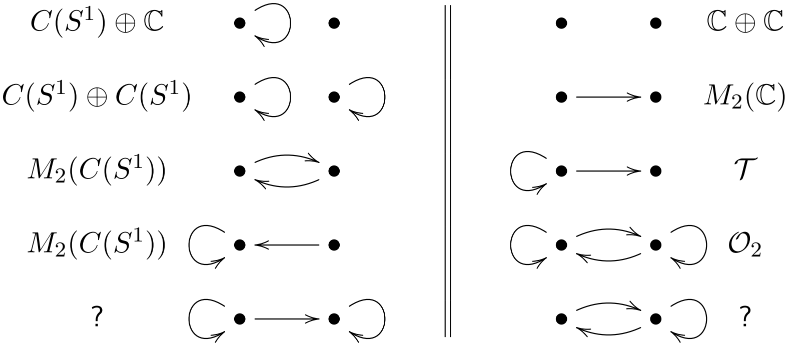
- *Any irreducible component which is a single cycle has only one vertex*
- *Any irreducible component which is not a single cycle has positive entries and at least two more vertices than there are summands in the Bowen-Franks group*

Key remarks

- Flow equivalence, i.e. existence of such SL block matrices, is decidable, but not practically so.
- The proof constructively replaces U and V by a sequence of row/column additions/subtractions.
- A Franks' standard form is not useful in the general case.

Outline

$n = 2$



$C(S^1) \oplus \mathbb{C}$
$C(S^1) \oplus C(S^1)$
$M_2(C(S^1))$
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?
$\mathbb{C} \oplus \mathbb{C}$
$M_2(\mathbb{C})$
$\mathcal{T}$
$\mathcal{O}_2$
?

Theorem (Cuntz/Krieger)

If E and F are essential and finite graphs, then

$$\mathsf{X}_E \sim_{\mathrm{FE}} \mathsf{X}_F \Longrightarrow C^*(E) \sim_{\mathrm{ME}} C^*(F)$$

Proposition

Let E be an essential and finite graph. If $E^\sharp$ arises from E by an out-splitting, we have

$$C^*(E^\sharp) \simeq C^*(E)$$

We outsplit E by partitioning

$$s^{-1}(v) = \mathcal{E}_v^1 \sqcup \mathcal{E}_v^2 \sqcup \cdots \sqcup \mathcal{E}_v^{n(v)}$$

Then we have

$$\begin{aligned}
(E^\sharp)^0 &= \{v^i \mid v \in E^0, i = 1, \dots, n(v)\} \\
(E^\sharp)^1 &= \{e^i \mid e \in E^1, i = 1, \dots, n(r(e))\} \\
r(e^i) &= r(e)^i \\
s(e^i) &= s(e)^{\mathcal{E}(e)} \text{ when } e \in \mathcal{E}_{s(e)}^{\mathcal{E}(e)}
\end{aligned}$$

Proposition

Let E be an essential and finite graph. If $E^\sharp$ arises from E by an out-spliting, we have

$$C^*(E^\sharp) \simeq C^*(E)$$

Proof

We define $\varphi : C^*(E) \to C^*(E^\sharp)$ by

$$\varphi(p_v) = \sum_{i=1}^{n(v)} p_{v^i} \qquad \varphi(s_e) = \sum_{i=1}^{n(r(e))} s_{e^i}$$

The map is easily seen to be surjective, and it is injective by the GIUT.

Proposition

Let E be an essential and finite graph. If $E_\sharp$ arises from E by an out-spliting, we have

$$C^*(E_\sharp) \sim_{\mathrm{ME}} C^*(E)$$

We in-split E by partitioning

$$r^{-1}(v) = \mathcal{E}_1^v \sqcup \mathcal{E}_2^v \sqcup \cdots \sqcup \mathcal{E}_{n(v)}^v$$

Then we have

$$\begin{aligned}
(E_\sharp)^0 &= \{v_i \mid v \in E^0, i = 1, \dots, n(v)\} \\
(E_\sharp)^1 &= \{e_i \mid e \in E^1, i = 1, \dots, n(s(e))\} \\
r(e_i) &= r(e)_{\mathcal{E}(e)} \text{ when } e \in \mathcal{E}_{\mathcal{E}(e)}^{r(e)} \\
s(e_i) &= s(e)_i
\end{aligned}$$

Proposition

Let E be an essential and finite graph. If $E_\sharp$ arises from E by an in-spliting, we have

$$C^*(E_\sharp) \sim_{\mathrm{ME}} C^*(E)$$

Proof

We define $\varphi : C^*(E) \to C^*(E_\sharp)$ by

$$\varphi(p_v) = p_{v_1} \qquad \varphi(s_e) = \sum_{f \in s^{-1}(r(e))} s_{e_1} s_{f_{\mathcal{E}(e)}} s^*_{f_1}$$

As before, it is injective by the GIUT, and the image is $p_\sharp C^*(E_\sharp) p_\sharp$ with

$$p_\sharp = \sum_{v \in E^0} p_{v_1}$$

Theorem (Brown-Green-Rieffel)

When $\mathfrak{A}$ and $\mathfrak{B}$ are separable C^*-algebras, the following are equivalent

1. $\mathfrak{A} \otimes \mathbb{K} \simeq \mathfrak{B} \otimes \mathbb{K}$
2. *There exists a C^*-algebra $\mathfrak{D}$ and orthogonal full projections $p, q \in M(\mathfrak{D})$ with*

$$p\mathfrak{D}p \simeq \mathfrak{A} \qquad q\mathfrak{D}q \simeq \mathfrak{B}$$

3. *There exists an $\mathfrak{A} - \mathfrak{B}$ imprimitivity bimodule*

We say that $\mathfrak{A}$ and $\mathfrak{B}$ are *Morita equivalent* and write $\mathfrak{A} \sim_{\mathrm{ME}} \mathfrak{B}$ in this case. Note all of

$$\mathbb{C}, M_2(\mathbb{C}), M_3(\mathbb{C}), \dots, \mathbb{K}$$

are Morita equivalent.

Proposition

Let E be an essential and finite graph. If $\widetilde{E}$ arises from E by an edge expansion, we have

$$C^*(E) \sim_{\mathrm{ME}} C^*(\widetilde{E})$$

Extending the edge f gives the graph

$$\begin{aligned}
(\widetilde{E})^0 &= \{v^0 \mid v \in E^0\} \cup \{w_1\} \\
(\widetilde{E})^1 &= \{e^0 \mid e \in E^1 \backslash \{f\}\} \cup \{f^0, f^1\} \\
r(e^0) &= r(e)^0 \text{ when } e \neq f \\
s(e^0) &= s(e)^0 \text{ when } e \neq f \\
s(f^0) &= s(f)^0 \\
r(f^0) &= s(f^1) = w^1 \\
r(f^1) &= r(f)^0
\end{aligned}$$

Proposition

Let E be an essential and finite graph. If $\widetilde{E}$ arises from E by an edge expansion, we have

$$C^*(E) \sim_{\mathrm{ME}} C^*(\widetilde{E})$$

Proof

We define $\varphi : C^*(E) \to C^*(\widetilde{E})$ by

$$\varphi(p_v) = p_{v^0} \qquad \varphi(s_e) = s_{e^0} \qquad \varphi(s_f) = s_{f^0} s_{f^1}$$

When we define $\beta_z \in \mathrm{Aut}(C^*(\widetilde{E}))$ by

$$\beta_z(p_{v^0}) = p_{v^0} \qquad \beta_z(p_{w^0}) = p_{w^0}$$
$$\beta_z(s_{e^0}) = z s_{e^0} \qquad \beta_z(s_{f^0}) = z s_{f^0} \qquad \beta_z(s_{f^1}) = s_{f^1}$$

we get $\beta_z \circ \varphi = \varphi \circ \gamma_z$, so we can again conclude that it is injective by the GIUT. The image is $(1 - p_{w^1}) C^*(\widetilde{E})(1 - p_{w^1})$.

	Morita equivalence	$*$-isomorphism	Gauge invariance
Out-splitting	$\surd$	$\surd$	$\surd$
In-splitting	$\surd$	-	$\surd$
Edge expansion	$\surd$	-	-

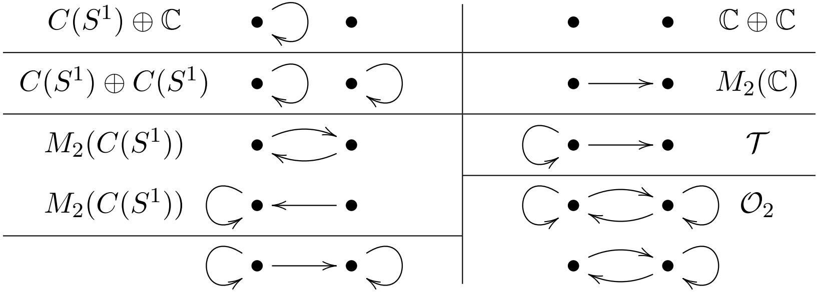
Flow equivalence is the coarsest equivalence relation on the set of edge shifts by essential graphs containing in-splitting, out-splitting. edge expansion, isomorphism of graphs.

Theorem (Cuntz/Krieger)

If E and F are essential and finite graphs, then

$$\mathsf{X}_E \sim_{\mathrm{FE}} \mathsf{X}_F \Longrightarrow C^*(E) \sim_{\mathrm{ME}} C^*(F)$$

$n = 2$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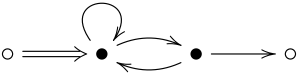
Outline



$$\mathsf{A}_E = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & \infty & 0 \end{bmatrix}$$

$$\mathsf{A}_E^\bullet = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} \quad \mathsf{A}_E^\circ = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \infty & 0 \end{bmatrix}$$

A subtler invariant

K-theory for C^*-algebras is invariant for Morita equivalence.

Formulas

With $\mathsf{A}_E^\bullet$ the regular part of the adjacency matrix of E and $I^\bullet$ the corresponding part of the identity matrix, we have

$$K_0(C^*(E)) = \operatorname{cok}(I^\bullet - \mathsf{A}_E^\bullet)^t$$
$$K_1(C^*(E)) = \ker(I^\bullet - \mathsf{A}_E^\bullet)^t$$

Key observation

$K_0(C^*(E))$ coincides with the Bowen-Franks group when E is essential and finite!

$$\mathsf{A}_E^{\bullet} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} \quad \mathsf{A}_E^{\circ} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \infty & 0 \end{bmatrix}$$

$$I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad I^{\bullet} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad I^{\circ} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$(I^{\bullet} - \mathsf{A}_E^{\bullet})^t = \begin{bmatrix} -1 & 0 \\ 1 & -1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 \\ 0 & -1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Outline

Definition

A *Kirchberg algebra* is

- simple
- purely infinite
- nuclear
- separable

Definition

A C^*-algebra is said to be AF (approximately finite) if it is the inductive limit of finite-dimensional C^*-algebras.

Trichotomy

Theorem

If a graph C^*-algebra has no non-trivial gauge invariant ideals, it is either

1. *an AF algebra;*
2. *a Kirchberg algebra; or*
3. *$C(\mathbb{T}) \otimes \mathbb{K}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$.*

The first case occurs when the graph has no cycles; the second when one vertex supports several cycles.

Theorem (Elliott)

When $\mathfrak{A}$ and $\mathfrak{B}$ are AF algebras, then

$$K_0(\mathfrak{A}) \simeq K_0(\mathfrak{B}) \Longleftrightarrow \mathfrak{A} \sim_{\mathrm{ME}} \mathfrak{B}$$

Theorem (Kirchberg-Phillips)

When $\mathfrak{A}$ and $\mathfrak{B}$ are Kirchberg algebras with the UCT, then

$$K_*(\mathfrak{A}) \simeq K_*(\mathfrak{B}) \Longleftrightarrow \mathfrak{A} \sim_{\mathrm{ME}} \mathfrak{B}$$

Theorem

If two graph C^*-algebras $C^*(E)$ and $C^*(F)$ have no non-trivial gauge invariant ideals, we have

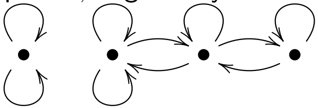
$$K_*(C^*(E)) \simeq K_*(C^*(F)) \Longleftrightarrow C^*(E) \sim_{\mathrm{ME}} C^*(F)$$

One needs to use the order of K_0 to distinguish between the three cases in the trichotomy, which is easy since

1. $\mathfrak{A}$ an AF algebra: $K_0(\mathfrak{A})_+ \neq K_0(\mathfrak{A})$, $K_1(\mathfrak{A}) = 0$.
2. $\mathfrak{A}$ a Kirchberg algebra: $K_0(\mathfrak{A})_+ = K_0(\mathfrak{A})$
3. $\mathfrak{A} = C(\mathbb{T}) \otimes \mathbb{K}(H)$: $K_0(\mathfrak{A})_+ \neq K_0(\mathfrak{A})$, $K_1(\mathfrak{A}) \neq 0$.

$\mathcal{O}_2$ versus $\mathcal{O}_2^-$

Consider the two graphs E, F given by



We have

$$\mathbb{Z}/(I - \mathsf{A}_E)\mathbb{Z} = 0 = \mathbb{Z}^2/(I - \mathsf{A}_F)\mathbb{Z}^2$$

but

$$\det(I - \mathsf{A}_E) = -1 \neq 1 = \det(I - \mathsf{A}_F).$$

Observation (Cuntz/Rørdam)

$E \not\sim_{\mathrm{FE}} F$, yet $C^*(E) \sim_{\mathrm{ME}} C^*(F)$.

Historical remark/motivation

The classification of simple graph C^*-algebras associated to essential and finite graphs by Rørdam predates the Kirchberg-Phillips theorem and provided several clues to its proof.

Outline

Moves

Move **(S)**

Remove a regular source, as

Move **(R)**

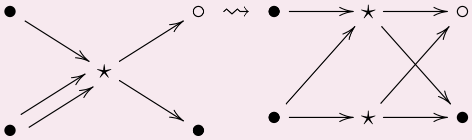
Reduce a configuration with a transitional regular vertex, as

or

Moves

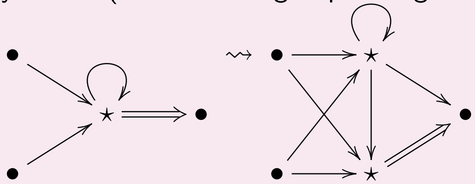
Move **(I)**

Insplit at regular vertex



Move (O)

Outsplit at any vertex (at most one group of edges infinite)



Move **(C)**

"Cuntz splice" on a vertex supporting two cycles

Theorem (Sørensen)

Let E and F be graphs with finitely many elements so that $C^*(E)$ and $C^*(F)$ are gauge simple. Then the following are equivalent

1. $C^*(E) \sim_{\mathrm{ME}} C^*(F)$
2. $K_*(C^*(E)) \simeq K_*(C^*(F))$
3. *There is a finite sequence of moves of type*

   (S),**(R)**,**(O)**,**(I)**,**(C)**

   and their inverses, leading from E to F.

Outline

A MASA $\mathfrak{D}$ in a C^*-algebra $\mathfrak{A}$ is a maximal abelian subalgebra (commutative and not contained in any larger such subalgebra). We reserve the notation $\mathfrak{D} \hookrightarrow \mathfrak{A}$ for this situation.

Example

Consider the elements in $\mathbb{K}(H)$ as infinite matrices given by some orthonormal basis for H. Then the entries in each diagonal matrix must tend to zero, and we get

$$c_0 \hookrightarrow \mathbb{K}$$

Example

With

$$\mathfrak{D}_E = C^*(s_\alpha s_\alpha^* \mid \alpha \text{ a path in } E) \subset C^*(E)$$

we get when every cycle has an exit that

$$\mathfrak{D}_E \hookrightarrow C^*(E)$$

Let now E again be essential and finite. In this case we have in fact $\mathfrak{D}_E \simeq C(\mathsf{X}_E)$.

	Morita equiv.	$*$-isom.	Gauge invariance	$\mathfrak{D}$-preserving
Out-splitting	$\surd$	$\surd$	$\surd$	$\surd$
In-splitting	$\surd$	-	$\surd$	$\surd$
Edge expansion	$\surd$	-	-	$\surd$
Cuntz splice	$\surd$	-	-	-

Observation

When $\mathsf{X}_E \sim_{\mathrm{FE}} \mathsf{X}_F$, we have

$$\begin{array}{ccc} C^*(E) \otimes \mathbb{K} & \xrightarrow{\simeq} & C^*(F) \otimes \mathbb{K} \\ \big\uparrow & & \big\uparrow \\ \mathfrak{D}_E \otimes c_0 & \xrightarrow[\simeq]{} & \mathfrak{D}_F \otimes c_0 \end{array}$$

Theorem (Matsumoto-Matui)

With E and F presenting irreducible SFTs, the following are equivalent

(i) $\mathsf{X}_E \sim_{\mathrm{FE}} \mathsf{X}_F$

(ii)
$$\begin{array}{ccc} C^*(E)\otimes\mathbb{K} & \xrightarrow{\ \simeq\ } & C^*(F)\otimes\mathbb{K} \\ \big\uparrow & & \big\uparrow \\ \mathfrak{D}_E\otimes c_0 & \xrightarrow[\ \simeq\ ]{} & \mathfrak{D}_F\otimes c_0 \end{array}$$

By the results already described, we "just" need to prove that (ii) implies that $\operatorname{sgn}\det(1-\mathsf{A}_E)=\operatorname{sgn}\det(1-\mathsf{A}_F)$. This goes via a result of Boyle/Handelman on ordered cohomology.