# Classification of $C^*$-algebras, flow equivalence of shift spaces, and graph and Leavitt path algebras

Søren Eilers
eilers@math.ku.dk

Department of Mathematical Sciences
University of Copenhagen

Lecture 2
May 12, 2015

# Content

# Outline

# Key definitions

Let $\mathfrak{a}$ be a finite set and equip $\mathfrak{a}^{\mathbb{Z}}$ with the product topology based on the discrete topology on $\mathfrak{a}$.

## Definition

A **shift space** is a subset $X$ of $\mathfrak{a}^{\mathbb{Z}}$ which is closed and closed under the **shift map**

$$\sigma : \mathfrak{a}^{\mathbb{Z}} \to \mathfrak{a}^{\mathbb{Z}} \qquad \sigma((x_i)) = (x_{i+1})$$

## Definition

When $E = (E^0, E^1, r, s)$ is a finite graph, $\mathsf{X}_E$ denotes the *edge shift*

$$\mathsf{X}_E = \{(e_i) \in (E^1)^{\mathbb{Z}} \mid r(e_i) = s(e_{i+1})\}$$

It is customary and convenient to think of $E = E_A$ as defined by an adjacency matrix $A$ and abbreviate $\mathsf{X}_A = \mathsf{X}_{E_A}$.

# Essential graphs

Obviously, sinks and sources do not contribute to the edge shifts, so we try to avoid these.

**Definition**

$E$ is *essential* if it contains no sinks and no sources.

| | | |
|---|---|---|
| 1 | 2 | 1 |
| 2 | 10 | 5 |
| 3 | 104 | 55 |
| 4 | 3044 | 1918 |

# Conjugacy

**Definition**

$X$ and $Y$ are *conjugate*, written $X \simeq Y$, if there exists a bijection $\varphi : X \to Y$ which is a homeomorphism and satisfies

$$\sigma \circ \varphi = \varphi \circ \sigma$$

The *shifts of finite type* (SFTs) are the shift spaces conjugate to edge shifts.

# Easy invariants

## Definition

A shift space $X$ is *irreducible* when for some $x \in X$,

$$\{\sigma^n(x) \mid n \in \mathbb{N}\}$$

is dense in $X$.

## Observation

Let $X$ and $Y$ be conjugate shift spaces.

- If $X$ is finite, so is $Y$.
- If $X$ is irreducible, so is $Y$.

Note that $\mathsf{X}_A$ is finite precisely when $E_A$ is a union of disjoint cycles, and that $\mathsf{X}_A$ is irreducible precisely when $E_A$ is strongly connected.

# Flow equivalence

Associated to any shift space there is a **suspension flow** given by product topology on

$$SX = \frac{X \times \mathbb{R}}{(x,t) \sim (\sigma(x), t+1)}$$

## Definition

$X$ and $Y$ are *flow equivalent* (written $X \sim_{\mathrm{FE}} Y$) when $SX$ and $SY$ are homeomorphic in a way preserving direction in $\mathbb{R}$.

# Symbol expansion

Fix $a \in \mathfrak{a}$ and $\star \notin \mathfrak{a}$ and define $\eta : \mathfrak{a}^{\mathbb{Z}} \to (\mathfrak{a} \cup \{\star\})^{\mathbb{Z}}$ as the map inserting a $\star$ after each $a$:

$$\cdots babbbaba \cdots \qquad \mapsto \qquad \cdots ba \star bbba \star ba \star \cdots$$

## Definition

The "$a \mapsto a\star$" symbol expansion of a shift space $X$ is the shift space

$$X_{a \mapsto a\star} = \eta(X) \cup \sigma(\eta(X)).$$

## Lemma

$X \sim_{\mathrm{FE}} X_{a \to a\star}$

## Proof idea

$$\varphi([x,t]) = \begin{cases} [\eta(x), 2t] & x_0 = a, t \in [0, 1/2] \\ [\sigma(\eta(x)), 2t-1] & x_0 = a, t \in [1/2, 1] \\ [\eta(x), t] & x_0 \neq a \end{cases}$$

# Key result

**Theorem (Parry-Sullivan)**

*Flow equivalence is the coarsest equivalence relation containing conjugacy and* $X \sim X_{a \to a\star}$

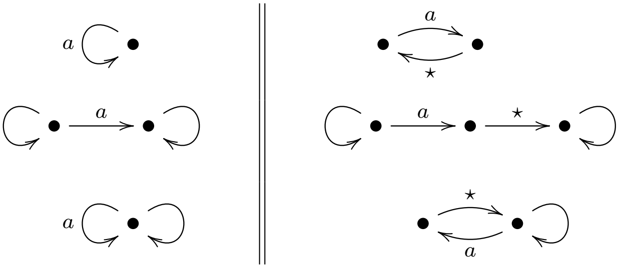
**Observation**

Let $X \sim_{\mathrm{FE}} Y$.

- If $X$ is finite, so is $Y$.
- If $X$ is irreducible, so is $Y$.

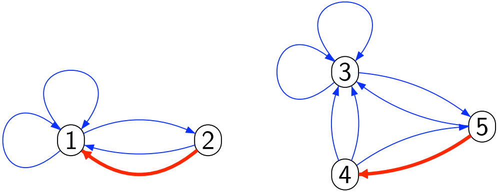
# Edge expansion

Note how symbol expansion takes the form of *edge expansion* for edge shifts:



# Outline

# State splitting




$$\begin{bmatrix}2 & 1\\2 & 0\end{bmatrix} = \begin{bmatrix}2 & 0 & 1\\1 & 1 & 0\end{bmatrix}\begin{bmatrix}1 & 0\\1 & 0\\0 & 1\end{bmatrix} \qquad \begin{bmatrix}2 & 0 & 1\\2 & 0 & 1\\1 & 1 & 0\end{bmatrix} = \begin{bmatrix}1 & 0\\1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}2 & 0 & 1\\1 & 1 & 0\end{bmatrix}$$

# Key result

## Definition

Two matrices $A$ and $B$ are *elementary equivalent* if there exist (possibly rectangular) matrices $D, E$ with entries in $\mathbb{N}_0$ so that

$$DE = A \qquad ED = B$$

## Definition

*Strong shift equivalence* is the coarsest equivalence relation containing elemenary equivalence.

## Theorem (Williams)

*Two edge shifts* $\mathsf{X}_A$ *and* $\mathsf{X}_B$ *given by essential matrices are conjugate precisely when* $A$ *and* $B$ *are strong shift equivalent.*

# Key result

## Theorem (Williams)

*Conjugacy is coarsest equivalence relation on the set of edge shifts by essential graphs containing in-splitting, out-splitting and isomorphism of graphs.*

## Corollary

Flow equivalence is the coarsest equivalence relation on the set of edge shifts by essential graphs containing

1. in-splitting
2. out-splitting
3. edge expansion
4. isomorphism of graphs

# Outline

## Definition

Let $A$ be the adjacency matrix of an essential graph with $n$ vertices. The *Bowen-Franks* invariant of $A$ is the pair

$$\mathrm{BF}(A) = [\mathbb{Z}^n/(\mathrm{Id} - A)\mathbb{Z}^n, \mathrm{sgn}(\det(\mathrm{Id} - A))]$$

## Observation

When $\mathsf{X}_A \sim_{\mathrm{FE}} \mathsf{X}_B$, $\mathrm{BF}(A) = \mathrm{BF}(B)$

# Flow classification of SFTs

**Theorem (Franks)**

*Let $\mathsf{X}_A$ and $\mathsf{X}_B$ be two irreducible and infinite SFTs given by graphs with essential adjacency matrices $A$ and $B$, respectively. The following conditions are equivalent.*

(i) $\mathsf{X}_A \sim_{\mathrm{FE}} \mathsf{X}_B$

(ii) $\mathrm{BF}(A) \simeq \mathrm{BF}(B)$

# Proof idea

## Lemma (Basic move)

*When* $A \geq 0$ *with* $a_{ij} > 0$ *we have that* $\mathsf{X}_A \sim_{\mathrm{FE}} \mathsf{X}_{A^{(ij)}}$ *where*

$$A^{(ij)} = \begin{bmatrix} a_{11} & \dots & a_{1j} & \dots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{i1} + a_{j1} & \dots & a_{ij} + a_{jj} - 1 & \dots & a_{in} + a_{jn} \\ \vdots & & \vdots & & \vdots \\ a_{n1} & \dots & a_{nj} & \dots & a_{nn} \end{bmatrix}$$

## Step 1

Outsplit to go

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \to \begin{bmatrix} 0 & 0 & 1 \\ a_{11} & a_{11} & a_{12}-1 \\ a_{21} & a_{21} & a_{22} \end{bmatrix}$$

## Step 2

Insplit to go

$$\begin{bmatrix} 0 & 0 & 1 \\ a_{11} & a_{11} & a_{12}-1 \\ a_{21} & a_{21} & a_{22} \end{bmatrix} \to \begin{bmatrix} 0 & 0 & 1 & 0 \\ a_{11} & a_{11} & 0 & a_{12}-1 \\ a_{21} & a_{21} & 0 & a_{22} \\ a_{21} & a_{21} & 0 & a_{22} \end{bmatrix}$$

## Step 3

Symbol reduce to go

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ a_{11} & a_{11} & 0 & a_{12}-1 \\ a_{21} & a_{21} & 0 & a_{22} \\ a_{21} & a_{21} & 0 & a_{22} \end{bmatrix} \rightarrow \begin{bmatrix} a_{11} & a_{11} & a_{12}-1 \\ a_{21} & a_{21} & a_{22} \\ a_{21} & a_{21} & a_{22} \end{bmatrix}$$

## Step 4

Out-amalgamate to go

$$\begin{bmatrix} a_{11} & a_{11} & a_{12}-1 \\ a_{21} & a_{21} & a_{22} \\ a_{21} & a_{21} & a_{22} \end{bmatrix} \rightarrow \begin{bmatrix} a_{11}+a_{21} & a_{12}+a_{22}-1 \\ a_{21} & a_{22} \end{bmatrix}$$

## Proposition

For any $A \geq 0$ there is a $B \geq 0$ such that

$$\mathsf{X}_A \sim_{\mathrm{FE}} \mathsf{X}_{I+B}$$

## Proof

If all $a_{jj} > 0$ we are done. If not, employ that

$$A^{(ij)} = \begin{bmatrix} a_{11} & \dots & a_{1j} & \dots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{i1} + a_{j1} & \dots & a_{ij} - 1 & \dots & a_{in} + a_{jn} \\ \vdots & & \vdots & & \vdots \\ a_{n1} & \dots & a_{nj} & \dots & a_{nn} \end{bmatrix}$$

to create a zero column, which may then be deleted.

## Proposition

If a row or column addition takes an irreducible matrix $B \geq 0$ to $B' \geq 0$, we have

$$\mathsf{X}_{I+B} \sim_{\mathrm{FE}} \mathsf{X}_{I+B'}$$

## Proof

Suppose row 2 of $B$ is added to row 1 to create $B'$. The first row of $I + B'$ is

$$\begin{bmatrix} 1 + b_{11} + b_{21} & b_{12} + b_{22} & b_{13} + b_{23} & \dots \end{bmatrix}$$

and the first two rows of $I + B$ are

$$\begin{bmatrix} 1 + b_{11} & b_{12} & b_{13} & \dots \\ b_{21} & 1 + b_{22} & b_{23} & \dots \end{bmatrix}$$

Note how this coincides with "basic move" when $b_{12} > 0$. In general, use irreducibility.

## Proposition

Let an irreducible matrix $B \geq 0$ be of size $n \times n$ with $n > 1$. Then

$$\mathsf{X}_{I+B} \sim_{\mathrm{FE}} \mathsf{X}_{I+C}$$

where we may assume that $C > 0$ of any size $m \geq n$.

## Proof

We may keep adding rows until all entries are $\geq N$ for any $N > 0$. New rows may be added as required by state splitting as soon as the entries are sufficiently large.

## Proposition

When $C > 0$ we have $\mathsf{X}_{I+C} \sim \mathsf{X}_{I+D}$ where the first column of $D$ is identically $d$, with

$$d = \gcd\{c_{ij}\} = \gcd\{d_{ij}\}$$

## Proof

Subsequent "column prepared row subtractions" and "row prepared column subtractions".

## Standard form 1

When $C > 0$ is a given $n \times n$-matrix with $\mathbb{Z}^n / C\mathbb{Z}^n = \sum_{i=1}^n \mathbb{Z}/d_i\mathbb{Z}$ where

$$d_1 \mid d_2 \mid \cdots \mid d_n$$

and $\det(-C) = (-1)^n \det(C) < 0$ we have that $\mathsf{X}_{I+C} \sim \mathsf{X}_{I+D}$ where

$$D = \begin{bmatrix} 0 & & \dots & 0 & d_n \\ d_1 & 0 & & 0 & 0 \\ 0 & d_2 & & 0 & 0 \\ & & \ddots & & \vdots \\ 0 & & \dots & d_{n-1} & 0 \end{bmatrix}$$

## Standard form 2

When $C > 0$ is a given $n \times n$-matrix with $\mathbb{Z}^n/C\mathbb{Z}^n = \sum_{i=1}^n \mathbb{Z}/d_i\mathbb{Z}$ where

$$d_1 \mid d_2 \mid \cdots \mid d_n$$

and $\det(-C) = (-1)^n \det(C) > 0$ we have that $\mathsf{X}_{I+C} \sim \mathsf{X}_{I+D}$ where

$$D = \begin{bmatrix} 0 & & \dots & d_{n-1} & d_{n-1} \\ d_1 & 0 & & 0 & 0 \\ 0 & d_2 & & 0 & 0 \\ & & \ddots & \vdots & \vdots \\ 0 & & \dots & d_{n-1} & d_{n-1} + d_n \end{bmatrix}$$

## Standard form 3

When $C > 0$ is a given $n \times n$-matrix with $\mathbb{Z}^n / C\mathbb{Z}^n = \sum_{i=1}^n \mathbb{Z}/d_i\mathbb{Z}$ where

$$d_1 \mid d_2 \mid \cdots \mid d_n$$

and $\operatorname{rank}(C) = k < n$ we have that $\mathsf{X}_{I+C} \sim \mathsf{X}_{I+D}$ where

$$D = \begin{bmatrix} 0 & & \dots & 0 & d_k & \dots & d_k \\ d_1 & 0 & & 0 & 0 & \dots & 0 \\ 0 & d_2 & & & 0 & & 0 \\ & & \ddots & \vdots & \vdots & & \\ & & & d_{k-1} & 0 & \dots & 0 \\ 0 & & \dots & 0 & d_k & \dots & d_k \\ \vdots & & & \vdots & \vdots & & \vdots \\ 0 & & \dots & 0 & d_k & \dots & d_k \end{bmatrix}$$

# Outline

We call a non-irreducible shift space reducible. Any reducible shift space can be analyzed into irreducible components which in turn define a partially ordered set where one component $C_1$ dominates another component $C_2$ when there is a path from some vertex in $C_1$ to some vertex in $C_2$.
When $X = \mathsf{X}_A$, we color those vertices that correspond to irreducible components that are single cycles and arrive at a colored partially ordered set $\mathcal{P}_A$.

**Observation**

When $\mathsf{X}_A \sim_{\mathrm{FE}} \mathsf{X}_B$, $\mathcal{P}_A \simeq \mathcal{P}_B$.

We can even associate the Bowen-Franks invariant to all the points in $\mathcal{P}_A$!

# Recall irreducible case

## Proposition

If a row or column addition takes an irreducible matrix $B \geq 0$ to $B' \geq 0$, we have

$$\mathsf{X}_{I+B} \sim_{\mathrm{FE}} \mathsf{X}_{I+B'}$$

## Proof

Suppose row 2 of $B$ is added to row 1 to create $B'$. The first row of $I + B'$ is

$$\begin{bmatrix} 1 + b_{11} + b_{21} & b_{12} + b_{22} & b_{13} + b_{23} & \dots \end{bmatrix}$$

and the first two rows of $I + B$ are

$$\begin{bmatrix} 1 + b_{11} & b_{12} & b_{13} & \dots \\ b_{21} & 1 + b_{22} & b_{23} & \dots \end{bmatrix}$$

Note how this coincides with "basic move" when $b_{12} > 0$...

### Proposition

If the addition of row or column $j$ to row or column $i$ takes an irreducible matrix $B \geq 0$ to $B' \geq 0$, we have

$$\mathsf{X}_{I+B} \sim_{\mathrm{FE}} \mathsf{X}_{I+B'}$$

when $B_{ij} > 0$

# Recall irreducible case

Assume that $B, B'$ are irreducible matrices both of size $n$. Then the following are equivalent

1. $\mathsf{X}_{I+B} \sim_{\mathrm{FE}} \mathsf{X}_{I+B'}$
2. $\mathrm{BF}(I+B) = \mathrm{BF}(I+B')$
3. There exist $SL$ matrices $U, V$ with

$$UBV = B'$$

## Theorem (Boyle-Huang, Boyle)

*Let $\mathsf{X}_A$ and $\mathsf{X}_B$ be reducible edge shifts with isomorphic colored partial order given by their irreducible components. Then $\mathsf{X}_A \sim_{\mathrm{FE}} \mathsf{X}_B$ in a way preserving the given isomorphism precisely when there exist block $SL$ matrices $U, V$ such that*

$$U(I - A')V = I - B'$$

*where $\mathsf{X}_{A'} \sim_{\mathrm{FE}} \mathsf{X}_A$ and $\mathsf{X}_{B'} \sim_{\mathrm{FE}} \mathsf{X}_B$ are prepared on the form*

- *Any irreducible component which is a single cycle has only one vertex*
- *Any irreducible component which is not a single cycle has positive entries and has at least two more vertices than there are summands in the Bowen-Franks group*