Classification of C^*-algebras, flow equivalence of shift spaces, and graph and Leavitt path algebras

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Content

Apologies/disclaimers/warnings

- I will be taking the easiest way in, avoiding all technicalities which may meaningfully be avoided.
- I am no expert on Leavitt path algebras (but plenty of people here are).
- The literature is
  - scattered;
  - sometimes non-existent;
  - inconsistent regarding notation.
- I need to exert stringent time discipline.

Outline

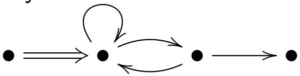
Definition

A graph is a tuple (E^0, E^1, r, s) with

$$r, s : E^1 \to E^0$$

and E^0 and E^1 countable sets.

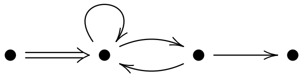
We think of $e \in E^1$ as an edge from $s(e)$ to $r(e)$ and often represent graphs visually



or by an adjacency matrix

$$\mathsf{A}_E = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \infty & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Infinite graphs



$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ \infty & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

- Note our use of the symbols $\Longrightarrow$ and ∞.
- A *finite graph* has $|E^0|, |E^1| < \infty$.
- A graph with $|E^0| < \infty$ but $|E^1| = \infty$ is infinite but has *finitely many vertices*.

Simple graphs

Definition

A graph is *simple* if it has no multiple edges

The simple graphs are precisely those whose adjacency matrices have entries in $\{0, 1\}$.
The number of simple graphs with n vertices grows quickly:

$n = 1$	2
$n = 2$	10
$n = 3$	104
$n = 4$	3044
$n = 5$	291968
$n = 6$	96928992
$n = 7$	112282908928

Singular and regular vertices

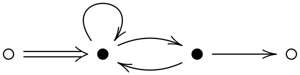
Definitions

Let E be a graph and $v \in E^0$.

- v is a *sink* if $|s^{-1}(\{v\})| = 0$
- v is a *source* if $|r^{-1}(\{v\})| = 0$
- v is an *infinite emitter* if $|s^{-1}(\{v\})| = \infty$
- v is a *infinite receiver* if $|r^{-1}(\{v\})| = \infty$

Definition

v is *singular* if v is a sink or an infinite emitter. v is *regular* if it is not singular.



Outline

C^*-algebras

Definition

A C^*-*algebra* is a complex Banach algebra with involution $a \mapsto a^*$ such that
$$\|aa^*\| = \|a\|^2$$

Key examples:

- $\mathbb{B}(\mathcal{H})$
- $\mathbb{K}(\mathcal{H})$
- $M_n(\mathbb{C})$
- $C(X)$, X compact Hausdorff
- $C_0(X)$, X locally compact Hausdorff

Rigidity

1. Any C^*-algebra is $*$-isomorphic to a sub-C^*-algebra of some $\mathbb{B}(\mathcal{H})$
2. Any commutative C^*-algebra is $*$-isomorphic to $C_0(X)$ or $C(X)$.
3. Any $*$-isomorphism is an isometry

Graph algebras

Definition

The *graph C^*-algebra* $C^*(E)$ is given as the universal C^*-algebra generated by $\{p_v : v \in E^0\}$ and $\{s_e : e \in E^1\}$ subject to:

- $p_v = p_v^2 = p_v^*$
- $s_e s_e^* s_e = s_e$
- $p_v p_w = 0$ when $v \neq w$
- $(s_e s_e^*)(s_f s_f^*) = 0$ when $e \neq f$
- $s_e^* s_e = p_{r(e)}$
- $s_e s_e^* \leq p_{s(e)}$
- $p_v = \sum_{s(e)=v} s_e s_e^*$ for every regular v

Graph algebras

Compressed definition

The *graph C^*-algebra* $C^*(E)$ is given as the universal C^*-algebra generated by mutually orthogonal projections $\{p_v : v \in E^0\}$ and partial isometries $\{s_e : e \in E^1\}$ with mutually orthogonal ranges subject to the Cuntz-Krieger relations

1. $s_e^* s_e = p_{r(e)}$
2. $s_e s_e^* \leq p_{s(e)}$
3. $p_v = \sum_{s(e)=v} s_e s_e^*$ for every regular v

$C^*(E)$ is unital when E has finitely many vertices.

Leavitt path algebras

Let k be a field.

Definition

The *Leavitt path algebra* $L_{\mathsf{k}}(E)$ is given as the universal k-algebra generated by mutually orthogonal idempotents $\{v : v \in E^0\}$ and elements $\{e, e^* : e \in E^1\}$ subject to the relations

1. $s(e)e = er(e) = e$
2. $r(e)e^* = e^*s(e) = e^*$
3. $e^*f = \delta_{e,f}r(e)$
4. $v = \sum_{s(e)=v} ee^*$ for every regular v

$L_{\mathsf{k}}(E)$ is unital when E has finitely many vertices.

Graph C^*-algebras versus LPAs

Theorem (Tomforde)

$C^*(E)$ contains a canonical dense copy of $L_{\mathbb{C}}(E)$

(i) $C^*(E) \simeq C^*(F)$

(ii) $L_{\mathbb{C}}(E) \simeq L_{\mathbb{C}}(F)$ as $*$-algebras

(iii) $\forall(\mathsf{k}, *) : L_{\mathsf{k}}(E) \simeq L_{\mathsf{k}}(F)$ as $*$-algebras

(iv) $L_{\mathbb{C}}(E) \simeq L_{\mathbb{C}}(F)$ as rings

(v) $\forall \mathsf{k} : L_{\mathsf{k}}(E) \simeq L_{\mathsf{k}}(F)$ as rings

Graph C^*-algebras versus LPAs

(i) $C^*(E) \simeq C^*(F)$

(ii) $L_{\mathbb{C}}(E) \simeq L_{\mathbb{C}}(F)$ as $*$-algebras

(iii) $\forall \mathsf{k} : L_{\mathsf{k}}(E) \simeq L_{\mathsf{k}}(F)$ as $*$-algebras

(iv) $L_{\mathbb{C}}(E) \simeq L_{\mathbb{C}}(F)$ as rings

(v) $\forall \mathsf{k} : L_{\mathsf{k}}(E) \simeq L_{\mathsf{k}}(F)$ as rings

(ii)$\Longrightarrow$ (i) by Tomforde's result, and clearly (iii)$\Longrightarrow$ (ii), (v)$\Longrightarrow$ (iv), (ii)$\Longrightarrow$ (iv), (iii)$\Longrightarrow$ (v)

Conjecture [Abrams-Tomforde]

(iv)$\Longrightarrow$ (i)

Outline

$E = \bullet$

$C^*(E) \simeq \mathbb{C}$

$\varphi : C^*(E) \to \mathbb{C}$ given by

$$\varphi(p_v) = 1$$

is a $*$-isomorphism.

Similarly, $L_{\mathsf{k}}(E) = \mathsf{k}$

$E =$ 

$C^*(E) \simeq C(S^1)$

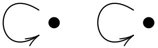
$\varphi : C^*(E) \to C(S^1)$ given by

$$\varphi(p_v) = 1 \qquad \varphi(s_e) = z$$

is a $*$-isomorphism.

Similarly, $L_{\mathsf{k}}(E) = \mathsf{k}[z, z^{-1}]$.

$n = 2$, Abelian cases

E	$C^*(E)$
	$\mathbb{C} \oplus \mathbb{C}$
	$C(S^1) \oplus \mathbb{C}$
	$C(S^1) \oplus C(S^1)$

The Cuntz-Krieger uniqueness theorem

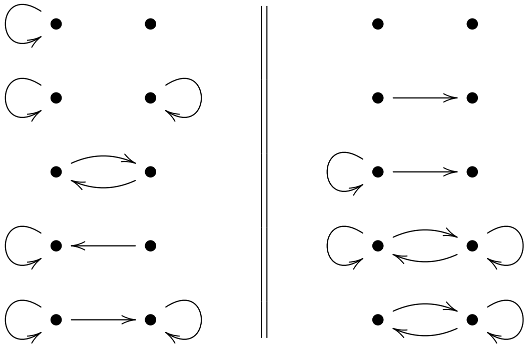
Theorem

Suppose $\varphi : C^*(E) \to \mathfrak{A}$ is a $*$-homomorphism with the property that

$$\forall v \in E^0 : \varphi(p_v) \neq 0$$

When E has the property that every cycle has an exit, φ is injective.

$n = 2$, applicability of CKUT



$E = v \xrightarrow{e} w$

$C^*(E) \simeq M_2(\mathbb{C})$

$\varphi : C^*(E) \to M_2(\mathbb{C})$ given by

$$\varphi(p_v) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\varphi(p_w) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\varphi(s_e) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

is a $*$-isomorphism.

$E = \overset{e}{\circlearrowright} v \xrightarrow{f} w$

The operator $S \in \mathbb{B}(\ell^2(\mathbb{N}_0))$ defined by

$$S(\xi_0, \xi_1, \xi_2, \xi_3, \dots) = (0, \xi_0, \xi_1, \xi_2, \xi_3, \dots)$$

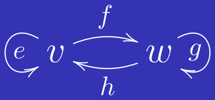
is called the unilateral shift. The *Toeplitz algebra* $C^*(S) = \mathcal{T}$ has universal properties.

$C^*(E) \simeq \mathcal{T}$

$\varphi : C^*(E) \to \mathbb{B}(\ell^2(\mathbb{N}_0))$ given by

$$\begin{aligned}
\varphi(p_v)(\xi_0, \xi_1, \xi_2, \xi_3, \dots) &= (0, \xi_1, \xi_2, \xi_3, \dots) \\
\varphi(p_w)(\xi_0, \xi_1, \xi_2, \xi_3, \dots) &= (\xi_0, 0, 0, 0, 0, \dots) \\
\varphi(s_e)(\xi_0, \xi_1, \xi_2, \xi_3, \dots) &= (0, \xi_0, 0, 0, 0, \dots) \\
\varphi(s_f)(\xi_0, \xi_1, \xi_2, \xi_3, \dots) &= (0, 0, \xi_1, \xi_2, \xi_3, \dots)
\end{aligned}$$

is injective and maps onto $C^*(S)$.

$E =$ 

Consider isometries $S_1, S_2 \in \mathbb{B}(\ell^2(\mathbb{N}_0))$ given by

$$S_1(\xi_0, \xi_1, \xi_2, \xi_3, \dots) = (0, \xi_0, 0, \xi_1, 0, \xi_2, 0, \xi_3, \dots)$$
$$S_2(\xi_0, \xi_1, \xi_2, \xi_3, \dots) = (\xi_0, 0, \xi_1, 0, \xi_2, 0, \xi_3, 0, \dots)$$

The Cuntz algebra $C^*(S_1, S_2) = \mathcal{O}_2$ has universal properties.

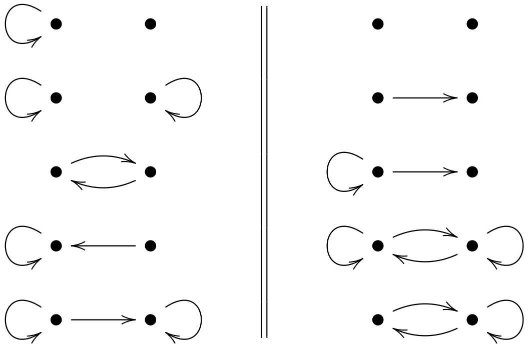
$C^*(E) \simeq \mathcal{O}_2$

$\varphi : C^*(E) \to \mathcal{O}_2$ given by

$$\begin{aligned} \varphi(p_v) &= S_1 S_1^* & \qquad \varphi(p_w) &= S_2 S_2^* \\ \varphi(s_e) &= S_1 S_1 & \qquad \varphi(s_f) &= S_1 S_2 \\ \varphi(s_g) &= S_2 S_2 & \qquad \varphi(s_h) &= S_2 S_1 \end{aligned}$$

is a $*$-isomorphism.

$n = 2$, applicability of CKUT



Gauge action

Observation

$$\gamma_z(p_v) = p_v \qquad \gamma_z(s_e) = zs_e$$

induces a **gauge action** $S^1 \mapsto \mathrm{Aut}(C^*(E))$

The action is strongly (i.e. point-norm) continuous.

The gauge invariant uniqueness theorem

Theorem

Suppose $\varphi : C^*(E) \to \mathfrak{A}$ *is a* $*$*-homomorphism with the property that*

$$\forall v \in E^0 : \varphi(p_v) \neq 0$$

When $\mathfrak{A}$ *also has a strongly continuous gauge action* β_z *which intertwines* φ *and* γ *in the sense that*

$$\forall z \in S^1 : \beta_z \circ \varphi = \varphi \circ \gamma_z$$

then φ *is injective.*

Key example: $\beta_z \in \mathrm{Aut}(C(S^1))$ with $\beta_z(f)(w) = f(zw)$.

$E = v \underset{f}{\overset{e}{\rightleftarrows}} w$

$C^*(E) \simeq M_2(C(S^1))$

$\varphi : C^*(E) \to M_2(C(S^1))$ given by

$$\varphi(p_v) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad \varphi(p_w) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\varphi(s_e) = \begin{bmatrix} 0 & z \\ 0 & 0 \end{bmatrix} \qquad \varphi(s_f) = \begin{bmatrix} 0 & 0 \\ z & 0 \end{bmatrix}$$

is a $*$-isomorphism by the GIUT.

$E = \overset{e}{\circlearrowleft} v \xleftarrow[f]{} w$

$C^*(E) \simeq M_2(C(S^1))$

$\varphi : C^*(E) \to M_2(C(S^1))$ given by

$$\varphi(p_v) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad \varphi(p_w) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\varphi(s_e) = \begin{bmatrix} z & 0 \\ 0 & 0 \end{bmatrix} \qquad \varphi(s_f) = \begin{bmatrix} 0 & 0 \\ z & 0 \end{bmatrix}$$

is a $*$-isomorphism by the GIUT.

$n = 2$

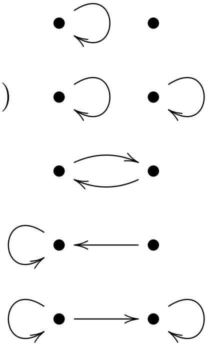
$C(S^1) \oplus \mathbb{C}$

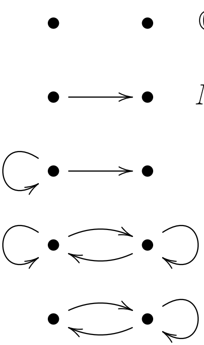
$C(S^1) \oplus C(S^1)$

$M_2(C(S^1))$

$M_2(C(S^1))$

?





$\mathbb{C} \oplus \mathbb{C}$

$M_2(\mathbb{C})$

$\mathcal{T}$

$\mathcal{O}_2$

?

Outline

Theorem (Brown-Green-Rieffel)

When $\mathfrak{A}$ and $\mathfrak{B}$ are separable C^*-algebras, the following are equivalent

1. $\mathfrak{A} \otimes \mathbb{K} \simeq \mathfrak{B} \otimes \mathbb{K}$
2. *There exists a C^*-algebra $\mathfrak{D}$ and orthogonal full projections $p, q \in M(\mathfrak{D})$ with*
$$p\mathfrak{D}p \simeq \mathfrak{A} \qquad q\mathfrak{D}q \simeq \mathfrak{B}$$
3. *There exists an $\mathfrak{A} - \mathfrak{B}$ imprimitivity bimodule*

We say that $\mathfrak{A}$ and $\mathfrak{B}$ are *Morita equivalent* and write $\mathfrak{A} \sim_{\mathrm{ME}} \mathfrak{B}$ in this case. Note all of

$$\mathbb{C}, M_2(\mathbb{C}), M_3(\mathbb{C}), \ldots, \mathbb{K}$$

are Morita equivalent.

Definition

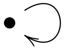
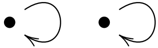
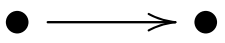
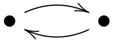
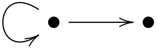
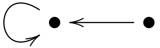
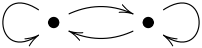
Let R and S be unital rings. We say that R and S are Morita equivalent if the category R-mod of left modules over R is equivalent to the category S-mod of left modules over S.

When R and S are Abelian, Morita equivalence reduces to isomorphism. One sees that all of

$$\mathsf{k}, M_2(\mathsf{k}), M_3(\mathsf{k}), \ldots$$

are Morita equivalent.

Identified algebras

$C(S^1) \oplus \mathbb{C}$			$\mathbb{C}$
$C(S^1) \oplus C(S^1)$			$M_2(\mathbb{C})$
$M_2(C(S^1))$			$\mathcal{T}$
$M_2(C(S^1))$			$\mathcal{O}_2$
$C(S^1)$			$\mathbb{C} \oplus \mathbb{C}$

Graph C^*-algebras versus LPAs

Let E, F be graphs with finitely many vertices and consider

(i) $C^*(E) \sim_{\mathrm{ME}} C^*(F)$

(ii) $L_{\mathbb{C}}(E) \sim_{\mathrm{ME}} L_{\mathbb{C}}(F)$

(iii) $\forall \mathsf{k} : L_{\mathsf{k}}(E) \sim_{\mathrm{ME}} L_{\mathsf{k}}(F)$

Conjecture [Abrams-Tomforde]

(ii) $\Longrightarrow$ (i)

Key questions

Let $\mathcal{G}$ denote the set of graphs with finitely many vertices.

Geometric classification

1. Which equivalence relation $\sim_{C^*}$ is induced on $\mathcal{G}$ by

$$C^*(E) \sim_{\mathrm{ME}} C^*(F)?$$

2. Which equivalence relation $\sim_{\mathrm{LPA}}$ is induced on $\mathcal{G}$ by

$$L_{\mathbb{C}}(E) \sim_{\mathrm{ME}} L_{\mathbb{C}}(F)?$$

Simple graphs

Let $\mathcal{G}_s[n]$ denote the set of simple graphs with n vertices.

n	$\lvert\mathcal{G}_s[n]\rvert$	$\lvert\mathcal{G}_s[n]/\sim_{C^*}\rvert$	$\lvert\mathcal{G}_s[n]/\sim_{\mathrm{LPA}}\rvert$
1	2	2	2
2	10	8	8
3	104	35	35
4	3044	206	?
5	291968	?	?

Outline

Let $\mathfrak{A}$ be separable. The set of prime ideals $\mathrm{Prim}(\mathfrak{A})$ in a C^*-algebra has a (possibly non-Hausdorff) hull-kernel topology. There is a $1-1$ correspondence between the ideals of $\mathfrak{A}$ and the open sets of $\mathrm{Prim}(\mathfrak{A})$. Key examples:

- $\mathrm{Prim}(\mathbb{K}(\mathcal{H})) = \mathrm{Prim}(M_n(\mathbb{C})) = \mathrm{Prim}(\mathcal{O}_2) = \{\star\}$
- $\mathrm{Prim}(\mathbb{B}(\mathcal{H})) = \{\star, \Box\}$ with $\{\Box\}$ the only non-open set
- $\mathrm{Prim}(C(X)) = X$
- $\mathrm{Prim}(\mathcal{T}) = \{\star\} \cup S^1$ with $\star$ dense and the usual topology on S^1.

Observation

When $\mathfrak{A} \sim_{\mathrm{ME}} \mathfrak{B}$, $\mathrm{Prim}(\mathfrak{A}) \simeq \mathrm{Prim}\,\mathfrak{B}$.

Theorem

When E is a finite graph, there is a $1-1$ correspondence between the gauge invariant ideals of $C^*(E)$ and subsets $V \subseteq E^0$ that are **hereditary** *and* **saturated** *sets of vertices V:*

- $s(e) \in V \implies r(e) \in V$
- $r(s^{-1}(v)) \subseteq V \implies v \in V$

Proposition

If $\mathfrak{I} \triangleleft \mathfrak{J} \triangleleft C^*(E)$ are gauge invariant ideals such that $\mathfrak{J}/\mathfrak{I}$ has no non-trivial gauge-invariant ideals yet is not simple, then

$$\mathfrak{J}/\mathfrak{I} \sim_{\mathrm{ME}} C(S^1)$$

Recipe for computing $\mathrm{Prim}(C^*(E))$ for E finite

1. Locate all hereditary and saturated subsets of E^0 (don't forget the empty set)
2. Extract those sets V that contain a largest such proper subset V_0
3. Organize these sets into a partially ordered $\mathcal{P}$ set using containment of sets as the order
4. Represent the partially ordered set as a Hasse diagram
5. Color those vertices with the property that $V \backslash V_0$ is a cycle with no exit

$\mathrm{Prim}(C^*(E))$ is obtained as the Alexandrov topology of $\mathcal{P}$ with a circle substituted at each colored vertex. Thus, the colored Hasse diagram is a Morita equivalence invariant for $C^*(E)$.

$n = 2$