Graph C^*-algebras and orbit equivalence

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Cuntz-Krieger algebras and flow equivalence

Theorem [Cuntz and Krieger ('80), Matsumoto and Matui ('13)]

Suppose A and B are irreducible $\{0,1\}$-matrices that are not permutation matrices. Then the following are equivalent:

1. $(\bar{X}_A, \bar{\sigma}_A)$ and $(\bar{X}_B, \bar{\sigma}_B)$ are flow equivalent.
2. $(\mathscr{O}_A \otimes \mathscr{K}, \mathscr{D}_A \otimes \mathscr{C})$ and $(\mathscr{O}_B \otimes \mathscr{K}, \mathscr{D}_B \otimes \mathscr{C})$ are isomorphic.

Cuntz-Krieger algebras and orbit equivalence

An essential part of the proof of the previous theorem is the following result.

Theorem [Matsumoto ('13), Matsumoto and Matui ('13)]

Suppose A and B are irreducible $\{0,1\}$-matrices that are not permutation matrices. Then the following are equivalent:

1. (X_A, σ) and (X_B, σ_B) are continuously orbit equivalent.
2. $\mathscr{G}_A$ and $\mathscr{G}_B$ are isomorphic.
3. $(\mathscr{O}_A, \mathscr{D}_A)$ and $(\mathscr{O}_B, \mathscr{D}_B)$ are isomorphic.

Graph algebras and orbit equivalence

Theorem [Brownlowe, Carlsen, and Whittaker]

Let E and F be graphs. Consider the following 4 statements.

(1) There is an isomorphism from $C^*(E)$ to $C^*(F)$ which maps $\mathscr{D}(E)$ onto $\mathscr{D}(F)$.

(2) The graph groupoids $\mathscr{G}_E$ and $\mathscr{G}_F$ are isomorphic as topological groupoids.

(3) The pseudogroups of E and F are isomorphic.

(4) E and F are orbit equivalent.

Then (1) $\iff$ (2), (3) $\iff$ (4) and (2) $\implies$ (3). If E and F satisfy condition (L), then (3) $\implies$ (2) and the 4 statements are equivalent.

Directed graphs

- A *directed graph* E is a quadruple (E^0, E^1, r, s) consisting of two sets E^0 and E^1 and two maps $r, s : E^1 \to E^0$.
- The elements of E^0 are called *vertices*.
- The elements of E^1 are called *edges*.
- If e is an edge, $s(e)$ is called the *source* of e, and $r(e)$ is called the *range* of e.
- If $s(e) = v$ and $r(e) = w$, then we say that v *emits* e, and that w *receives* e.
- If $v \in E^0$, then we let $vE^1 = \{e \in E^n : s(e) = v\}$ and $E^1 v = \{e \in E^n : r(e) = v\}$.

Graph C^*-algebras

Let E be a graph. The C^*-algebra $C^*(E)$ of the graph E is defined as the universal C^*-algebra generated by a family $(s_e, p_v)_{e\in E^1, v\in E^0}$ consisting of partial isometries $(s_e)_{e\in E^1}$ with mutually orthogonal range projections and mutually orthogonal projections $(p_v)_{v\in E^0}$ satisfying

1. $s_e^* s_e = p_{r(e)}$ for all $e \in E^1$,
2. $s_e s_e^* \leq p_{s(e)}$ for all $e \in E^1$,
3. $p_v = \sum_{e\in vE^1} s_e s_e^*$ for all $v \in E^0_{\mathrm{reg}}$.

Paths

- A *path of length n* in a directed graph E is a sequence $\mu = \mu_1\mu_2\ldots\mu_n$ of edges in E such that $r(\mu_i) = s(\mu_{i+1})$ for $i \in \{1, 2, \ldots, n-1\}$.
- We write $|\mu|$ for the length n of a path.
- We denote by E^n the set of paths of length n, and let $E^* = \bigcup_{n=0}^{\infty} E^n$.
- We extend the range and source maps to E^* by setting $s(\mu) = s(\mu_1)$ and $r(\mu) = r(\mu_n)$ when $|\mu| \geq 1$, and $s(\mu) = r(\mu) = \mu$ when $\mu \in E^0$.
- If $\mu, \nu \in E^*$ and $r(\mu) = s(\nu)$, then we write $\mu\nu$ for the path $\mu_1 \ldots \mu_{|\mu|}\nu_1 \ldots \nu_{|\nu|}$.

The C^*-subalgebra $\mathscr{D}(E)$

- For $\mu \in E^*$, we let $s_\mu = s_{\mu_1} \dots s_{\mu_{|\mu|}}$ when $|\mu| \geq 1$, and $s_\mu = p_\mu$ when $\mu \in E^0$.
- We let $\mathscr{D}(E)$ denote the C^*-subalgebra of $C^*(E)$ generated by $\{s_\mu s_\mu^* \mid \mu \in E^*\}$.
- $\mathscr{D}(E)$ is abelian and its spectrum is homeomorphic to ∂E by a homeomorphism $h_E : \partial E \to \mathrm{Spec}(\mathscr{D}(E))$ satisfying

$$h_E(x)(s_\mu s_\mu^*) = \begin{cases} 1 & \text{if } x \in Z(\mu), \\ 0 & \text{if } x \notin Z(\mu). \end{cases}$$

Graph algebras and orbit equivalence

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(1) There is an isomorphism from $C^*(E)$ to $C^*(F)$ which maps $\mathscr{D}(E)$ onto $\mathscr{D}(F)$.

(2) The graph groupoids $\mathscr{G}_E$ and $\mathscr{G}_F$ are isomorphic as topological groupoids.

(3) The pseudogroups of E and F are isomorphic.

(4) E and F are orbit equivalent.

Then (1) $\iff$ (2), (3) $\iff$ (4) and (2) $\implies$ (3). If E and F satisfy condition (L), then (3) $\implies$ (2) and the 4 statements are equivalent.

Infinite paths

- An *infinite path* in a directed graph E is an infinite sequence $x = x_1 x_2 \dots$ of edges in E such that $s(x_i) = r(x_{i+1})$ for $i \in \{1, 2, \dots\}$.
- We denote by E^∞ the set of infinite paths in E.
- We extend the range map to E^∞ by setting $r(x) = r(x_1)$.
- If $\mu \in E^*$, $x \in E^\infty$ and $s(\mu) = r(x)$, then we write μx for the path $\mu_1 \dots \mu_{|\mu|} x_1 x_2 \dots$ (if $\mu \in E^0$, then $\mu x = x$).

The boundary path space

- We let $E^0_{\text{reg}} = \{v \in E^0 : vE^1 \text{ is finite and nonempty}\}$ and $E^0_{\text{sing}} = E^0 \setminus E^0_{\text{reg}}$.
- The *boundary path space* of E is the space $\partial E := E^\infty \cup \{\mu \in E^* : r(\mu) \in E^0_{\text{sing}}\}$.
- For $\mu \in E^*$, we let $Z(\mu) = \{\mu x : x \in \partial E,\ s(\mu) = r(x)\}$.
- Given $\mu \in E^*$ and a finite subset $F \subseteq r(\mu)E^1$ we let $Z(\mu \setminus F) = Z(\mu) \setminus (\bigcup_{e \in F} Z(\mu e))$.
- We equip ∂E with the topology generated by $\{Z(\mu \setminus F) : u \in E^*, F \text{ is a finite subset of } r(\mu)E^1\}$.
- ∂E then becomes a totally disconnected locally compact Hausdorff space.
- $Z(\mu \setminus F)$ is open and compact for all $\mu \in E^*$ and all finite subsets F of $r(\mu)E^1$.
- ∂E is compact if and only if E^0 is finite.

The shift map

- For $n \in \mathbb{N}$, let $\partial E^{\geq n} = \{x \in \partial E : |x| \geq n\}$.
- Then $\partial E^{\geq n}$ is an open subset of ∂E.
- We define the *shift map* on E to be the map $\sigma_E : \partial E^{\geq 1} \to \partial E$ given by $\sigma_E(x_1 x_2 x_3 \cdots) = x_2 x_3$ for $x_1 x_2 x_3 \cdots \in \partial E^{\geq 2}$ and $\sigma_E(e) = r(e)$ for $e \in \partial E \cap E^1$.
- For $n \geq 1$, we let σ_E^n be the n-fold composition of σ_E with itself.
- We let σ_E^0 denote the identity map on ∂E.
- Then σ_E^n is a local homeomorphism for all $n \in \mathbb{N}$.
- When we write $\sigma_E^n(x)$, we implicitly assume that $x \in \partial E^{\geq n}$.
- The *orbit* of an $x \in \partial E$ is the set $\bigcup_{n \in \mathbb{N}} \bigcup_{m=0}^{|x|} (\sigma_E^n)^{-1}(\sigma_E^m(\{x\}))$.

The groupoid of a graph

- Let E be a graph.
- Let $\mathscr{G}_E = \{(x, m-n, y) : x, y \in \partial E,\ m, n \in \mathbb{N},$ and $\sigma_E^m(x) = \sigma_E^n(y)\}$.
- We define a partial defined product on $\mathscr{G}_E$ by $(x, k, y)(w, l, z) = (x, k+l, z)$ if $y = w$ and the product is undefined otherwise; and an inverse map $(x, k, y)^{-1} = (y, -k, x)$.
- With these operations $\mathscr{G}_E$ becomes a groupoid.
- The unit space $\mathscr{G}_E^0$ of $\mathscr{G}_E$ is $\{(x, 0, x) : x \in \partial E\}$ which we will freely identify with ∂E via the map $(x, 0, x) \mapsto x$. We then have that the range and source maps $r, s : \mathscr{G}_E \to \partial E$ are given by $r(x, k, y) = x$ and $s(x, k, y) = y$.

The groupoid of a graph

- When $m,n \in \mathbb{N}$, U is an open subset of $\partial E^{\geq m}$ such that the restriction of σ_E^m to U is injective, V is an open subset of $\partial E^{\geq n}$ such that the restriction of σ_E^n to V is injective, and $\sigma_E^m(U) = \sigma_E^n(V)$, we let $Z(U,m,n,V) := \{(x,k,y) \in \mathscr{G}_E : x \in U,\ k = m-n,\ y \in V,\ \sigma_E^m(x) = \sigma_E^n(y)\}$.
- For $\mu,\nu \in E^*$ with $r(\mu) = r(\nu)$, let $Z(u,v) := Z(Z(\mu),|\mu|,|\nu|,Z(\nu))$.
- Then $\mathscr{G}_E$ is a locally compact, Hausdorff, étale topological groupoid with the topology given by the basis $\{Z(U,m,n,V) : m,n \in \mathbb{N},\ U$ is an open subset of $\partial E^{\geq m}$ such that the restriction of σ_E^m to U is injective, V is an open subset of $\partial E^{\geq n}$ such that the restriction of σ_E^n to V is injective, $\sigma_E^m(U) = \sigma_E^n(V)\}$.

The groupoid of a graph

- We furthermore have that each $Z(\mu,\nu)$ is compact and open, and that the topology ∂E inherits when we consider it as a subset of $\mathscr{G}_E$ by identifying it with $\{(x,0,x) : x \in \partial E\}$ agrees with the topology described previously.
- Notice that $\{Z(U,|\mu|,|\nu|,V) : \mu,\nu \in E^*$, U is a clopen subset of $Z(\mu)$, V is a clopen subset of $Z(\nu)$, $\sigma_E^{|\mu|}(U) = \sigma_E^{|\nu|}(V)\}$ is a basis for the topology of $\mathscr{G}_E$.
- $\mathscr{G}_E$ is topological amenable, so the reduced and universal C^*-algebras of $\mathscr{G}_E$ are equal.

Graph groupoids and graphs

Proposition

Let E be a graph. Then there is a unique isomorphism from $C^*(E)$ to the C^*-algebra $C^*(\mathscr{G}_E)$ of $\mathscr{G}_E$ that, for each $v \in E^0$, maps p_v to the indicator function $1_{Z(v,v)}$ of the compact open set $Z(v,v)$, and, for each $e \in E^1$, maps s_e to the indicator function $1_{Z(e,r(e))}$ of the compact open set $Z(e,r(e))$. This isomorphism maps $\mathscr{D}(E)$ onto $C_0(\mathscr{G}_E^0)$.

Proposition

Let E and F graphs. If $\mathscr{G}_E$ and $\mathscr{G}_F$ are isomorphic as topological groupoids, then there is an isomorphism from $C^*(E)$ to $C^*(F)$ which maps $\mathscr{D}(E)$ onto $\mathscr{D}(F)$.

Graph algebras and orbit equivalence

Theorem [Brownlowe, Carlsen, and Whittaker]

Let E and F be graphs. Consider the following 4 statements.

(1) There is an isomorphism from $C^*(E)$ to $C^*(F)$ which maps $\mathscr{D}(E)$ onto $\mathscr{D}(F)$.

(2) The graph groupoids $\mathscr{G}_E$ and $\mathscr{G}_F$ are isomorphic as topological groupoids.

(3) The pseudogroups of E and F are isomorphic.

(4) E and F are orbit equivalent.

Then (1) $\iff$ (2), (3) $\iff$ (4) and (2) $\implies$ (3). If E and F satisfy condition (L), then (3) $\implies$ (2) and the 4 statements are equivalent.

Orbit equivalence

- The *orbit* of $x \in \partial E$ is the set
$$\mathrm{orb}(x) = \bigcup_{n \in \mathbb{N}} \bigcup_{m=0}^{|x|} (\sigma_E^n)^{-1}(\sigma_E^m(\{x\})).$$
- So the orbit of x is the smallest subset $\mathrm{orb}(x)$ of ∂E which contains x and satisfies $y \in \mathrm{orb}(x) \implies \sigma_E(y) \in \mathrm{orb}(x)$ and $\sigma_E(y) \in \mathrm{orb}(x) \implies y \in \mathrm{orb}(x)$.
- Suppose $h : \partial E \to \partial F$ is a homeomorphism such that $h(\mathrm{orb}(x)) = \mathrm{orb}(h(x))$ for all $x \in \partial E$. Then there is for each $x \in \partial E$ nonnegative integers $k(x)$ and $l(x)$ such that $\sigma_F^{k(x)}(h(\sigma_E(x))) = \sigma_F^{l(x)}(h(x))$.
- Similarly, there is for each $y \in \partial F$ nonnegative integers $k'(y)$ and $l'(y)$ such that $\sigma_E^{k'(y)}(h^{-1}(\sigma_F(y))) = \sigma_E^{l'(y)}(h^{-1}(y))$.
- If $k(x)$, $l(x)$, $k'(y)$, and $l'(y)$ can be choosen such that $k, l : \partial E^{\geq 1} \to \mathbb{N}$ and $k', l' : \partial F^{\geq 1} \to \mathbb{N}$ are continuous, then we say that E and F are *orbit equivalent*.

Continuously orbit equivalence

Let E and F be graphs. We say that E and F are *continuously orbit equivalent* if there exists a homeomorphism $h : \partial E \to \partial F$ and continuous functions $k_1, l_1 : \partial E^{\geq 1} \to \mathbb{N}$ and $k_1', l_1' : \partial F^{\geq 1} \to \mathbb{N}$ such that

$$\sigma_F^{k_1(x)}(h(\sigma_E(x))) = \sigma_F^{l_1(x)}(h(x))$$

and

$$\sigma_E^{k_1'(y)}(h^{-1}(\sigma_F(y))) = \sigma_E^{l_1'(y)}(h^{-1}(y)),$$

for all $x \in \partial E^{\geq 1}, y \in \partial F^{\geq 1}$.

Graph algebras and orbit equivalence

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The pseudogroup of a graph

- We let $\mathscr{P}_E$ be the set of homeomorphisms $\alpha : V_\alpha \to U_\alpha$ where U_α and V_α are open subsets of ∂E such that there exist continuous functions $m, n : V_\alpha \to \mathbb{N}$ such that $\sigma_E^{m(x)}(x) = \sigma_E^{n(x)}(\alpha(x))$ for all $x \in V_\alpha$.
- $\mathscr{P}_E$ forms an inverse semigroup with product defined by $\alpha\beta : \beta^{-1}(V_\alpha) \to \alpha(V_\alpha \cap U_\beta)$, $(\alpha\beta(x)) = \alpha(\beta(x))$ for $x \in \beta^{-1}(V_\alpha)$.

Pseudogroups of a graphs and orbit equivalence

Suppose that E and F are two graphs and that there exists a homeomorphism $h : \partial E \to \partial F$. Let U and V be open subsets of ∂E and let $\alpha : V \to U$ be a homeomorhism. We then let $h \circ \alpha \circ h^{-1}$ denote the homeomorphism from $h(V)$ to $h(U)$ given by $h \circ \alpha \circ h^{-1}(x) = h(\alpha(h^{-1}(x)))$. We let $h \circ \mathscr{P}_E \circ h^{-1} = \{h \circ \alpha \circ h^{-1} : \alpha \in \mathscr{P}_E\}$. We say that the pseudogroups of E and F are isomorphic if there is a homeomorphism $h : \partial E \to \partial F$ such that $h \circ \mathscr{P}_E \circ h^{-1} = \mathscr{P}_F$.

Proposition

Let E and F be two graphs. Then E and F are orbit equivalent if and only if the pseudogroups of E and F are isomorphic.

Graph algebras and orbit equivalence

Theorem [Brownlowe, Carlsen, and Whittaker]

Let E and F be graphs. Consider the following 4 statements.

(1) There is an isomorphism from $C^*(E)$ to $C^*(F)$ which maps $\mathscr{D}(E)$ onto $\mathscr{D}(F)$.

(2) The graph groupoids $\mathscr{G}_E$ and $\mathscr{G}_F$ are isomorphic as topological groupoids.

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Then (1) $\iff$ (2), (3) $\iff$ (4) and (2) $\implies$ (3). If E and F satisfy condition (L), then (3) $\implies$ (2) and the 4 statements are equivalent.

The pseudogroup of an étale groupoid

- Let $\mathscr{G}$ be an étale groupoid.
- Define a *bisection* to be a subset A of $\mathscr{G}$ such that the restriction of the source map of $\mathscr{G}$ to A and the restriction of the range map of $\mathscr{G}$ to A both are injective.
- The set of all open bisections of $\mathscr{G}$ form an inverse semigroup $\mathscr{S}$ with product defined by $AB = \{\gamma\gamma' : (\gamma, \gamma') \in (A \times B) \cap \mathscr{G}^{(2)}\}$ (where $\mathscr{G}^{(2)}$ denote the set of compassable pairs of $\mathscr{G}$), and the inverse of A defined to be the image of A under the inverse map of $\mathscr{G}$.
- Each $A \in \mathscr{S}$ defines a unique homeomorphism $\alpha_A : s(A) \to r(A)$ such that $\alpha(s(\gamma)) = r(\gamma)$ for $\gamma \in A$.
- The set $\{\alpha_A : A \in \mathscr{S}\}$ of partial homeomorphisms on $\mathscr{G}^0$ is the pseudogroup of $\mathscr{G}$.

The pseudogroup of $\mathscr{G}_E$

Let E be a graph. It is not difficult to check that the pseudogroup of $\mathscr{G}_E$ is equal to $\mathscr{P}_E$.

Thus we get:

Proposition

Let E and F graphs. If $\mathscr{G}_E$ and $\mathscr{G}_F$ are isomorphic as topological groupoids, then $\mathscr{P}_E$ and $\mathscr{P}_F$ are isomorphic.

Graph algebras and orbit equivalence

Theorem [Brownlowe, Carlsen, and Whittaker]

Let E and F be graphs. Consider the following 4 statements.

(1) There is an isomorphism from $C^*(E)$ to $C^*(F)$ which maps $\mathscr{D}(E)$ onto $\mathscr{D}(F)$.

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Cycles

- A *cycle* is a path $\mu \in E^*$ for which $|\mu| \geq 1$ and $s(\mu) = r(\mu)$.
- An *exit* for a cycle μ is an edge $e \in E^1$ such that $s(e) = s(\mu_i)$ and $e \neq \mu_i$ for some $i \in \{1, 2, \ldots, |\mu|\}$.
- A graph is said to satisfy *condition (L)* if every cycle has an exit.

Topological principal groupoids

An étale groupoid is said to be *topologically principal* if the set of points of $\mathscr{G}^0$ with trivial isotropy group is dense (the isotropy group of $x \in \mathscr{G}^0$ is the group $\{\gamma \in \mathscr{G} : s(\gamma) = r(\gamma) = x\}$).

Proposition

Let E be a graph. Then the following are equivalent:

1. The groupoid $\mathscr{G}_E$ is topologically principal.
2. E satisfies condition (L).
3. There exists no isolated points $x \in \partial E$ which are periodic (i.e. $\sigma^n(x) = x$ for some $n > 0$).
4. $\mathscr{D}(E)$ is a MASA in $C^(E)$.

Graph algebras and orbit equivalence

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The groupoid of germs

- Let $\mathscr{P}$ be a pseudogrope on a topological space X.
- The *groupoid of germs* of $\mathscr{P}$ is $\mathscr{G}_{\mathscr{P}} = \{[x,\alpha,y] : \alpha \in \mathscr{P}, y \in \mathrm{dom}(\alpha), x \in \mathrm{ran}(\alpha)\}$ where $[x,\alpha,y] = [x,\beta,y]$ if and only if there exists an open subset V such that $y \in V \subseteq \mathrm{dom}(\alpha) \cap \mathrm{dom}(\beta)$ and $\alpha(z) = \beta(z)$ for all $z \in V$.
- The product on $\mathscr{G}_{\mathscr{P}}$ is defined by $[x,\alpha,y][y,\beta,z] = [x,\alpha\beta,z]$ and the inverse by $[x,\alpha,y]^{-1} = [y,\alpha^{-1},x]$.
- The topology of $\mathscr{G}_{\mathscr{P}}$ is generated by sets $Z(U,\alpha,V) := \{[x,\alpha,y] : x \in U,\ y \in V\}$ where $\alpha \in \mathscr{P}$, V is an open subset of $\mathrm{dom}(\alpha)$, and U is an open subset of $\mathrm{ran}(\alpha)$.

The groupoid of germs

Renault has shown that if $\mathscr{G}$ is Hausdorff and topological principal étale groupoid, then the groupoids of germs of the pseudogroup of $\mathscr{G}$ is isomorphic to $\mathscr{G}$.

Thus we get:

Proposition

Let E and F graphs satisfying condition (L). If $\mathscr{P}_E$ and $\mathscr{P}_F$ are isomorphic, then $\mathscr{G}_E$ and $\mathscr{G}_F$ are isomorphic as topological groupoids.

Graph algebras and orbit equivalence

Theorem [Brownlowe, Carlsen, and Whittaker]

Let E and F be graphs. Consider the following 4 statements.

(1) There is an isomorphism from $C^*(E)$ to $C^*(F)$ which maps $\mathscr{D}(E)$ onto $\mathscr{D}(F)$.

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The normalizer of $\mathscr{D}(E)$

- Let E be a graph.
- The *normalizer* of $\mathscr{D}(E)$ is the set $N(\mathscr{D}(E)) := \{n \in C^*(E) : ndn^*, n^*dn \in \mathscr{D}(E)$ for all $d \in \mathscr{D}(E)\}$.
- If $n \in N(\mathscr{D}(E))$, then $nn^*, n^*n \in \mathscr{D}(E)$.
- For $n \in N(\mathscr{D}(E))$ let $\mathrm{dom}(n) := \{x \in \partial E : h_E(x)(n^*n) > 0\}$ and $\mathrm{ran}(n) := \{x \in \partial E : h_E(x)(nn^*) > 0\}$.
- There is a unique homeomorphism $\alpha_n : \mathrm{dom}(n) \to \mathrm{ran}(n)$ such that $h_E(x)(n^*dn) = h_E(\alpha_n(x))(d)h_E(x)(n^*n)$ for all $d \in \mathscr{D}(E)$.

Isolated points in ∂E

- Let ∂E_{iso} be the set of isolated points in ∂E.
- If $x \in \partial E_{\text{iso}}$, then the characteristic function $1_{\{x\}}$ of $\{x\}$ belongs to $C_0(\partial E)$.
- Let p_x denote the unique element of $\mathscr{D}(E)$ satisfying that $h_E(y)(p_x)$ is 1 if $y = x$ and zero otherwise.
- We say that $x \in \partial E$ is *eventually periodic* if there are $m, n \in \mathbb{N}$, $m \neq n$ such that $\sigma_E^m(x) = \sigma_E^n(x)$.

Lemma

Let $x \in \partial E_{\text{iso}}$. If x is not eventually periodic, then $p_x C^*(E) p_x = p_x \mathscr{D}(E) p_x = \mathbb{C} p_x$. If x is eventually periodic, then $p_x C^*(E) p_x$ is isomorphic to $C(\mathbb{T})$ and $p_x \mathscr{D}(E) p_x = \mathbb{C} p_x$.

The extended Weyl groupoid of $(C^*(E), \mathscr{D}(E))$

- If $x_1, x_2 \in \partial E$, $n_1, n_2 \in N(\mathscr{D}(X))$, $x_1 \in \mathrm{dom}(n_1)$, and $x_2 \in \mathrm{dom}(n_2)$, then we write $(n_1, x_1) \sim (n_2, x_2)$ if either $x_1 = x_2 \notin \partial E_{\mathrm{iso}}$ and there is an open set U such that $x_1 \in U \subseteq \mathrm{dom}(n_1) \cap \mathrm{dom}(n_2)$ and $\alpha_{n_1}(y) = \alpha_{n_2}(y)$ for all $y \in U$; or $x_1 = x_2 \in \partial E_{\mathrm{iso}}$, $\alpha_{n_1}(x_1) = \alpha_{n_2}(x_2)$, and $[(p_{x_1} n_1^* n_2 p_{x_1} n_2^* n_1 p_{x_1})^{-1/2} p_{x_1} n_1^* n_2 p_{x_1}]_1 = 0$;.
- Then $\sim$ is an equivalence relation on $\{(n, x) : n \in N(\mathscr{D}(E)),\ x \in \mathrm{dom}(n)\}$.

The extended Weyl groupoid of $(C^*(E), \mathscr{D}(E))$

- We let $[(n,x)]$ denote the equivalence class of (n,x), and we let $\mathscr{G}_{(C^*(E),\mathscr{D}(E))} = \{[(n,x)] : n \in N(\mathscr{D}(E)),\ x \in \mathrm{dom}(n)\}$.
- We define a partial defined product on $\mathscr{G}_{(C^*(E),\mathscr{D}(E))}$ by $[(n_1,x_1)][(n_2,x_2)] = [(n_1 n_2, x_2)]$ if $\alpha_{n_2}(x_2) = x_1$ (the product is undefined otherwise) and an inverse map $[(n,x)]^{-1} = [(n^*, \alpha_n(x))]$.
- Then $\mathscr{G}_{(C^*(E),\mathscr{D}(E))}$ equipped with these operations is a groupoid.
- We equipe $\mathscr{G}_{(C^*(E),\mathscr{D}(E))}$ with the topology generated by $\{\{[(n,x)] : x \in \mathrm{dom}(n)\} : n \in N(\mathscr{D}(E))\}$.

The extended Weyl groupoid of $(C^*(E), \mathscr{D}(E))$

Proposition

Let E be a graph. Then $\mathscr{G}_{(C^*(E),\mathscr{D}(E))}$ is a topological groupoid, and $\mathscr{G}_E$ and $\mathscr{G}_{(C^*(E),\mathscr{D}(E))}$ are isomorphic as topological groupoids.

Proposition

Let E and F graphs. If there is an isomorphism from $C^*(E)$ to $C^*(F)$ which maps $\mathscr{D}(E)$ onto $\mathscr{D}(F)$, then $\mathscr{G}_E$ and $\mathscr{G}_F$ are isomorphic as topological groupoids.

Graph algebras and orbit equivalence

Theorem [Brownlowe, Carlsen, and Whittaker]

Let E and F be graphs. Consider the following 4 statements.

(1) There is an isomorphism from $C^*(E)$ to $C^*(F)$ which maps $\mathscr{D}(E)$ onto $\mathscr{D}(F)$.

(2) The graph groupoids $\mathscr{G}_E$ and $\mathscr{G}_F$ are isomorphic as topological groupoids.

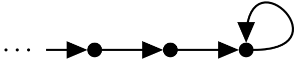
(3) The pseudogroups of E and F are isomorphic.

(4) E and F are orbit equivalent.

Then (1) $\iff$ (2), (3) $\iff$ (4) and (2) $\implies$ (3). If E and F satisfy condition (L), then (3) $\implies$ (2) and the 4 statements are equivalent.

Examples

1. Let E be the graph • and let F be the graph 
   Then $\partial E = \{\star\} = \partial F$, so E and F are orbit equivalent, but $C^*(E) \cong \mathbb{C} \ncong C(\mathbb{T}) \cong C^*(F)$.
2. Let E be the graph 

   and let F be the graph 
   Then $\partial E = \mathbb{N} = \partial F$, so E and F are orbit equivalent, but $C^*(E) \cong \mathscr{K} \ncong \mathscr{K} \otimes C(\mathbb{T}) \cong C^*(F)$.