Proof. Let \( z = (z_1, z_2, \ldots, z_m) \) be a vector of formal variables. For \( v \in \mathbb{N}^m \) (the natural numbers start in zero) we write \( z^v = z_1^{v_1} z_2^{v_2} \ldots z_m^{v_m} \). For a subset \( S \subset \mathbb{N}^m \) we write \( S(z) = \sum_{i \in S} z^i \) and call \( S(z) \) the formal series of the set \( S \). The proof rests on the following marvelous theorem.

**Theorem 1.** Let \( \Phi \) be an \( m \times m \) matrix with integer coefficients. Let us denote by \( S \) the set of solutions of the system \( \Phi x = 0 \) such that \( x \in \mathbb{N}^m \). The formal series \( S(z) \) is a rational function. Its denominator has the form \( \prod_i (1 - z^{\alpha_i}) \).

By some elementary analysis, we can extend the previous theorem to sets not necessarily described by a system of homogeneous diophantine equations. We can add to the system \( \Phi x = 0 \) inequalities of the form \( ax \leq 0 \). The resulting system is transformed into a homogeneous system of equalities introducing slack variables. The generating function of the original set is obtained setting to 1 the formal variables corresponding to the slack variables. Thus, the formal series of the new set will still be rational. We can also add inequations to the system since \( \{ x \in S : ax \neq 0 \} = S \setminus \{ x \in S : ax = 0 \} \). Finally, by the principle of inclusions and exclusions, we can take unions of sets defined by homogeneous systems of equations, inequalities and inequations. Thus, if the set \( S \) is described by a combination of equalities, inequalities and inequations concatenated by \( \lor \)s (ORs) and \( \land \)s (ANDs) then \( S(z) \) is rational.

Let us identify the \( n \times n \) chessboard with the set \([0, n-1]^2\). Let \( F = \{(\alpha_i, \beta_i), i = 1, \ldots, kw\} \) be a figure of type \((n, k, w)\). The fact that there are exactly \( w \) squares on the same row as \((\alpha_1, \beta_1)\) can be expressed by the system

\[
\bigvee_{P_{1,w}} \left( \alpha_1 = \alpha_s \land \alpha_1 \neq \alpha_{s'}, \forall s \in P_{1,w}, \forall s' \notin P_{1,w} \right),
\]

where \( P_{1,w} \) runs through all subsets of \( \{2, \ldots, kw\} \) with exactly \( w - 1 \) elements. We add similar equations for the columns and diagonals passing through \((\alpha_1, \beta_1)\) and similarly for every other square of the figure. To account for the fact that the figure is contained in a finite chessboard of size \( n \) we add an extra variable \( \gamma \) and the inequalities

\[
\alpha_i \leq \gamma, \beta_i \leq \gamma, \quad i = 1, \ldots, kw.
\]

Let \( S \) be the set of vectors \((\alpha_1, \beta_1, \ldots, \alpha_{kw}, \beta_{kw}, \gamma) \in \mathbb{N}^{2kw+1}\) satisfying the preceding system. By Theorem 1 and the remark below it the formal series \( S(z) \) is a rational function. Setting the formal variables corresponding to \( \{(\alpha_i, \beta_i), i = 1, \ldots, kw\} \) equal to 1 we get that

\[
\sum_{n=0}^{\infty} W(n, k, w) z^{n-1}
\]

is also a rational function. \( \square \)