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Confidence limits and prediction limits for a Weibull distribution based on the generalized variable approach

K. Krishnamoorthy[a,*], Yin Lin[a], Yanping Xia[b]

[a]Department of Mathematics, University of Louisiana at Lafayette, Lafayette, LA 70504, USA
[b]Department of Mathematics, Southeast Missouri State University, Cape Girardeau, MO 63701, USA

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ABSTRACT

In this article, we consider the problems of constructing confidence interval for a Weibull mean and setting prediction limits for future samples. Specifically, we construct upper prediction limits that include at least l of m samples from a Weibull distribution at each of r locations. The methods are based on the concept of *generalized variable approach*. The procedures can be easily extended to the type II censored samples, and they can be used to find approximate inferential procedures for type I censored samples. The proposed methods are conceptually simple and easy to use. The results are illustrated using some practical examples.

1. Introduction

In this article, we consider the problems of estimating a Weibull mean and setting upper tolerance limits for future samples. In general, the statistical problems involving Weibull distributions are not simple as the natural maximum likelihood estimators (MLEs) do not have closed form, and they have to be obtained numerically. Therefore, exact analytical procedures were obtained only for a few problems (e.g., Lawless, 1973, 1978) which are also not simple to apply. Even though the MLEs have no explicit form, the distributions of certain pivotal quantities based on them are parameter free. These distributional results were obtained by Thoman et al. (1969), and are given in Lemma 1 in the following section. These results for the MLEs allow us to find the distributions of some pivotal quantities empirically from which inferential procedures for Weibull parameters and other related problems can be developed. Using this approach, Thoman et al. (1969) obtained pivotal quantities for some one-sample problems and Thoman and Bain (1969) extended the results to some two-sample problems. Thoman et al. (1970) developed methods for setting confidence limits for reliability and for constructing one-sided tolerance limits. These procedures are exact except for simulation errors.

The solutions for the problems that we will address in this article are based on the concept of *generalized variable* (GV) approach due to Tsui and Weerahandi (1989) and Weerahandi (1993). The GV approach is useful to develop a so called *generalized pivotal quantity* (GPQ) which is used to construct confidence intervals for a parametric function of interest. Unlike the ordinary pivotal quantity, GPQ is a function of observed statistics and random variables whose distributions are free of unknown parameters. An appealing feature of the GV approach is that a GPQ for a function of parameters can be readily obtained by substitution if GPQs for

* Corresponding author. Tel.: +1 337 482 5283; fax: +1 337 482 5346.
E-mail address: krishna@louisiana.edu (K. Krishnamoorthy).

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individual parameters are available. This GV approach has been used successfully to obtain solutions to several complex problems. For more details and other references, we refer to Hannig et al. (2006). These authors have noted that the GV procedures are a special case of fiducial inference procedures and are asymptotically exact in many situations.

In this article, we provide generalized inferential procedures for the mean of a Weibull distribution, and for setting upper prediction limits that include at least l of m future observations from each of r locations. It appears that the inferential procedures for the mean were seldom addressed, except for the paper by Colosimo and Ho (1999) which provides an asymptotic Wald method for setting confidence limits for the mean. The mean of a failure time distribution is referred to as the mean time to failure (MTTF), and is also used as a measure of reliability. Hisada and Arizino (2002) have pointed out that a hypothesis test regarding the MTTF is used to determine whether or not a production lot is acceptable. These authors have noted that tests for the mean were often carried out assuming that the shape parameter is known, and in this case, the problem simplifies to testing the scale parameter (see the mean expression in Section 5.1) for which an exact method is available (Thoman et al., 1969). We will show in the sequel that a generalized inferential method for a Weibull mean can be easily obtained without any assumption on the shape parameter.

The prediction problem mentioned earlier arises in monitoring and control problems where the future samples that will be collected periodically during the operation of a process, are compared with some past samples to determine on each sampling occasion if a change in the process has occurred. This type of process monitoring is also practiced in groundwater quality monitoring in the vicinity of hazardous waste management facilities (HWMF). For example, to monitor ground water quality, a series of m samples (measurements of an analyte or pollutant) from each of r monitoring wells in the vicinity of a HWMF are often compared with an upper prediction limit based on a sample of n measurements obtained from one or more upgradient sampling locations of the facility. If at least l samples out of these m samples from each of r location are less than the upper prediction limit, then the facility is considered to be within compliance. If this requirement is not met, then considerably more intensive monitoring of an extended list of contaminants is to be instituted and continued indefinitely (see Davis and McNichols, 1987). Bhaumik and Gibbons (2006) noted that such an upper predictions limit has applications in the areas of molecular genetics and industrial quality control. Davis and McNichols (1987) addressed this problem assuming normality. Bhaumik and Gibbons (2006) and Krishnamoorthy et al. (2007) proposed approximate methods for constructing upper prediction limits for the aforementioned purpose assuming a gamma distribution. We shall show that a Weibull distribution also well fits the *vinyl chloride data* that were used for illustration in Bhaumik and Gibbons' paper, and outline a GV procedure for constructing an upper prediction limit as stated at the beginning of this paragraph.

Some special cases of the above prediction problem have applications in lifetime data analysis. For example, when $r = 1$ and $l = 1$, the problem of interest is to predict the smallest observation in a single future sample of m observations. In this case, a lower prediction limit is desired. Lawless (1973) considered this problem assuming a Weibull distribution. He also noted that this prediction problem arises while estimating warrantable life, or "safe" life, for a number of production units, based on the results of previous life tests on the units. For a specific example, see the introduction section of Lawless (1973). The prediction problem in the special case of $r = 1$ and $l = m$ has been addressed in the paper by Antle and Rademaker (1972). Specifically, the problem of interest here is to find an upper prediction limit for the largest observation in a future sample of m observations. Antle and Rademaker noted that such an upper prediction limit is required to predict the maximum flood level, rainfall, temperature, and wind gusts.

The GV procedure that will be considered in this article is also applicable if the samples are singly type II censored because, pivotal quantities for the MLEs are also valid in this case; however, they are not valid when samples are type I censored. Nevertheless, we will show in Section 7 that the procedure for type II censored data can be used as an approximation for type I censored samples. We also note that our procedure can also be applied when the samples are type I singly left censored. This type of data arises while measuring workplace contaminants or pollution levels in an environment where the amount of a contaminant can not be determined if it is below a threshold value (usually referred to as *detection limit*). Here, a detection limit x_0 (of a device or a method) is specified, and measurements that fall below x_0 are not included in the sample. For a good exposition of the problems and data analysis with non-detectable values, see the book by Helsel (2005).

The rest of the article is organized as follows. In the following section, we give some preliminary results concerning the MLEs. In Section 3, we describe the GV approach in a general setup. The GPQs for the shape and scale parameters are given in Section 4. The GV procedures for constructing confidence interval for a mean is given in Section 5. In Section 6, we outline an empirical procedure to find upper prediction limit for future samples. A GV methods for estimating a Weibull mean when the samples are type I censored are outlined in Section 7. Applicability of the GV method to some other problems and some concluding remarks are given in Section 8.

2. Preliminaries

Let X follow a Weibull distribution with scale parameter b and shape parameter c. The probability density function of X is given by

$$f(x|b,c) = \frac{c}{b}\left(\frac{x}{b}\right)^{c-1} \exp\left\{-\left[\frac{x}{b}\right]^{c}\right\}, \quad x > 0, \ b > 0, \ c > 0. \tag{1}$$

Let $x_1, \ldots, x_n$ be a sample from a Weibull(b, c) distribution. The MLEs for the complete and censored cases can be obtained from Cohen (1965) as follows. For the complete case, the MLE $\widehat{c}$ of c is the solution to the equation

$$\frac{1}{\widehat{c}} - \frac{\sum_{i=1}^{n} x_i^{\widehat{c}} \ln(x_i)}{\sum_{i=1}^{n} x_i^{\widehat{c}}} + \frac{1}{n}\sum_{i=1}^{n} \ln(x_i) = 0 \tag{2}$$

and the MLE of b is given by $\widehat{b} = (\sum_{i=1}^{n} x_i^{\widehat{c}}/n)^{1/\widehat{c}}$.

For a type II censored sample, let $x_{(1)} \leqslant \cdots \leqslant x_{(k)}$ denote the failure times of k items that were recorded. The MLE for c can be obtained as the solution to

$$\frac{1}{\widehat{c}} - \frac{\sum_{i=1}^{n} x_{i*}^{\widehat{c}} \ln(x_{i*})}{\sum_{i=1}^{n} x_{i*}^{\widehat{c}}} + \frac{1}{k}\sum_{i=1}^{k} \ln(x_{i*}) = 0, \tag{3}$$

where $x_{i*} = x_{(i)}$ for $i = 1, \ldots, k$ and $x_{i*} = x_{(k)}$ for $i = k+1, \ldots, n$. The MLE of b is given by $\widehat{b} = (\sum_{i=1}^{n} x_{i*}^{\widehat{c}}/k)^{1/\widehat{c}}$. The same formulas are used to find the MLEs when the samples are type I censored except that we take $x_{i*} = x_0$ for $i = k+1, \ldots, n$, where x_0 is the censoring time (mission time).

As the procedures that we will develop in the sequel heavily depend on empirical distributions of the MLEs, it is worth noting an iterative method of solving the likelihood Eq. (2). Let $x_1, \ldots, x_n$ be a sample of observations from a Weibull(b, c) distribution. Let $y_i = \ln(x_i),\ i = 1, \ldots, n$. Menon (1963) showed that the estimator

$$\widehat{c}_u = \frac{\pi}{\sqrt{6}}\left(\frac{\sum_{i=1}^{n}(y_i - \bar{y})^2}{n-1}\right)^{-1/2} \tag{4}$$

is asymptotically unbiased with $N(c, 1.1c^2/n)$ distribution. Using this unbiased estimator as an initial value, the Newton–Raphson iterative method can be applied to find the root of Eq. (2).

In the following lemma, we give some distributional results of the MLEs by Thoman et al. (1969). These results are also valid for type II singly censored samples, but they are not valid if the samples are type I censored.

Lemma 1. *Let $\widehat{b}^*$ and $\widehat{c}^*$ be the MLEs that are based on a sample of n observations from a Weibull$(1, 1)$ distribution. That is, $\widehat{c}^*$ is the solution of the likelihood Eq. (2) with $x_1, \ldots, x_n$ being a sample from a Weibull$(1, 1)$ distribution, and $\widehat{b}^* = (\sum_{i=1}^{n} x_i^{\widehat{c}^*}/n)^{1/\widehat{c}^*}$. Let $\widehat{b}$ and $\widehat{c}$ be the MLEs that are based on a sample of n observations from a Weibull(b, c) distribution. Then, $\widehat{c}/c$ is distributed as $\widehat{c}^*$ and $\widehat{c}\ln(\widehat{b}/b)$ is distributed as $\widehat{c}^* \ln(\widehat{b}^*)$. That is, the distributions of $\widehat{c}/c$ and $\widehat{c}\ln(\widehat{b}/b)$ do not depend on b and c, and these are pivotal quantities.*

3. Generalized pivotal quantities

Let X be a random variable whose distribution depends on a parameter of interest θ, and a nuisance parameter δ. Let x denote the observed value of X. A GPQ for θ is a random quantity denoted by $T_\theta(X; x)$ and satisfies the following:

(i) The distribution of $T_\theta(X; x)$ is free of any unknown parameters, (C1)
(ii) the value of $T_\theta(X; x)$ at $X = x$, i.e., $T_\theta(x; x)$ is free of the nuisance parameter δ. In most cases, $T_\theta(x; x) = \theta$.

Appropriate percentiles of $T_\theta(X; x)$ form a confidence interval for θ. Specifically, if T_α denotes the 100α percentage point of $T_\theta(X; x)$, then $(T_{\alpha/2}, T_{1-\alpha/2})$ is a $1-\alpha$ generalized confidence interval for θ. Because, for a given x, the distribution of $T_\theta(X; x)$ does not depend on any unknown parameters, its percentiles can be found.

In the above setup, suppose we are interested in testing the hypotheses

$$\mathrm{H}_0 : \theta \leqslant \theta_0 \ \text{ vs. } \ \mathrm{H}_a : \theta > \theta_0 \tag{5}$$

for a specified θ_0. The generalized test variable (GTV), denoted by $T_\theta^t(X; x)$, is defined as follows.

(i) The value of $T_\theta^t(X; x)$ at $X = x$ is free of any unknown parameters,
(ii) the distribution of $T_\theta^t(X; x)$ is stochastically monotone (i.e., stochastically increasing or stochastically decreasing) in θ for any fixed x and δ,
(iii) the distribution of $T_{\theta_0}^t(X; x)$ is free of any unknown parameters. (C2)

Let $T_0 = T_{\theta_0}^t(x; x)$, the value of $T_\theta^t(X; x)$ at $(X, \theta) = (x, \theta_0)$. When the conditions in (C2) hold, the generalized p-value for testing the hypotheses in (5) is defined as $P[T_{\theta_0}(X; x) \geqslant T_0]$ if $T_\theta^t(X; x)$ is stochastically increasing in θ, and is $P[T_{\theta_0}^t(X; x) \leqslant T_0]$ if $T_\theta^t(X; x)$ is stochastically decreasing in θ. In many situations, $T_\theta^t(X; x) = T_\theta(X; x) - \theta$. The test based on the generalized p-value consists of rejecting H_0 when the generalized p-value is small, say, less than a nominal level α. However, the size and power of such a test may depend on the nuisance parameters.

For more details on generalized p-values and generalized confidence intervals, we refer the readers to Weerahandi (1995, Chapters 5 and 6).

4. GPQS for Weibull parameters

Let $\widehat{b}_0$ and $\widehat{c}_0$ be the observed values of the MLEs based on a sample of n observations from a Weibull$(b,\ c)$ distribution. A GPQ for the scale parameter b is given by

$$G_b = \left(\frac{b}{\widehat{b}}\right)^{\widehat{c}/\widehat{c}_0}, \quad \widehat{b}_0 = \left(\frac{1}{\widehat{b}^*}\right)^{\widehat{c}^*/\widehat{c}_0}\widehat{b}_0. \tag{6}$$

We shall now verify that G_b satisfies the two conditions in (C1) of Section 3. (i) The value of G_b at $(\widehat{b},\widehat{c}) = (\widehat{b}_0,\widehat{c}_0)$ is b. (ii) For a given $(\widehat{b}_0,\widehat{c}_0)$, we see from the second equation of (6) and Lemma 1 that the distribution of G_b does not depend on any unknown parameters. Thus, G_b is a valid GPQ for b, and its quantiles, namely, $G_{b;\alpha/2}$ and $G_{b;1-\alpha/2}$ form a $1-\alpha$ confidence interval for b. The GTV for testing b based on G_b is given by $G_b^t = G_b - b$.

A GPQ for c can be obtained as

$$G_c = \frac{c}{\widehat{c}}, \quad \widehat{c}_0 = \frac{\widehat{c}_0}{\widehat{c}^*}. \tag{7}$$

It can be easily checked that the G_c satisfies conditions (C1) and (C2), respectively.

As mentioned earlier, a useful feature of the GV approach is that a GPQ for a function of b and c can be obtained by simply plugging their GPQs in the function. Specifically, a GPQ for a function $h(b,c)$ is given by $h(G_b,G_c)$, and a GTV is given by $h(G_b,G_c) - h(b,c)$.

5. Inference for the mean

5.1. Confidence intervals for the mean

The mean of a Weibull(b,c) distribution is given by $\eta = b\ln\Gamma(1+1/c)$. The Wald confidence intervals are developed on the basis of asymptotic normality of the MLEs. In particular, $(\widehat{b},\widehat{c})$ has an asymptotic bivariate normal distribution with mean (b,c) and the covariance matrix determined by the Fisher information matrix. It appears that the distribution of $\ln\widehat{\eta}$ is better approximated by a normal than that of $\widehat{\eta}$, and so asymptotically

$$\frac{\ln\widehat{\eta} - \ln\eta}{\sqrt{\mathrm{Var}(\ln\widehat{\eta})}} \sim N(0,1),$$

where $\widehat{\eta} = \widehat{b}\Gamma(1+1/\widehat{c})$. An estimate of the variance (using the delta method) is given by

$$\widehat{\mathrm{Var}}(\widehat{\eta}) = \widehat{\mathrm{Var}}(\widehat{c})\left(\left.\frac{\partial\eta}{\partial c}\right|_{\widehat{b},\widehat{c}}\right)^2 + 2\widehat{\mathrm{Cov}}(\widehat{c},\widehat{b})\left(\left.\frac{\partial\eta}{\partial c}\right|_{\widehat{b},\widehat{c}}\right)\left(\left.\frac{\partial\eta}{\partial b}\right|_{\widehat{b},\widehat{c}}\right) + \widehat{\mathrm{Var}}(\widehat{b})\left(\left.\frac{\partial\eta}{\partial b}\right|_{\widehat{b},\widehat{c}}\right)^2 \tag{8}$$

and $\mathrm{Var}(\ln\widehat{\eta}) \simeq \mathrm{Var}(\widehat{\eta})/\widehat{\eta}^2$. The formulas for the variance and covariance estimates of the MLEs can be found in Cohen (1965). Thus, the Wald confidence intervals for η is given by

$$\exp\left(\ln\widehat{\eta} \pm z_{\alpha/2}\sqrt{\widehat{\mathrm{Var}}(\ln\widehat{\eta})}\right), \tag{9}$$

where $z_{\alpha/2}$ is the upper $\alpha/2$ quantile of a standard normal distribution. Notice that the Wald interval is also not easy to compute as the variance and covariance estimates involve computation of the gamma and digamma functions.

A GPQ for the mean $\eta = b\Gamma(1+1/c)$ can be obtained by replacing the parameters b and c, respectively, by their GPQs in (6) and (7), and is given by

$$G_\eta = G_b\Gamma(1+1/G_c) = \left(\frac{1}{\widehat{b}^*}\right)^{\widehat{c}^*/\widehat{c}_0}\widehat{b}_0\Gamma(1+\widehat{c}^*/\widehat{c}_0), \tag{10}$$

where $\widehat{b}_0$ and $\widehat{c}_0$ are observed values of $\widehat{b}$ and $\widehat{c}$, respectively. For a given $\widehat{b}_0$ and $\widehat{c}_0$, the percentiles of G_η can be estimated using Monte Carlo simulation. Appropriate percentiles of G_η form a confidence interval for η. The generalized p-value for testing $H_0: \eta \leqslant \eta_0$ vs. $H_a: \eta > \eta_0$ is given by $P(G_\eta \leqslant \eta_0)$.

Table 1
Coverage probabilities of 95% one-sided confidence limits for a Weibull mean.

n	Method	Lower limit					Upper limit				
		c					c				
		0.5	1	2	3.5	5	0.5	1	2	3.5	5
10	GV	0.94	0.95	0.94	0.94	0.95	0.95	0.95	0.94	0.94	0.95
	Wald	0.98	0.95	0.93	0.91	0.91	0.86	0.89	0.93	0.93	0.95
15	GV	0.95	0.94	0.95	0.94	0.94	0.95	0.96	0.95	0.96	0.96
	Wald	0.98	0.95	0.93	0.92	0.93	0.87	0.90	0.94	0.95	0.95
35	GV	0.95	0.95	0.95	0.95	0.95	0.96	0.95	0.95	0.95	0.95
	Wald	0.97	0.96	0.94	0.94	0.93	0.91	0.93	0.94	0.95	0.95
50	GV	0.95	0.95	0.95	0.95	0.95	0.95	0.95	0.95	0.95	0.95
	Wald	0.97	0.96	0.94	0.94	0.93	0.92	0.93	0.95	0.95	0.95

$b = 1$.

Table 2
Numbers of millions of revolutions of 23 ball bearings before failure.

17.88	28.92	33.00	41.52	42.12	45.60	48.48	51.84
51.96	54.12	55.56	67.80	68.64	68.64	68.88	84.12
93.12	98.64	105.12	105.84	127.92	128.04	173.40	

5.2. Coverage studies

To judge the accuracy of the GV procedure, we estimated the coverage probabilities of the 95% Wald and generalized confidence intervals using Monte Carlo simulation. As both confidence intervals are scale invariant, without loss of generality, we can take b to be 1 for evaluation purpose. The simulation for estimating the coverage probabilities of the generalized confidence intervals was carried out along the lines of Algorithm 2 of Krishnamoorthy and Mathew (2003). The estimated coverage probabilities of 95% one-sided limits are given in Table 1 for some sample sizes ranging from 10 to 50, and values of c ranging from 0.5 to 5. It is clear from Table 1 that the coverage probabilities of the generalized confidence intervals are in good agreement with the nominal confidence level for all the cases considered and so the GV procedure is expected to be very satisfactory even for small samples. The Wald one-sided lower limits are very conservative and one-sided upper limits are too liberal for small values of c. On the other hand, the upper limits are liberal for small values of c, and satisfactory for large values of c. The Wald confidence intervals are satisfactory only when n is at least 35, and c is not small.

Example 1. The data in Table 2 represent the number of million revolutions before failure for each of 23 ball bearings. The data were analyzed by Thoman et al. (1969) and others using a Weibull distribution. The MLEs $\widehat{c}_0 = 2.102$ and $\widehat{b}_0 = 81.874$.

The MLE $\widehat{\eta}$ for the Weibull mean $\widehat{b}_0\Gamma(1 + 1/\widehat{c}_0) = 72.51$. Substituting the above MLEs in GPQ (10), and using a simulation consisting 10,000 runs, we computed a 95% lower confidence limit for η as 60.72, and 95% upper confidence limit as 87.62. To construct the Wald confidence interval, we computed $\widehat{\mathrm{Var}}(\ln\widehat{\eta}) = 0.01097$. Using this variance estimate in (9), we computed the 95% Wald lower limit as 61.04 and the upper limit as 86.15. These two intervals are not appreciably different.

Similarly, for testing $H_0 : \eta \leqslant 56$ vs. $H_a : \eta > 56$, we estimated the generalized p-value as 0.012. To compute the p-value of the Wald test, note that the observed value is $(\ln\widehat{\eta} - \ln\eta_0)/\sqrt{\widehat{\mathrm{Var}}(\ln\widehat{\eta})} = 2.4679$ and so the p-value is $1 - \Phi(2.4679) = 0.007$, where Φ is the standard normal cumulative distribution function. Both tests indicate that the true mean is significantly larger than 56 at the level 0.05.

6. One-sided prediction limits for at least *l* of *m* observations at each of *r* locations

We shall first describe a GV approach to find a prediction interval for a future observation x from a Weibull(b,c) distribution. The problem of predicting a single future observation has been well addressed in the literature (e.g., Fertig et al., 1980). We describe the GV approach for this special case so that the readers can follow the GV approach for the general case easily. Let $x_1, \ldots, x_n$ be a sample from a Weibull(b,c) distribution, and let $y_i = \ln(x_i)$ so that we can regard $y_1, \ldots, y_n$ as a sample from an extreme-value distribution with the location parameter $\mu = \ln(b)$ and the scale parameter $\sigma = 1/c$. The MLEs of μ and σ are given by $\widehat{\mu} = \ln(\widehat{b})$ and $\widehat{\sigma} = 1/\widehat{c}$. Furthermore, $(y_i - \widehat{\mu})/\widehat{\sigma}$, $i = 1, \ldots, n$ are ancillary statistics and their distribution does not depend on μ or σ (see Lawless, 2003, p. 568). Using this result, a GPQ for y can be obtained as

$$\widehat{\mu}_0 + \frac{y - \widehat{\mu}}{\widehat{\sigma}}\widehat{\sigma}_0, \tag{11}$$

where $(\widehat{\mu}, \widehat{\sigma})$ are the MLEs based on a random sample of size n from an extreme-value (μ, σ) distribution and $(\widehat{\mu}_0, \widehat{\sigma}_0)$ is an observed value of $(\widehat{\mu}, \widehat{\sigma})$. The variables $\widehat{\mu}$, $\widehat{\sigma}$ and y are mutually independent. Let $q = y - \widehat{\mu}/\widehat{\sigma}$ and q_p denotes the pth quantile of q. Then $\widehat{\mu}_0 + q_{1-\alpha}\widehat{\sigma}_0$ is a $1-\alpha$ upper prediction limit for a future observation $y = \ln(x)$. As the distribution of q does not depend on any unknown parameters, its percentiles can be estimated using Monte Carlo simulation. Specifically, q is distributed as $(y^* - \widehat{\mu}^*)/\widehat{\sigma}^*$,where $y^* \sim$extreme-value$(0,1)$, and $\widehat{\mu}^*$ and $\widehat{\sigma}^*$ are the MLEs based on a sample from an extreme-value$(0,1)$ distribution, and so the percentiles of q can be obtained using Monte Carlo simulation. Notice that the $1-\alpha$ generalized prediction interval based on (11) is a realization of the random interval $\widehat{\mu} + q_{1-\alpha}\widehat{\sigma}$, and for a $y_0 \sim$ extreme $-$ value(μ, σ) independently of $\widehat{\mu}$ and $\widehat{\sigma}$, $P(\widehat{\mu} + q_{1-\alpha}\widehat{\sigma} \geqslant y_0) = P((y_0 - \widehat{\mu})/\widehat{\sigma} \leqslant q_{1-\alpha}) = 1-\alpha$. Thus, the coverage probability of the prediction interval is $1-\alpha$ for all (μ, σ) or (b, c).

A $1-\alpha$ prediction interval for a future observation y is given by $(\widehat{\mu} + q_{\alpha/2}\widehat{\sigma}, \widehat{\mu} + q_{1-\alpha/2}\widehat{\sigma})$, where q_p denotes the pth quantile of $(y^* - \widehat{\mu}^*)/\widehat{\sigma}^*$, and y^*, $\widehat{\mu}^*$ and $\widehat{\sigma}^*$ are as defined in the preceding paragraph.

We shall now give an empirical method of constructing a $1-\alpha$ UPL so that it will include at least l of m samples from each of r locations. In view of the prediction limits given in Davis and McNichols (1987) and Bhaumik and Gibbons (2006), we shall consider the UPL of the form $\widehat{\mu} + u_{n,r,m,l}\widehat{\sigma}$, where $u_{n,r,m,l}$ is the factor to be determined so that the coverage probability is $1-\alpha$. Let $\widehat{\mu}^*$ and $\widehat{\sigma}^*$ be the MLEs based on a sample $y_1^*, \ldots, y_n^*$ from an extreme-value$(0,1)$ distribution, and $y_{11}^*, \ldots, y_{1m}^*, y_{21}^*, \ldots, y_{2m}^*, \ldots, y_{r1}^*, \ldots, y_{rm}^*$ be independent samples from an extreme-value $(0,1)$ distribution. Assume that y_i^*'s and y_{ij}^*'s are all mutually independent. Let $y_{i(l)}^*$ be the lth order statistic, $i = 1, \ldots, r$, $y_{r,m,l}^* = \max_{1 \leqslant i \leqslant r} y_{i(l)}^*$ and $u = (y_{r,m,l}^* - \widehat{\mu}^*)/\widehat{\sigma}^*$.The $1-\alpha$ quantile of u is the desired factor for constructing UPL. A $1-\alpha$ UPL that will include at least l of m observations from each of r locations is given by

$$\exp(\widehat{\mu} + u_{n,r,m,l}\widehat{\sigma}). \tag{12}$$

As the distribution of u does not depend on any parameter, its percentiles can be obtained using Monte Carlo simulation as explained in the following algorithm. Furthermore, using the argument similar to the one in the preceding paragraph, it can be shown that the GV procedure for constructing UPL is exact except for simulation error.

Algorithm 1.

For a given n, r, m, l and $1-\alpha$:
For $j = 1, N$
  Generate $y_1^*, \ldots, y_n^*$ from an extreme-value$(0,1)$ distribution
  Compute the MLEs $\widehat{\mu}^*$ and $\widehat{\sigma}^*$
  Generate $y_{i1}^*, \ldots, y_{im}^*$ from an extreme-value$(0,1)$ distribution, $i = 1, \ldots, r$
  Find the lth order statistic $y_{i(l)}$, $i = 1, \ldots, r$
  Set $y_{r,m,l}^* = \max_{1 \leqslant i \leqslant r} y_{i(l)}^*$
  Set $u_j = (y_{r,m,l}^* - \widehat{\mu}^*)/\widehat{\sigma}^*$
(end j loop)

The $100(1-\alpha)$th percentile of the u_j's is a Monte Carlo estimate of the factor $u_{n,r,m,l}$.

We evaluated the factors $u_{n,r,m,l}$ for computing 95% UPL for a Weibull distribution and presented them in Table 6. The factors are given for values of n, r, m and l as given in Davis and McNichols (1987). As an illustration, suppose one wants to find 95% UPL based on a sample of size 15 that will include at least 2 of 3 future observations at each of eight locations. Here $n = 15, r = 8, m = 3$ and $l = 2$, and the required factor from Table 6 is 1.447.

Remark 1. As mentioned in the introduction, in some applications it is desired to find an upper prediction limit for the largest observation in a future sample of m observations. In this case, the factor $u_{n,1,m,m}$ can be used to find such an upper prediction limit. Algorithm 1 can also be used to find a lower prediction limit for the smallest observation in a future sample of m observations. The necessary factor for this problem is the 100α percentile of the u_j's (with $r = 1$ and $l = 1$) generated by Algorithm 1. We also note that in some applications, a two-sided prediction interval that includes at least l of m samples from each of r locations may be desired. The problem of constructing such an interval seems to be difficult, and it is yet to be investigated. In particular, the applicability of the GV approach for this two-sided prediction interval problem is not clear.

Example 2. To illustrate the construction of a UPL, we use the data given in Table 1 of Bhaumik and Gibbons (2006). The data, reproduced here in Table 3, represent vinyl chloride concentrations collected from clean upgradient monitoring wells. A Q–Q plot by Bhaumik and Gibbons showed an excellent fit of these data to a gamma distribution. We also found that a Weibull model also fits the data very well (see Fig. 1). We shall construct various Weibull UPLs and compare them with those given in Bhaumik and Gibbons (2006) and Krishnamoorthy et al. (2007) that were calculated assuming a gamma distribution.

The values of the factor $u_{n,r,m,l}$ for constructing 95% UPLs for a few combinations of r, m and l are given in Table 4. In order to compare the Weibull based UPLs with those of Bhaumik and Gibbons (2006) and those of Krishnamoorthy et al. (2007), we chose the same combinations of r, l and m as given in their papers. The Weibull based prediction limits, along with those based

Table 3
Vinyl chloride data from clean upgradient ground-water monitoring wells in ($\mu g/L$).

5.1	2.4	0.4	0.5	2.5	0.1	6.8	1.2	0.5	0.6
5.3	2.3	1.8	1.2	1.3	1.1	0.9	3.2	1.0	0.9
0.4	0.6	8.0	0.4	2.7	0.2	2.0	0.2	0.5	0.8
2.0	2.9	0.1	4.0						

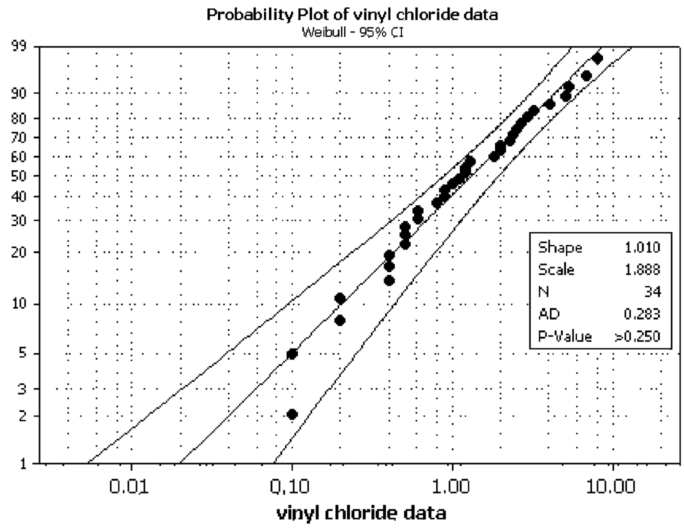

Fig. 1. Probability plot of vinyl chloride data.

Table 4
95% Upper prediction limits for the vinyl chloride data.

r	m	l	$u_{n,r,m,l}$	Weibull UPL	KMM[a]	Bhaumik–Gibbons[b]
1	2	1	0.461	2.974	2.893	2.931
10	2	1	1.079	5.483	5.203	5.224
10	3	1	0.659	3.618	3.479	3.521
10	3	2	1.296	6.797	6.369	6.330

$n=34$, $\widehat{\mu}=0.6335$, and $\widehat{\sigma}=0.990$; MLEs $\widehat{\mu}=\ln(\widehat{b})=0.6335$ and $\widehat{\sigma}=1/\widehat{c}=0.990$.

[a]KMM—approximate limits by Krishnamoorthy et al. (2007) approach.

[b]Approximate limits by Bhaumik and Gibbons (2006); both papers assume a gamma distribution.

on a gamma distribution are given in Table 4. We observe from this table that the UPLs based on a Weibull model are slightly larger than those based on a gamma model for all the cases considered. It should be noted that the UPLs based on the Weibull distribution are exact except for simulation errors whereas the ones based on a gamma distribution are approximate.

7. Censored samples

If the samples are type II censored, then the pivotal quantities based on the MLEs are still valid, and so the GV approaches can be extended in a straightforward manner to make inferences about Weibull parameters or for other problems addressed in the preceding sections. The properties of the GV procedures for type II censored samples should be similar to those of the complete sample case. So, we shall illustrate the GV procedure for constructing a confidence interval for a Weibull mean when the sample is type I censored. Recall that in type I censoring, failures are recorded until a mission time, and so the number of failed items k in a random sample is a random variable. Because the distribution of k depends on the parameters, the pivotal quantities given in Lemma 1 are no longer valid for the type I censored samples. However, we can use the pivotal quantities for type II censored samples as approximations for the type I censored samples. This type of approximation was noted by Schmee et al. (1985) for the normal case. This approximation is expected to work for a large sample because $x_{(k)}$ is likely to be in the neighborhood of the mission time, and so the MLEs based on type I or type II censored samples are the same.

Table 5
Coverage probabilities of 95% confidence intervals based on type I censored samples for a Weibull mean.

n	P_{x_0}	c									
		0.5	1	1.5	2	2.5	3	3.5	4.0	4.5	5
20	0.10	0.93	0.95	0.95	0.94	0.95	0.95	0.94	0.94	0.95	0.95
	0.30	0.93	0.94	0.94	0.95	0.94	0.94	0.95	0.94	0.95	0.95
	0.60	0.95	0.95	0.94	0.93	0.95	0.94	0.93	0.95	0.95	0.95
30	0.10	0.94	0.96	0.95	0.95	0.94	0.95	0.95	0.94	0.95	0.95
	0.30	0.95	0.94	0.95	0.95	0.94	0.94	0.95	0.94	0.95	0.95
	0.60	0.94	0.94	0.95	0.94	0.95	0.95	0.95	0.95	0.95	0.95

$b = 1$; P_{x_0} = proportion of survivors.

Table 6
Factors $u_{n,r,m,l}$ for 95% UPL as a Function of n, r, m, and l

$n = 5$				$n = 8$				$n = 10$			
r	l	m	$u_{n,r,m,l}$	r	l	m	$u_{n,r,m,l}$	r	l	m	$u_{n,r,m,l}$
1	1	2	0.859	1	1	2	0.653	1	1	2	0.600
1	1	3	0.352	1	1	3	0.212	1	1	3	0.151
1	1	4	0.029	1	2	4	0.619	1	2	4	0.554
1	2	4	0.862	1	2	5	0.322	1	2	5	0.275
1	2	5	0.514	1	2	6	0.103	1	2	6	0.051
1	2	6	0.265	2	1	2	0.944	2	1	2	0.860
2	1	2	1.257	2	1	3	0.469	2	1	3	0.418
2	1	3	0.678	2	1	4	0.153	2	1	4	0.099
2	1	4	0.312	2	2	4	0.859	2	2	4	0.777
2	1	5	0.058	2	2	5	0.554	2	2	5	0.474
2	2	5	0.811	2	2	6	0.308	2	2	6	0.259
2	2	6	0.526	4	1	2	1.212	4	1	2	1.105
4	1	2	1.621	4	1	3	0.708	4	1	3	0.622
4	1	3	1.003	4	1	4	0.367	4	1	4	0.316
4	1	4	0.593	4	1	5	0.143	4	1	5	0.080
4	1	5	0.322	4	2	4	1.085	4	2	4	0.974
4	1	6	0.106	4	2	5	0.744	4	2	5	0.663
4	2	6	0.785	4	2	6	0.508	4	2	6	0.442
8	1	2	2.006	8	1	2	1.477	8	1	2	1.335
8	1	3	1.295	8	1	3	0.937	8	1	3	0.835
8	1	4	0.870	8	1	4	0.579	8	1	4	0.511
8	1	5	0.560	8	1	5	0.321	8	1	5	0.271
8	1	6	0.335	8	1	6	0.126	8	1	6	0.072
8	2	4	1.831	8	2	4	1.309	8	2	4	1.167
8	2	5	1.369	8	2	5	0.945	8	2	5	0.834
8	2	6	1.020	8	2	6	0.691	8	2	6	0.592
16	1	2	2.395	16	1	2	1.721	16	1	2	1.561
16	1	3	1.604	16	1	3	1.145	16	1	3	1.029
16	1	4	1.135	16	1	4	0.783	16	1	4	0.676
16	1	5	0.804	16	1	5	0.512	16	1	5	0.420
16	1	6	0.550	16	1	6	0.307	16	1	6	0.226
16	2	5	1.619	16	2	5	1.132	16	2	5	1.002
16	2	6	1.258	16	2	6	0.858	16	2	6	0.750

$n = 15$				$n = 20$				$n = 30$			
r	l	m	$u_{n,r,m,l}$	r	l	m	$u_{n,r,m,l}$	r	l	m	$u_{n,r,m,l}$
1	1	2	0.536	1	1	2	0.500	1	1	2	0.452
1	1	3	0.101	1	1	3	0.074	1	2	3	0.764
1	2	3	0.862	1	2	3	0.828	1	2	4	0.399
1	2	4	0.480	1	2	4	0.431	1	2	5	0.135
1	2	5	0.201	1	2	5	0.166	2	1	2	0.675
1	3	6	0.409	1	3	6	0.378	2	1	3	0.271
2	1	2	0.766	2	1	3	0.303	2	2	2	1.626
2	1	3	0.327	2	1	3	0.297	2	2	3	0.950
2	2	4	0.670	2	2	3	1.010	2	2	4	0.572
2	2	5	0.392	2	2	4	0.625	2	2	5	0.306
2	2	6	0.176	2	2	5	0.345	2	3	6	0.500
4	1	2	0.989	2	2	6	0.138	4	1	2	0.869
4	1	3	0.537	2	3	6	0.540	4	1	3	0.451
4	1	4	0.222	4	1	2	0.932	4	1	4	0.158
4	2	3	1.269	4	1	3	0.494	4	2	3	1.128

Table 6 (*Continued*).

$n=5$				$n=8$				$n=10$			
r	l	m	$u_{n,r,m,l}$	r	l	m	$u_{n,r,m,l}$	r	l	m	$u_{n,r,m,l}$
4	2	4	0.844	4	2	3	1.197	4	2	4	0.734
4	2	5	0.555	4	2	4	0.791	4	2	5	0.461
4	2	6	0.335	4	2	5	0.505	4	2	6	0.242
8	1	2	1.174	4	2	6	0.288	4	3	6	0.633
8	1	3	0.722	8	1	2	1.107	8	1	2	1.045
8	1	4	0.406	8	1	3	0.670	8	1	3	0.617
8	1	5	0.179	8	1	4	0.362	8	1	4	0.326
8	2	3	1.447	8	1	5	0.133	8	2	3	1.266
8	2	4	1.013	8	2	4	0.937	8	2	4	0.875
8	2	5	0.712	8	2	5	0.648	8	2	5	0.596
8	2	6	0.485	8	2	6	0.427	8	2	6	0.379
16	1	2	1.361	8	3	6	0.827	8	3	6	0.758
16	1	3	0.887	16	1	2	1.276	16	1	2	1.196
16	1	4	0.559	16	1	3	0.821	16	1	3	0.759
16	1	5	0.326	16	1	4	0.513	16	1	4	0.458
16	1	6	0.143	16	1	5	0.280	16	1	5	0.234
16	2	3	1.621	16	1	6	0.097	16	2	3	1.407
16	2	4	1.154	16	2	4	1.078	16	2	4	0.998
16	2	5	0.847	16	2	5	0.785	16	2	5	0.716
16	2	6	0.615	16	2	6	0.553	16	2	6	0.500

7.1. Generalized variable procedures for type I censored samples

The inferential procedures for the type I censored samples with censoring time x_0 can be obtained as follows. Let $x_{(1)} < \cdots < x_{(k)}$ be a set of failure times recorded (before the termination time x_0) from a sample of n test items. Let $\widehat{c}$ be the MLE determined by Eq. (3) with $x_{i*} = x_{(i)}, i = 1, \ldots, k$ and $x_{i*} = x_0, i = k+1, \ldots, n$, and $\widehat{b} = (\sum_{i=1}^n x_{i*}^{\widehat{c}}/k)^{1/\widehat{c}}$. Then, $\widehat{c}/c$ is approximately distributed as $\widehat{c}^*$ and $\widehat{c}\ln(\widehat{b}/b)$ is approximately distributed as $\widehat{c}^*\ln(\widehat{b}^*)$, where $\widehat{c}^*$ satisfies

$$\frac{1}{\widehat{c}^*} - \frac{\sum_{i=1}^n z_{i*}^{\widehat{c}^*}\ln(z_{i*})}{\sum_{i=1}^n z_{i*}^{\widehat{c}^*}} + \frac{1}{k}\sum_{i=1}^k \ln(z_{i*}) = 0 \tag{13}$$

and $\widehat{b}^* = (\sum_{i=1}^n x_{i*}^{\widehat{c}^*}/k)^{1/\widehat{c}^*}$. In the above equation, $z_{i*} = z_{(i)}, i = 1, \ldots, k$ and $z_{i*} = z_{(k)}, i = k+1, \ldots, n$, where $z_1, \ldots, z_n$ is a random sample from a Weibull$(1,1)$ distribution. Using these approximate distributional results, one can obtain solutions to all the problems discussed earlier.

7.2. Coverage studies for the type I censored case

To appraise the accuracy of the GV procedures on the basis of the approximate distributions given in the preceding paragraph, we evaluated coverage probabilities of confidence intervals for a Weibull mean and of confidence intervals for comparing two Weibull means. The simulation study was carried out as in Section 5.1, and the coverage probabilities of 95% confidence intervals for a Weibull(b,c) are reported in Table 5. The estimated coverage probabilities clearly indicate that the GV procedure for constructing confidence interval works satisfactorily when proportion of censored data P_{x_0} is no more than 0.6. Indeed, our evaluation studies (not reported here) showed that the GV procedure works satisfactorily as long as P_{x_0} is 0.65 or less. We also observed that the GV procedure was liberal for large values of P_{x_0}.

8. Concluding remarks

We have provided GV procedures for making inferences for some standard problems involving Weibull distributions. The GV approach is in general easy to use and is expected to produce satisfactory results even for small samples as indicated by our simulation studies. The GV approach can be readily applied to find one-sided tolerance limits and to find confidence intervals for a survival probability. Indeed, our investigation (see Lin, 2009) showed that the results for these problems coincide with those of the Monte Carlo approach by Thoman et al. (1970). We have also showed that the GV procedures can be used to make approximate inferences for a Weibull mean with complete or censored samples. Inferential procedures for other problems when the data are type I censored can be obtained similarly. For example, GV approach can be used to find one-sided confidence limits for quantiles (equivalently, one-sided tolerance limits), to find confidence intervals for a survival probability, and to find upper prediction limits (as in Section 6) based on type I censored samples.

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