One-Sided Tolerance Limits in Balanced and Unbalanced One-Way Random Models Based on Generalized Confidence Intervals

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We consider the problem of deriving one-sided tolerance intervals in the one-way random model with balanced as well as unbalanced data, under the usual normality assumptions. The problems investigated deal with the computation of such intervals for the observable random variable, as well as the unobservable random effect in the one-way random model. The tolerance limits are derived using the concept of a generalized confidence interval. Some approximations are derived for the tolerance limits, and their performance is investigated by simulation. The simulation results show that the proposed tolerance limits are quite satisfactory for practical use.

KEY WORDS: Confidence; Content; Generalized pivotal quantity; Lower tolerance limit; Noncentral t; Upper tolerance limit.

1. INTRODUCTION

The problem of constructing tolerance limits in the one-way random-effects model has been investigated by several authors (see, e.g., Mee and Owen 1983; Mee 1984; Bhaumik and Kulkarni 1991, 1996; Vangel 1992); for details on earlier work on the problem, see these articles. The problem has important engineering applications. In particular, Vangel (1992) described an application where tolerance limits are required on the tensile strength of composite materials used in aircraft components. Tolerance intervals are also relevant in the analysis of data on occupational exposure to contaminants. In the context of samples from a univariate normal population, such applications have been discussed by Tuggle (1982) and Lyles and Kupper (1996). The relevance of the one-way random model for analyzing occupational exposure data was pointed out in Lyles, Kupper, and Rappaport (1997a, b), and the tolerance interval problem is quite relevant in this situation. Specifically, if an upper tolerance limit based on a sample of exposure measurements is less than a specified standard, then it is likely that most future measurements will be lower than the standard, and hence exposure monitoring might be reduced or terminated until a process change occurs.

Among the available procedures for computing one-sided tolerance limits in the one-way balanced random-effects model, Vangel's (1992) procedure appears to be the most satisfactory. It should also be noted that the aforementioned articles, except that by Bhaumik and Kulkarni (1991), deal only with balanced data. Furthermore, all of the cited articles deal with the computation of a tolerance limit for the values of an observable random variable following the one-way random model. In some applications, one may be interested in tolerance limits for the unobservable random effect. Important work in this direction was done by Wang and Iyer (1994); an example that they considered involves the determination of sulfur content in bottles of coal. The problem of interest was the construction of a tolerance interval for the distribution of the true sulfur content for the population of bottles, ignoring measurement errors. Wang and Iyer (1994) constructed two-sided tolerance limits for the true sulfur content, that is, for the random effect in a one-way random model, ignoring measurement errors. Their article, dealing with balanced data, also contains several other examples.

This article explores the computation of a one-sided tolerance limit for the observable random variable or the unobservable random effect in a one-way random model with balanced as well as unbalanced data. As is well known, the problem of computing a one-sided tolerance limit reduces to that of computing a one-sided confidence limit for the percentile of the relevant probability distribution. In the case of the one-way random model, the main difficulty in solving this problem is that the variance ratio is unknown. Some authors have dealt with this problem by first computing the one-sided confidence limit assuming that the variance ratio is known, and then replacing the unknown variance ratio by a confidence limit (see, e.g., Mee and Owen 1983; Bhaumik and Kulkarni 1991). However, we obtain a solution using the concept of a *generalized confidence interval* due to Weerahandi (1993) (see also Weerahandi 1995). Extensive simulation results show that our generalized confidence interval approach, and some approximations that we have developed, provide satisfactory tolerance limits. We have investigated by simulation two aspects of the tolerance limit: the actual confidence level achieved by the tolerance interval and the expected value of the tolerance limit. For an upper tolerance limit, the smaller the expected value, the better. In terms of the actual confidence level, the numerically obtained tolerance limit is satisfactory in all situations. The approximation that we have developed is satisfactory in many cases, especially when the intraclass correlation is at least .50. In the case of balanced data, comparison with two approximate tolerance limits

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due to Vangel (1992) shows that his cubic polynomial approximation is quite satisfactory in most situations, whereas his other approximation can be quite conservative, especially when the number of levels of the random effect becomes large.

2. SOME PRELIMINARIES

2.1 The Model and the Tolerance Interval Problem

Let X_{ij} denote observations following the one-way random model given by

$$X_{ij} = \mu + \tau_i + e_{ij}, \qquad j = 1, 2, \ldots, n_i, \quad i = 1, 2, \ldots, k, \quad (1)$$

where μ is a fixed unknown parameter and the τ_i's and e_{ij}'s are independent random variables with $\tau_i \sim N(0, \sigma_\tau^2)$ and $e_{ij} \sim N(0, \sigma_e^2)$. Thus $X_{ij} \sim N(\mu, \sigma_\tau^2 + \sigma_e^2)$, with $\text{cov}(X_{ij}, X_{i'j'}) = \sigma_\tau^2 \delta_{ii'} + \sigma_e^2 \delta_{ii'} \delta_{jj'}$. We address the following tolerance interval problems:

(a) An upper tolerance limit for the observable random variable X, where $X \sim N(\mu, \sigma_\tau^2 + \sigma_e^2)$, and

(b) An upper tolerance limit for the unobservable random variable $\mu + \tau$, where $\tau \sim N(0, \sigma_\tau^2)$.

Problem (b) was addressed by Wang and Iyer (1994), who constructed two-sided tolerance limits based on balanced data. All of the other articles mentioned in Section 1 deal with problem (a).

A (p, γ) upper (lower) tolerance limit is a statistic for which at least $100p\%$ of the population of an underlying random variable is less than (greater than) the tolerance limit with $100\gamma\%$ confidence. The quantities p and γ are referred to as the *content* and the *confidence*. It is easily seen that (a) a (p, γ) upper tolerance limit for $N(\mu, \sigma_\tau^2 + \sigma_e^2)$ is simply a $100\gamma\%$ upper confidence limit for $\mu + z_p\sqrt{\sigma_\tau^2 + \sigma_e^2}$, where z_p denotes the $100p$th percentile of the standard normal distribution, and (b) a (p, γ) upper tolerance limit for $N(0, \sigma_\tau^2)$ is simply a $100\gamma\%$ upper confidence limit for $\mu + z_p\sigma_\tau$. We use the concept of a generalized confidence interval for obtaining the aforementioned upper confidence limits. A brief introduction to the generalized confidence limit follows.

2.2 Generalized Confidence Intervals

The concept of a *generalized confidence interval* is due to Weerahandi (1993); we refer to his book (Weerahandi 1995) for a detailed discussion along with numerous examples. The setup is as follows. Consider a random variable Y (scalar or vector) whose distribution depends on a scalar parameter of interest θ and a nuisance parameter η. Let y denote the observed value of Y. To construct a generalized confidence interval for θ, we first define a *generalized pivotal quantity*, $T(Y; y, \theta, \eta)$, which is a function of the random variable Y, its observed value y, and the parameters θ and η. $T(Y; y, \theta, \eta)$ is required to satisfy the following conditions:

a. For fixed y, the distribution of $T(Y; y, \theta, \eta)$ is free of unknown parameters.

b. The observed value of $T(Y; y, \theta, \eta)$, namely $T(y; y, \theta, \eta)$, is simply θ. (2)

Now let $T_{1-\alpha}$ denote the 100αth percentile of $T(Y; y, \theta, \eta)$. Then $T_{1-\alpha}$ is a generalized upper confidence limit for θ and $\{\theta : \theta \le T_{1-\alpha}\}$ is a $100(1-\alpha)\%$ generalized confidence interval for θ.

Note that in the foregoing generalized confidence interval, the data enter into the picture through $T_{1-\alpha}$, which obviously depends on y. The generalized confidence interval approach has been successfully applied to obtain confidence intervals for a number of problems (e.g., the Behrens–Fisher problem) for which traditional approaches are difficult to apply. In particular, the concept is applicable to confidence intervals on functions of variance components in mixed- and random-effects models (see Weerahandi 1995 for details).

3. AN UPPER TOLERANCE LIMIT FOR $N(\mu, \sigma_\tau^2 + \sigma_e^2)$ BASED ON BALANCED DATA

Consider the model (1) for balanced data, that is,

$$X_{ij} = \mu + \tau_i + e_{ij}, \qquad j = 1, 2, \ldots, n, \quad i = 1, 2, \ldots, k. \quad (3)$$

Define $\bar{X}_{i.} = \sum_{j=1}^n X_{ij}/n$, $\bar{X}_{..} = \sum_{i=1}^k \sum_{j=1}^n X_{ij}/(kn)$, $SS_\tau = n\sum_{i=1}^k (\bar{X}_{i.} - \bar{X}_{..})^2$, and $SS_e = \sum_{i=1}^k \sum_{j=1}^n (X_{ij} - \bar{X}_{i.})^2$. Then $\bar{X}_{..}$, SS_τ, and SS_e are independently distributed with

$$Z = \sqrt{kn}\frac{(\bar{X}_{..} - \mu)}{\sqrt{n\sigma_\tau^2 + \sigma_e^2}} \sim N(0, 1),$$

$$U_\tau = SS_\tau/(n\sigma_\tau^2 + \sigma_e^2) \sim \chi_{k-1}^2, \quad (4)$$

and

$$U_e = SS_e/\sigma_e^2 \sim \chi_{k(n-1)}^2,$$

where χ_r^2 denotes central chi-squared distribution with r degrees of freedom (df).

To define a generalized pivotal quantity, let $\bar{x}_{..}$, ss_τ, and ss_e denote the observed values of the random variables $\bar{X}_{..}$, SS_τ, and SS_e. The observed values $\bar{x}_{..}$, ss_τ and ss_e are treated as fixed. The generalized pivotal quantity, say T_1, that we define is a function of the random variables $\bar{X}_{..}$, SS_τ, and SS_e; their observed values $\bar{x}_{..}$, ss_τ, and ss_e; and the parameters

$$\begin{aligned}
T_1 &= \bar{x}_{..} - \frac{\sqrt{kn}(\bar{X}_{..} - \mu)}{\sqrt{SS_\tau}}\sqrt{\frac{ss_\tau}{kn}} \\
&\quad + z_p\left[\left(\frac{\sigma_e^2 + n\sigma_\tau^2}{nSS_\tau}ss_\tau - \frac{\sigma_e^2}{nSS_e}ss_e\right) + \frac{\sigma_e^2}{SS_e}ss_e\right]^{1/2} \\
&= \bar{x}_{..} - \frac{Z}{\sqrt{U_\tau}}\sqrt{\frac{ss_\tau}{kn}} + \frac{z_p}{\sqrt{n}}\left[\frac{ss_\tau}{U_\tau} + (n-1)\frac{ss_e}{U_e}\right]^{1/2} \\
&= \bar{x}_{..} + H,
\end{aligned}$$

where

$$H = -\frac{Z}{\sqrt{U_\tau}}\sqrt{\frac{ss_\tau}{kn}} + \frac{z_p}{\sqrt{n}}\left[\frac{ss_\tau}{U_\tau} + (n-1)\frac{ss_e}{U_e}\right]^{1/2} \quad (5)$$

and z_p denotes the $100p$th percentile of the standard normal distribution. From the second expression in (5), we see that for fixed $\bar{x}_{..}$, ss_τ, and ss_e, the distribution of T_1 is free of any unknown parameters. It can be readily verified that the observed value of T_1 [obtained by replacing $\bar{X}_{..}$, SS_τ, and SS_e with $\bar{x}_{..}$, ss_τ, and ss_e, in the first expression in (5)] is $\mu + z_p\sqrt{\sigma_\tau^2 + \sigma_e^2}$.

In other words, T_1 satisfies the two conditions in (2). Thus if $T_{1\gamma}$ denotes the 100γth percentile of T_1, then $T_{1\gamma}$ is a $100\gamma\%$ generalized upper confidence limit for $\mu + z_p\sqrt{\sigma_\tau^2 + \sigma_e^2}$, and hence is also a (p, γ) generalized upper tolerance limit for $N(\mu, \sigma_\tau^2 + \sigma_e^2)$.

We now explain the computation of $T_{1\gamma}$. From the second expression in (5), it is clear that $T_{1\gamma}$ can be easily obtained by simulation. Recall once again that once the data are obtained, $\bar{x}_{..}$, ss_τ, and ss_e are fixed numbers. Thus, generate the values of the independent random variables Z, U_τ, and U_e numerous times and compute the corresponding values of T_1 using the second expression in (5). The 100γth percentile of the T_1 values so generated is an estimate of $T_{1\gamma}$.

It is also possible to compute $T_{1\gamma}$ by numerical integration. For this, we note from (5) that

$$T_{1\gamma} = \bar{x}_{..} + \text{the } 100\gamma\text{th percentile of } H.$$

After straightforward algebra, H in (5) can be written as

$$H = \sqrt{\frac{ss_\tau}{kn(kn-1)Y}}\,\frac{Z + z_p\sqrt{k}(1 + \frac{(n-1)Yss_e}{(1-Y)ss_\tau})^{1/2}}{\sqrt{(U_\tau + U_e)/(kn-1)}},$$

where $Y = U_\tau/(U_\tau + U_e)$ has a beta distribution with parameters $(k-1)/2$ and $k(n-1)/2$. Note that Y is independent of $U_\tau + U_e \sim \chi^2_{kn-1}$. Thus conditionally, given Y, H has the representation

$$H = \sqrt{\frac{ss_\tau}{kn(kn-1)Y}}\,t_{kn-1}(\delta(Y)),$$

$$\text{where} \qquad \delta(Y) = z_p\sqrt{k}\left(1 + \frac{(n-1)Yss_e}{(1-Y)ss_\tau}\right)^{1/2}$$

and $t_{kn-1}(\delta(Y))$ denotes a noncentral t random variable with $kn-1$ df and noncentrality parameter $\delta(Y)$. Let c be the solution of the equation

$$\frac{\Gamma((kn-1)/2)}{\Gamma((k-1)/2, k(n-1)/2)} \times \int_0^1 P\big(t_{kn-1}(\delta(y)) \le c\sqrt{y}\big)y^{\frac{k-1}{2}-1}(1-y)^{\frac{k(n-1)}{2}-1}\,dy = \gamma.$$

Then $T_{1\gamma} = \bar{x}_{..} + c\sqrt{\frac{ss_\tau}{kn(kn-1)}}$.

3.1 An Approximation

To approximate $T_{1\gamma}$, note that H in (5) can also be expressed as

$$\begin{aligned} H &= \sqrt{\frac{ss_\tau}{k(k-1)n}}\left[\frac{-Z + z_p(k + (k-1)\frac{ss_e}{ss_\tau}\frac{U_\tau/(k-1)}{U_e/(k(n-1))})^{1/2}}{\sqrt{U_\tau/(k-1)}}\right] \\ &= \sqrt{\frac{ss_\tau}{k(k-1)n}}\left[\frac{-Z + z_p(k + (k-1)\frac{ss_e}{ss_\tau}F)^{1/2}}{\sqrt{U_\tau/(k-1)}}\right] \\ &= \sqrt{\frac{ss_\tau}{k(k-1)n}}\left[\frac{-Z + z_p\sqrt{k}(1 + (n-1)\frac{F}{f})^{1/2}}{\sqrt{U_\tau/(k-1)}}\right], \end{aligned} \tag{6}$$

where $F = \frac{U_\tau/(k-1)}{U_e/(k(n-1))}$ follows a central F distribution with $(k-1, k(n-1))$ df, and $f = \frac{ss_\tau/(k-1)}{ss_e/k(n-1)}$ is the observed mean squared ratio. We approximate H in (6) by replacing the random variable F with $F_{(k-1),k(n-1)}(1-\gamma)$, the $100(1-\gamma)$th percentile of F. An intuitive explanation for such an approximation is given in Remark 1 later. Using such an approximation and doing straightforward algebra, we get

$$H \overset{d}{\sim} \sqrt{\frac{ss_\tau}{k(k-1)n}}\left[\frac{Z + \delta_1}{\sqrt{U_\tau/(k-1)}}\right], \tag{7}$$

where $\overset{d}{\sim}$ denotes "approximately distributed" and

$$\delta_1 = z_p\left(k + (k-1)\frac{ss_e}{ss_\tau}F_{(k-1),k(n-1)}(1-\gamma)\right)^{1/2}. \tag{8}$$

This approximation to H has a noncentral t distribution with $k-1$ df and noncentrality parameter δ_1. Hence from (5) and (7), it follows that the 100γth percentile of T_1 is approximately equal to $\bar{x}_{..} + \sqrt{\frac{ss_\tau}{k(k-1)n}}t_{k-1}(\delta_1, \gamma)$, where $t_{k-1}(\delta_1, \gamma)$ denotes the 100γth percentile of the noncentral t distribution with $k-1$ df and noncentrality parameter δ_1. Hence the approximate (p, γ) upper tolerance limit is

$$\bar{x}_{..} + t_{k-1}(\delta_1, \gamma)\sqrt{\frac{ss_\tau}{k(k-1)n}}, \tag{9}$$

where δ_1 is given in (8).

Note that to compute the tolerance limit (9), we need the percentiles of the noncentral t distribution. These percentiles can be computed using IMSL subroutine TNIN. Free calculators that compute the noncentral t percentiles are available. For example, the software *StatCalc* can be downloaded for free from *http://www.etext.net/catalog/*; an online calculator is available at *http://calculators.stat.ucla.edu/*. Also, the *Applied Statistics* algorithm AS243 due to Lenth (1989) can be used to evaluate the noncentral t cumulative distribution function; this algorithm can be downloaded from *http://lib.stat.cmu.edu*.

Remark 1. An intuitive justification for (7) is as follows. Note that we are interested in computing the 100γth percentile of

$$W = \left[Z + z_p\left(k + (k-1)\frac{ss_e}{ss_\tau}F\right)^{1/2}\right]\Big/\sqrt{U_\tau/(k-1)}.$$

The idea is to compute the 100γth percentile of this quantity after replacing the random variable F by a suitable percentile. It is clear that if we replace F by its 100γth percentile itself, say $F_{(k-1),k(n-1)}(\gamma)$, then the 100γth percentile of

$$\left[Z + z_p\left(k + (k-1)\frac{ss_e}{ss_\tau}F_{(k-1),k(n-1)}(\gamma)\right)^{1/2}\right]\Big/\sqrt{U_\tau/(k-1)}$$

will be much larger than the 100γth percentile of the original random variable W given earlier. Thus we need to replace the random variable F by a quantity smaller than $F_{(k-1),k(n-1)}(\gamma)$. Our simulation results show that $F_{(k-1),k(n-1)}(1-\gamma)$ is an appropriate choice.

Remark 2. Our approximation is exact in the limit of large σ_τ^2; this can be seen by letting the observed mean squared ratio become infinite in (8) and (9).

Remark 3. A (p, γ) lower tolerance limit for $N(\mu, \sigma_\tau^2 + \sigma_e^2)$ can be obtained similarly, and has the approximate expression $\bar{x}_{..} - t_{k-1}(\delta_1, \gamma)\sqrt{\frac{ss_\tau}{k(k-1)n}}$.

3.2 Results of a Simulation Study

We compared the following three tolerance limits by simulation:

$$\begin{aligned} &\tilde{G}_1, \ \bar{x}_{\bullet\bullet} + t_{k-1}(\delta_1, \gamma)\sqrt{\tfrac{ss_\tau}{k(k-1)n}}, \text{ the approximate tolerance limit given in (9),} \\ &\tilde{V}, \ \text{Vangel's (1992) tolerance limit given in his equation (26),} \\ &G_1, \ \text{the tolerance limit } T_{1\gamma}, \text{ the } 100\gamma\text{th percentile of } T_1 \text{ in (5), obtained by simulation.} \end{aligned} \tag{10}$$

The simulation results are reported in Table 1 for various values of k, n, and the intraclass correlation $\rho = \sigma_\tau^2/(\sigma_\tau^2 + \sigma_e^2)$. Note that the simulated confidence level of the tolerance interval is the confidence level of the confidence interval for $\mu + z_p\sqrt{\sigma_\tau^2 + \sigma_e^2}$. Without loss of generality, we set $\mu = 0$ and $\sigma_\tau^2 + \sigma_e^2 = 1$, so that $\mu + z_p\sqrt{\sigma_\tau^2 + \sigma_e^2} = z_p$. We let $\hat{\gamma}_1$, $\hat{\gamma}_2$, and $\hat{\gamma}_3$ denote the estimated confidence level based on the tolerance limits $\tilde{G}_1$, $\tilde{V}$, and G_1 in (10). Along with the estimated confidence levels, Table 1 reports the expected values of $\tilde{G}_1$, $\tilde{V}$, and G_1, given below the $\hat{\gamma}_i$ values. We can thus check how large the expected values are compared with z_p. The results in Table 1 correspond to a (.90, .95) tolerance interval, so that $z_p = z_{.90} = 1.282$. From the numerical results in Table 1, it is clear that the approximation $\tilde{G}_1$ is not satisfactory (i.e., the corresponding confidence levels are below the nominal level of .95) when k and/or n is big and ρ is small. However, the approximation appears to be satisfactory when $\rho \geq .5$, that is, when $\sigma_\tau^2 \geq \sigma_e^2$, which is a realistic condition in many applications. Vangel's (1992) approximate tolerance limit $\tilde{V}$ is quite satisfactory except when k and/or n is big, in which case it can be quite conservative. When k is small, the tolerance limit G_1 is somewhat conservative for small values of ρ. Overall, G_1 appears to be a satisfactory tolerance limit. The differences among the confidence levels obviously show up in the expected values as well. We note from Table 1 that the expected values increase with ρ.

In addition to the tolerance limit $\tilde{V}$ considered in Table 1, Vangel (1992) also gave a cubic polynomial approximation to the tolerance limit; see his equation (32). For a few values of k and n, the coefficients of the cubic polynomial are given in Vangel's table 2 for $(p, \gamma) = (.90, .95)$, and in his table 3 for $(p, \gamma) = (.99, .95)$. We have also compared the performance of the tolerance limits $\tilde{G}_1$, $\tilde{V}$, and G_1, with the approximate tolerance limit given in equation (32) of Vangel (1992), denoted by V. Toward this end, numerical results similar to those in Table 1 are given in Table 2. We have selected a few values of k and n for which the coefficients necessary to compute V are available from Vangel (1992). From the numerical results in Table 2, we conclude that Vangel's tolerance limit V exhibits excellent performance, except when n is very big and ρ is small. (For the choice $n = 10^3$ in our Table 2, we used the coefficients corresponding to $n = \infty$ from Vangel's table 2.) Our tolerance limit, G_1, is conservative for small values of ρ; otherwise, it also performs quite well.

The simulation results are based on 2,500 sets of tolerance limits calculated for each combination of parameter values. Nested within this simulation, $T_{1\gamma}$ was itself approximated by a simulation using 5,000 draws of the random variables Z, U_τ, and U_e.

Table 1. Monte Carlo Estimates of the Confidence Levels and Expected Values for the (p, γ) Upper Tolerance Limits $\tilde{G}_1$, $\tilde{V}$, and G_1 in (10) for $N(\mu, \sigma_\tau^2 + \sigma_e^2)$ Based on Balanced Data With $\mu = 0$, $\rho = \sigma_\tau^2/(\sigma_\tau^2 + \sigma_e^2)$, $\sigma_\tau^2 + \sigma_e^2 = 1$, and $(p, \gamma) = (.90, .95)$

k	n	ρ	$\hat{\gamma}_1$ $(E(\tilde{G}_1))$	$\hat{\gamma}_2$ $(E(\tilde{V}))$	$\hat{\gamma}_3$ $(E(G_1))$
3	2	0	.94 (3.95)	.96 (3.53)	.99 (4.70)
		.10	.94 (4.13)	.96 (3.68)	.99 (4.80)
		.25	.95 (4.40)	.96 (3.91)	.99 (4.88)
		.50	.95 (4.78)	.95 (4.32)	.99 (5.24)
		.95	.95 (5.42)	.94 (5.34)	.96 (5.47)
3	5	0	.93 (2.73)	.96 (2.42)	.98 (2.97)
		.10	.93 (3.14)	.95 (2.75)	.98 (3.37)
		.25	.94 (3.60)	.94 (3.20)	.98 (3.74)
		.50	.94 (4.32)	.92 (3.99)	.97 (4.43)
		.95	.95 (5.35)	.94 (5.31)	.95 (5.30)
15	10	0	.76 (1.37)	.96 (1.49)	.97 (1.50)
		.10	.86 (1.45)	.96 (1.58)	.96 (1.56)
		.25	.90 (1.57)	.96 (1.69)	.95 (1.65)
		.50	.93 (1.74)	.96 (1.83)	.95 (1.80)
		.95	.95 (2.01)	.95 (2.02)	.95 (2.00)
4	100	0	.74 (1.34)	1.00 (1.75)	.97 (1.44)
		.10	.91 (1.78)	1.00 (2.27)	.96 (1.89)
		.25	.93 (2.27)	.99 (2.70)	.96 (2.32)
		.50	.94 (2.92)	.98 (3.21)	.95 (2.99)
		.95	.95 (3.75)	.96 (3.78)	.96 (3.75)
7	2	0	.91 (2.05)	.96 (2.17)	.98 (2.35)
		.10	.92 (2.11)	.96 (2.20)	.98 (2.39)
		.25	.93 (2.21)	.95 (2.26)	.97 (2.44)
		.50	.94 (2.37)	.95 (2.39)	.97 (2.44)
		.95	.95 (2.63)	.95 (2.63)	.94 (2.59)
35	25	0	.65 (1.30)	1.00 (2.75)	.94 (1.36)
		.10	.82 (1.35)	1.00 (2.55)	.94 (1.41)
		.25	.88 (1.42)	1.00 (2.34)	.96 (1.48)
		.50	.93 (1.53)	1.00 (2.09)	.95 (1.57)
		.95	.95 (1.70)	.97 (1.75)	.94 (1.71)

NOTE: $\hat{\gamma}_1$, $\hat{\gamma}_2$, and $\hat{\gamma}_3$ are the estimated confidence levels, and the expected values are given in parentheses.

Table 2. Monte Carlo Estimates of the Confidence Levels and Expected Values for the (p, γ) Upper Tolerance Limits $\tilde{G}_1$, $\tilde{V}$, and G_1 in (10), and Vangel's (1992) Cubic Polynomial Approximation V for $N(\mu, \sigma_\tau^2 + \sigma_e^2)$ Based on Balanced Data With $\mu = 0$, $\rho = \sigma_\tau^2/(\sigma_\tau^2 + \sigma_e^2)$, $\sigma_\tau^2 + \sigma_e^2 = 1$, and $(p, \gamma) = (.90, .95)$

k	n	ρ	$\hat{\gamma}_1$ $(E(\tilde{G}_1))$	$\hat{\gamma}_2$ $(E(\tilde{V}))$	$\hat{\gamma}_3$ $(E(G_1))$	$\hat{\gamma}_4$ $E(V)$
3	4	0	.93	.96	.99	0.95
			(2.98)	(2.62)	(3.26)	(3.04)
		.10	.94	.96	.99	.95
			(3.31)	(2.88)	(3.58)	(3.36)
		.25	.95	.95	.98	.96
			(3.79)	(3.33)	(3.96)	(3.83)
		.50	.95	.93	.97	.96
			(4.41)	(4.03)	(4.42)	(4.44)
		.95	.95	.94	.95	.95
			(5.30)	(5.24)	(5.30)	(5.31)
3	10	0	.91	.96	.98	.96
			(2.15)	(2.02)	(2.34)	(2.25)
		.10	.93	.95	.98	.96
			(2.70)	(2.48)	(2.88)	(2.75)
		.25	.94	.93	.97	.96
			(3.34)	(3.12)	(3.39)	(3.36)
		.50	.95	.93	.97	.96
			(4.25)	(4.09)	(4.21)	(4.26)
		.95	.95	.95	.95	.95
			(5.38)	(5.36)	(5.38)	(5.39)
3	10^3	0	.69	1.00	.97	.78
			(1.30)	(5.36)	(1.34)	(1.32)
		.10	.93	1.00	.96	.95
			(2.18)	(5.55)	(2.26)	(2.22)
		.25	.94	1.00	.96	.96
			(2.99)	(5.65)	(3.08)	(3.01)
		.50	.95	1.00	.95	.96
			(3.99)	(5.70)	(4.05)	(3.99)
		.95	.95	.95	.95	.95
			(5.29)	(5.48)	(5.32)	(5.30)
10	2	0	.90	.96	.97	.94
			(1.82)	(1.96)	(2.06)	(1.91)
		.10	.91	.96	.97	.94
			(1.87)	(1.98)	(2.08)	(1.95)
		.25	.92	.95	.97	.95
			(1.95)	(2.02)	(2.11)	(2.01)
		.50	.94	.95	.96	.95
			(2.07)	(2.11)	(2.16)	(2.10)
		.95	.95	.95	.94	.95
			(2.26)	(2.27)	(2.27)	(2.27)
10	10	0	.79	.96	.97	.95
			(1.41)	(1.55)	(1.56)	(1.53)
		.10	.87	.96	.97	.95
			(1.53)	(1.66)	(1.66)	(1.62)
		.25	.91	.96	.96	.95
			(1.68)	(1.81)	(1.78)	(1.75)
		.50	.93	.96	.96	.95
			(1.91)	(2.01)	(1.97)	(1.95)
		.95	.95	.95	.95	.95
			(2.25)	(2.26)	(2.26)	(2.25)
10	10^3	0	.53	1.00	.96	.90
			(1.28)	(9.88)	(1.31)	(1.30)
		.10	.86	1.00	.95	.95
			(1.43)	(7.74)	(1.51)	(1.50)
		.25	.92	1.00	.95	.95
			(1.61)	(6.33)	(1.69)	(1.67)
		.50	.94	1.00	.95	.95
			(1.87)	(4.72)	(1.92)	(1.91)
		.95	.95	.99	.95	.95
			(2.26)	(2.52)	(2.26)	(2.26)

NOTE: $\hat{\gamma}_1$, $\hat{\gamma}_2$, $\hat{\gamma}_3$, and $\hat{\gamma}_4$ are the estimated confidence levels, and the expected values are given in parentheses.

4. AN UPPER TOLERANCE LIMIT FOR $N(\mu, \sigma_\tau^2 + \sigma_e^2)$ BASED ON UNBALANCED DATA

We now consider the model (1) with unbalanced data. In the case of unbalanced data, we have $SS_e = \sum_{i=1}^k \sum_{j=1}^{n_i} (X_{ij} - \bar{X}_{i\bullet})^2$, where $\bar{X}_{i\bullet} = \sum_{j=1}^{n_i} X_{ij}/n_i$. Furthermore, $U_e = \frac{SS_e}{\sigma_e^2} \sim \chi^2_{N-k}$, where $N = \sum_{i=1}^k n_i$. Now define

$$\tilde{n} = \frac{1}{k}\sum_{i=1}^k n_i^{-1}, \qquad \bar{\bar{X}} = \frac{1}{k}\sum_{i=1}^k \bar{X}_{i\bullet}, \qquad \text{and}$$
$$SS_{\bar{x}} = \sum_{i=1}^k (\bar{X}_{i\bullet} - \bar{\bar{X}})^2. \tag{11}$$

Then

$$\bar{\bar{X}} \sim N\left(\mu, \frac{\sigma_\tau^2 + \tilde{n}\sigma_e^2}{k}\right).$$

By direct calculation, it can be verified that $E(SS_{\bar{x}}) = (k-1)(\sigma_\tau^2 + \tilde{n}\sigma_e^2)$. We use the result that

$$U_{\bar{x}} = \frac{SS_{\bar{x}}}{\sigma_\tau^2 + \tilde{n}\sigma_e^2} \sim \chi^2_{k-1} \qquad \text{approximately.} \tag{12}$$

The foregoing approximate distribution is due to Thomas and Hultquist (1978) and was also used by Bhaumik and Kulkarni (1991, 1996). The upper tolerance limit that we derive is based on the random variables $\bar{\bar{X}}$, $SS_{\bar{x}}$, and SS_e. Let $\bar{\bar{x}}$, $ss_{\bar{x}}$, and ss_e denote the corresponding observed values. Following (5), define

$$\begin{aligned} T_2 &= \bar{\bar{x}} - \frac{\sqrt{k}(\bar{\bar{X}} - \mu)}{\sqrt{SS_{\bar{x}}}}\sqrt{\frac{ss_{\bar{x}}}{k}} \\ &\quad + z_p\left[\left(\frac{\sigma_\tau^2 + \tilde{n}\sigma_e^2}{SS_{\bar{x}}} ss_{\bar{x}} - \frac{\tilde{n}\sigma_e^2}{SS_e} ss_e\right) + \frac{\sigma_e^2}{SS_e} ss_e\right]^{1/2} \\ &\overset{d}{\sim} \bar{\bar{x}} - \frac{Z}{\sqrt{\chi^2_{k-1}}}\sqrt{\frac{ss_{\bar{x}}}{k}} \\ &\quad + z_p\left[\frac{ss_{\bar{x}}}{\chi^2_{k-1}} + (1 - \tilde{n})\frac{ss_e}{\chi^2_{N-k}}\right]^{1/2}, \end{aligned} \tag{13}$$

where χ^2_{k-1} and χ^2_{N-k} denote independent chi-squared random variables with $k-1$ and $N-k$ df, and $Z = \sqrt{k}(\bar{\bar{X}} - \mu)/\sqrt{\sigma_\tau^2 + \tilde{n}\sigma_e^2} \sim N(0, 1)$. Using the first expression in (13), it is easy to verify that the observed value of T_2 is $\mu + z_p\sqrt{\sigma_\tau^2 + \sigma_e^2}$. Even though the second expression in (13) has a distribution free of any unknown parameters, the actual distribution of T_2 does depend on unknown parameters. However, as an approximation, we use the second expression in (13). The 100γth percentile of the random variable given in the second expression in (13) is our (p, γ) upper tolerance limit for $N(\mu, \sigma_\tau^2 + \sigma_e^2)$ when the data are unbalanced.

For the unbalanced case, an approximation similar to that in (9) also can be developed. The derivation is similar to that of (9), and the approximate tolerance limit is given by

$$\tilde{G}_2 = \bar{\bar{x}} + t_{k-1}(\delta_2, \gamma)\sqrt{\frac{ss_{\bar{x}}}{k(k-1)}},$$

where

$$\delta_2 = z_p\left(k + \frac{k(k-1)(1-\tilde{n})}{N-k}\frac{ss_e}{ss_{\bar{x}}}F_{k-1,N-k}(1-\gamma)\right)^{1/2}. \quad (14)$$

Also, let

$G_2 =$ the 100γth percentile of the random variable in the second expression in (13). (15)

In the balanced case, $N = nk$, $\tilde{n} = \frac{1}{n}$, and $ss_{\bar{x}} = \frac{ss_\tau}{n}$. Using these, it is readily verified that (14) reduces to the approximation in (9) when we have balanced data.

The approach taken by Bhaumik and Kulkarni (1991) is to derive the tolerance limit when the variance ratio $R = \sigma_\tau^2/\sigma_e^2$ is known, and then replace the unknown variance ratio by a confidence limit. Their recommendation is to use an upper confidence limit for R when $\tilde{n} < 1$ and a lower confidence limit for R when $\tilde{n} > 1$. However, it should be noted that $\tilde{n}$ is always less than 1, and hence R should be replaced by an upper confidence limit. Unfortunately, Bhaumik and Kulkarni (1991) proivded no guidelines regarding the choice of confidence level to use. It turns out that if we replace R by its $100(1-\gamma)\%$ upper confidence limit constructed by Thomas and Hultquist (1978), then the Bhaumik and Kulkarni tolerance limit actually coincides with our approximation $\tilde{G}_2$ in (14). To see this, we note that if $R = \sigma_\tau^2/\sigma_e^2$ is known, then the (p,γ) upper tolerance limit derived by Bhaumik and Kulkarni (1991) is given by

$$\bar{\bar{x}} + t_{k-1}(\delta_{BK}, \gamma)\sqrt{\frac{ss_{\bar{x}}}{k(k-1)}},$$

where

$$\delta_{BK} = z_p\sqrt{\frac{k(R+1)}{R+\tilde{n}}}.$$

The $100(1-\gamma)\%$ upper confidence limit for R due to Thomas and Hultquist (1978) is

$$\tilde{n}\left[\frac{N-k}{\tilde{n}(k-1)}\frac{ss_{\bar{x}}}{ss_e}\frac{1}{F_{k-1,N-k}(1-\gamma)} - 1\right].$$

Let $\hat{\delta}_{BK}$ denote δ_{BK}, with R replaced by the foregoing upper limit. Then it can be readily verified that $\hat{\delta}_{BK}$ coincides with δ_2 given in (14), and hence the tolerance limit $\bar{\bar{x}} + t_{k-1}(\hat{\delta}_{BK}, \gamma)\sqrt{\frac{ss_{\bar{x}}}{k(k-1)}}$ coincides with $\tilde{G}_2$ in (14).

We now report simulation results on the actual confidence level of the tolerance interval based on the approximate tolerance limit $\tilde{G}_2$ in (14). We also report the expected value of $\tilde{G}_2$. Note that in the unbalanced case, the actual distribution of $(\sigma_\tau^2 + \tilde{n}\sigma_e^2)/SS_{\bar{x}}$ will depend on unknown parameters. Because both $\bar{\bar{X}}$ and $SS_{\bar{x}}$ are functions of $\bar{X}_{i\cdot}$, we simulate values of the independent random variables $\bar{X}_{i\cdot} \sim N(\mu, \sigma_\tau^2 + \frac{\sigma_e^2}{n_i})$ (i = 1, 2, ..., k) and $SS_e \sim \sigma_e^2\chi^2_{N-k}$. The simulated confidence levels, along with the expected values of $\tilde{G}_2$, are given in Table 3 for p = .90; γ = .90, .95, and .99; for a few unbalanced designs, and for a few values of the intraclass correlation coefficient $\rho = \sigma_\tau^2/(\sigma_\tau^2 + \sigma_e^2)$, where we have also chosen $\mu = 0$ and $\sigma_\tau^2 + \sigma_e^2 = 1$. Obviously, there are too many parameter combinations in the unbalanced case. For the parameter combinations considered in Table 3, our approximation $\tilde{G}_2$ appears to be quite satisfactory. Recall that in the balanced case, the approximate tolerance limit in (9) was not always satisfactory. Consequently, the tolerance interval based on $\tilde{G}_2$ in (14) will have confidence levels below γ for some values of k, n_i, and ρ. In this case, we recommend the tolerance limit G_2 in (15), obtained as the 100γth percentile of the second expression in (13). To show this, we carried out some limited simulations in the unbalanced case with $k = 12$. We made the following choices for $\mathbf{n} = (n_1, n_2, \ldots, n_{12})'$:

(a) $\mathbf{n} = (3, 15, 30, 14, 2, 3, 13, 22, 8, 6, 9, 11)'$,
(b) $\mathbf{n} = (3, 4, 3, 4, 2, 3, 3, 2, 2, 2, 2, 2)'$, (16)
(c) $\mathbf{n} = (13, 40, 7, 14, 22, 30, 3, 2, 12, 2, 21, 4)'$.

Table 3. Monte Carlo Estimates of the Confidence Level and Expected Value of the (p,γ) Upper Tolerance Limit $\tilde{G}_2$ in (14) for $N(\mu, \sigma_\tau^2 + \sigma_e^2)$ Based on Unbalanced Data With $\mu = 0$, $\rho = \sigma_\tau^2/(\sigma_\tau^2 + \sigma_e^2)$, $\sigma_\tau^2 + \sigma_e^2 = 1$, and $p = .90$

n_1	n_2	n_3	n_4	ρ	γ = .90	.95	.99
3	3	2		0	.89	.94	.99
					(2.61)	(3.59)	(7.83)
				.10	.89	.94	.99
					(2.73)	(3.78)	(8.30)
				.25	.89	.95	.99
					(2.95)	(4.15)	(9.21)
				.50	.90	.95	.99
					(3.26)	(4.63)	(10.39)
				.95	.90	.95	.99
					(3.74)	(5.40)	(12.28)
4	2	5		0	.88	.94	.99
					(2.42)	(3.29)	(7.06)
				.10	.89	.95	.99
					(2.60)	(3.57)	(7.78)
				.25	.89	.95	.99
					(2.84)	(3.96)	(8.73)
				.50	.89	.95	.99
					(3.20)	(4.54)	(10.15)
				.95	.90	.95	.99
					(3.71)	(5.35)	(12.14)
2	3	3	2	0	.87	.93	.99
					(2.16)	(2.68)	(4.51)
				.10	.88	.94	.99
					(2.25)	(2.82)	(4.79)
				.25	.88	.94	.99
					(2.37)	(3.00)	(5.15)
				.50	.89	.94	.99
					(2.59)	(3.32)	(5.78)
				.95	.90	.95	.99
					(2.89)	(3.77)	(6.68)
5	2	12	4	0	.85	.92	.98
					(1.89)	(2.27)	(3.63)
				.10	.87	.93	.99
					(2.01)	(2.46)	(4.04)
				.25	.89	.94	.99
					(2.22)	(2.76)	(4.66)
				.50	.90	.95	.99
					(2.49)	(3.17)	(5.48)
				.95	.90	.95	.99
					(2.90)	(3.78)	(6.70)

NOTE: The expected values are given in parentheses.

Table 4. Monte Carlo Estimates of the Confidence Levels and Expected Values of the (p, γ) Upper Tolerance Limits $\tilde{G}_2$ and G_2 in (14) and (15) for $N(\mu, \sigma_\tau^2 + \sigma_e^2)$ Based on Unbalanced Data With $k = 12$, the n_i's given in (16), $\mu = 0$, $\rho = \sigma_\tau^2/(\sigma_\tau^2 + \sigma_e^2)$, $\sigma_\tau^2 + \sigma_e^2 = 1$, and $(p, \gamma) = (.90, .95)$

n in (16) (a)			**n** in (16) (b)			**n** in (16) (c)		
ρ	$\hat{\gamma}_1$ $(E(\tilde{G}_2))$	$\hat{\gamma}_2$ $(E(G_2))$	ρ	$\hat{\gamma}_1$ $(E(\tilde{G}_2))$	$\hat{\gamma}_2$ $(E(G_2))$	ρ	$\hat{\gamma}_1$ $(E(\tilde{G}_2))$	$\hat{\gamma}_2$ $(E(G_2))$
0	.83	.96	0	.89	.96	0	.83	.96
	(1.47)	(1.59)		(1.67)	(1.85)		(1.48)	(1.59)
.10	.88	.96	.10	.91	.96	.10	.88	.95
	(1.55)	(1.66)		(1.73)	(1.89)		(1.56)	(1.64)
.25	.91	.95	.25	.93	.95	.25	.91	.95
	(1.67)	(1.74)		(1.82)	(1.91)		(1.68)	(1.76)
.50	.94	.95	.50	.94	.95	.50	.94	.96
	(1.86)	(1.89)		(1.93)	(2.01)		(1.85)	(1.89)
.95	.95	.94	.95	.95	.95	.95	.95	.94
	(2.13)	(2.11)		(2.13)	(2.16)		(2.14)	(2.13)

NOTE: $\hat{\gamma}_1$ and $\hat{\gamma}_2$ are the estimated confidence levels, and the expected values are given in parentheses.

The results, given in Table 4, clearly show that $\tilde{G}_2$ is unsatisfactory for smaller values of ρ, whereas G_2 is quite satisfactory for all of the cases considered.

5. AN UPPER TOLERANCE LIMIT FOR $N(\mu, \sigma_\tau^2)$ BASED ON BALANCED DATA

The (p, γ) upper tolerance limit for $N(\mu, \sigma_\tau^2)$ is a $100\gamma\%$ upper confidence limit for $\mu + z_p\sigma_\tau$. Using the notations in Section 3, we now define the generalized pivotal quantity

$$\begin{aligned} T_3 &= \bar{x}_{\cdot\cdot} - \frac{\sqrt{kn}(\bar{X}_{\cdot\cdot} - \mu)}{\sqrt{SS_\tau}} \sqrt{\frac{ss_\tau}{kn}} \\ &\quad + z_p \left[\frac{\sigma_e^2 + n\sigma_\tau^2}{nSS_\tau} ss_\tau - \frac{\sigma_e^2}{nSS_e} ss_e \right]_+^{1/2} \\ &= \bar{x}_{\cdot\cdot} - \frac{Z}{\sqrt{U_\tau}} \sqrt{\frac{ss_\tau}{kn}} + \frac{z_p}{\sqrt{n}} \left[\frac{ss_\tau}{U_\tau} - \frac{ss_e}{U_e} \right]_+^{1/2}, \end{aligned} \tag{17}$$

where the random variables Z, U_τ, and U_e are as defined in (4) and for any scalar c, $c_+ = \max(c, 0)$. $T_{3\gamma}$, the 100γth percentile of T_3, is our tolerance limit for $N(\mu, \sigma_\tau^2)$.

As in the previous sections, we now develop an approximation for $T_{3\gamma}$. Toward that end, note that

$$\begin{aligned} &\frac{Z}{\sqrt{U_\tau}} \sqrt{\frac{ss_\tau}{kn}} + \frac{z_p}{\sqrt{n}} \left[\frac{ss_\tau}{U_\tau} - \frac{ss_e}{U_e} \right]_+^{1/2} \\ &\quad = \sqrt{\frac{ss_\tau}{k(k-1)n}} \left[\frac{Z + z_p \left\{ k - \frac{k-1}{n-1} \frac{ss_e}{ss_\tau} \frac{U_\tau/(k-1)}{U_e/k(n-1)} \right\}_+^{1/2}}{\sqrt{U_\tau/(k-1)}} \right]. \end{aligned}$$

Arguing as before, the approximate tolerance limit, say $\tilde{G}_3$, is given by

$$\tilde{G}_3 = \bar{x}_{\cdot\cdot} + t_{k-1}(\delta_3, \gamma) \sqrt{\frac{ss_\tau}{k(k-1)n}}, \tag{18}$$

where

$$\delta_3 = z_p \left\{ k - \frac{k-1}{n-1} \frac{ss_e}{ss_\tau} F_{k-1,k(n-1)}(1-\gamma) \right\}_+^{1/2}.$$

Let

$$Q_2 = \mu + z_p\sigma_\tau \qquad \text{and} \qquad G_3 = T_{3\gamma}, \tag{19}$$

where $T_{3\gamma}$ denotes the 100γth percentile of T_3 in (17). Table 5 gives the estimated confidence levels $\hat{\gamma}_1$ and $\hat{\gamma}_2$ of the tolerance intervals based on the tolerance limits $\tilde{G}_3$ and G_3, along with the expected values of $\tilde{G}_3$ and G_3, for a few balanced one-way random models. It is clear that the tolerance limits $\tilde{G}_3$ and G_3 are both quite satisfactory.

6. AN UPPER TOLERANCE LIMIT FOR $N(\mu, \sigma_\tau^2)$ BASED ON UNBALANCED DATA

For the unbalanced one-way random model, define

$$\begin{aligned} T_4 &= \bar{\bar{x}} - \frac{\sqrt{k}(\bar{\bar{X}} - \mu)}{\sqrt{SS_{\bar{x}}}} \sqrt{\frac{ss_{\bar{x}}}{k}} \\ &\quad + z_p \left[\left(\frac{\sigma_\tau^2 + \tilde{n}\sigma_e^2}{SS_{\bar{x}}} ss_{\bar{x}} - \frac{\tilde{n}\sigma_e^2}{SS_e} ss_e \right) \right]_+^{1/2} \\ &\overset{d}{\sim} \bar{\bar{x}} - \frac{Z}{\sqrt{\chi_{k-1}^2}} \sqrt{\frac{ss_{\bar{x}}}{k}} + z_p \left[\frac{ss_{\bar{x}}}{\chi_{k-1}^2} - \tilde{n} \frac{ss_e}{\chi_{N-k}^2} \right]_+^{1/2}, \end{aligned} \tag{20}$$

where the notations are as in (13). The 100γth percentile of the second expression in (20) is our tolerance limit for $N(\mu, \sigma_\tau^2)$. An approximation for this percentile is given by

$$G_4 = \bar{\bar{x}} + t_{k-1}(\delta_4, \gamma) \sqrt{\frac{ss_{\bar{x}}}{k(k-1)}}, \tag{21}$$

where

$$\delta_4 = z_p \left\{ k - \frac{\tilde{n}k(k-1)}{N-k} \frac{ss_e}{ss_{\bar{x}}} F_{k-1,N-k}(1-\gamma) \right\}_+^{1/2}.$$

The numerical results in Table 6 show that the foregoing approximation is quite satisfactory.

Table 5. Monte Carlo Estimates of the Confidence Levels and Expected Values of the (p, γ) Upper Tolerance Limits $\tilde{G}_3$ and G_3 in (18) and (19) for $N(\mu, \sigma_\tau^2)$ Based on Balanced Data With $\mu = 0$, $\sigma_e^2 = 1.0$, $Q = \mu + z_p\sigma_\tau$, $\rho = \sigma_\tau^2/(\sigma_\tau^2 + \sigma_e^2)$, and $(p, \gamma) = (.90, .95)$

k	n	ρ (Q)	$\hat{\gamma}_1$ ($E(\tilde{G}_3)$)	$\hat{\gamma}_2$ ($E(G_3)$)
3	2	0	.99	.98
		(0)	(3.69)	(3.48)
		.20	.95	.95
		(.64)	(4.61)	(4.36)
		.40	.95	.94
		(1.05)	(5.78)	(5.55)
		.50	.95	.95
		(1.28)	(6.59)	(6.46)
		.95	.95	.94
		(5.59)	(23.97)	(23.53)
3	5	0	.98	.98
		(0)	(2.34)	(2.22)
		.20	.95	.95
		(.64)	(3.60)	(3.56)
		.40	.95	.94
		(1.05)	(5.01)	(5.01)
		.50	.95	.94
		(1.28)	(5.94)	(5.83)
		.95	.95	.95
		(5.59)	(23.86)	(23.78)
15	10	0	.99	1.00
		(0)	(.48)	(.43)
		.20	.96	.95
		(.64)	(1.12)	(1.11)
		.40	.95	.96
		(1.05)	(1.74)	(1.73)
		.50	.95	.95
		(1.28)	(2.10)	(2.08)
		.95	.95	.95
		(5.59)	(8.86)	(8.91)
4	100	0	.98	.98
		(0)	(.35)	(.34)
		.20	.95	.95
		(.64)	(1.95)	(1.94)
		.40	.95	.95
		(1.05)	(3.16)	(3.21)
		.50	.95	.96
		(1.28)	(3.86)	(3.91)
		.95	.95	.94
		(5.59)	(16.69)	(15.79)
7	2	0	.99	.99
		(0)	(1.63)	(1.46)
		.20	.96	.94
		(.64)	(2.09)	(1.98)
		.40	.96	.95
		(1.05)	(2.69)	(2.64)
		.50	.95	.94
		(1.28)	(3.10)	(3.02)
		.95	.95	.96
		(5.59)	(11.64)	(11.72)
35	25	0	.99	.99
		(0)	(.22)	(.19)
		.20	.96	.93
		(.64)	(.89)	(.88)
		.40	.95	.95
		(1.05)	(1.42)	(1.42)
		.50	.95	.95
		(1.28)	(1.73)	(1.73)
		.95	.95	.96
		(5.59)	(7.50)	(7.53)

NOTE: $\hat{\gamma}_1$ and $\hat{\gamma}_2$ are the estimated confidence levels, and the expected values are given in parentheses.

Table 6. Monte Carlo Estimates of the Confidence Level and Expected Value of the (p, γ) Upper Tolerance Limit G_4 in (21) for $N(\mu, \sigma_\tau^2)$ Based on Unalanced Data With $\mu = 0$, $\sigma_e^2 = 1.0$, $Q = \mu + z_p\sigma_\tau$, and $\rho = \sigma_\tau^2/(\sigma_\tau^2 + \sigma_e^2)$, $p = .90$

				ρ	γ		
n_1	n_2	n_3	n_4	(Q)	.90	.95	.99
3	3	2		0	.96	.98	1.00
				(0)	(2.13)	(3.22)	(7.58)
				.20	.90	.95	.99
				(.64)	(2.84)	(4.22)	(9.81)
				.40	.91	.96	.99
				(1.05)	(3.76)	(5.53)	(12.72)
				.50	.90	.95	.99
				(1.28)	(4.35)	(6.37)	(14.61)
				.95	.90	.95	.99
				(5.59)	(16.62)	(24.07)	(54.83)
4	2	5		0	.96	.98	1.00
				(0)	(1.93)	(2.92)	(6.86)
				.20	.90	.95	.99
				(.64)	(2.67)	(3.96)	(9.20)
				.40	.90	.95	.99
				(1.05)	(3.60)	(5.28)	(12.13)
				.50	.90	.95	.99
				(1.28)	(4.26)	(6.22)	(14.26)
				.95	.90	.95	.99
				(5.59)	(16.58)	(23.98)	(54.54)
2	3	3	2	0	.97	.99	1.00
				(0)	(1.67)	(2.30)	(4.30)
				.20	.90	.95	.99
				(.64)	(2.19)	(2.96)	(5.43)
				.40	.90	.95	.99
				(1.05)	(2.90)	(3.87)	(7.00)
				.50	.90	.95	.99
				(1.28)	(3.39)	(4.49)	(8.09)
				.95	.90	.95	.99
				(5.59)	(12.82)	(16.75)	(29.74)
5	2	12	4	0	.97	.99	1.00
				(0)	(1.27)	(1.76)	(3.31)
				.20	.90	.95	.99
				(.64)	(1.95)	(2.61)	(4.76)
				.40	.89	.95	.99
				(1.05)	(2.71)	(3.59)	(6.47)
				.50	.90	.95	.99
				(1.28)	(3.21)	(4.24)	(7.59)
				.95	.90	.95	.99
				(5.59)	(12.90)	(16.86)	(29.93)

NOTE: The expected values are given in parentheses.

7. SOME EXAMPLES

Example 1. This, is the first example discussed by Vangel (1992), concerns tensile strength measurements made on five batches ($k = 5$) of composite materials. Each batch consists of $n = 5$ specimens. The data were given by Vangel (1992, table 4); the summary statistics are $\bar{x}_{\bullet\bullet} = 388.36$, $ss_\tau = 4163.4$, and $ss_e = 1578.4$. We compute (.90, .95) lower tolerance limits for the tensile strength. Recall that the quantities given in (10) are upper tolerance limits. Let $\tilde{G}_1^*$, $\tilde{V}^*$, and G_1^*, denote the corresponding lower tolerance limits. They have the following values: $\tilde{G}_1^* = 338.18$, $\tilde{V}^* = 338.04$, and $G_1^* = 337.74$. We used 10,000 simulations to obtain G_1^*. Because k is small, we expect the approximate tolerance limit $\tilde{G}_1^*$ to be quite satisfactory. Note that $\tilde{G}_1^*$ is also the largest among of the three.(Because we are computing a lower tolerance limit, the larger the tolerance limit, the better.)

Example 2. We now consider an example discussed by Wang and Iyer (1994), and later by Bhaumik and Kulkarni (1996). In this example, two measurements were obtained on the sulfur content of each of four bottles of coal, so that $k = 4$, $n = 2$. Using the data given in table 1 of Wang and Iyer (1994), we have $\bar{x}_{\bullet\bullet} = 4.64375$, $ss_\tau = .0105375$, and $ss_e = .0164500$. Suppose that we are interested in a (.99, .95) upper tolerance limit for the distribution of the "true" sulfur content, that is, the distribution of $\mu + \tau_i$. The approximate tolerance limit $\tilde{G}_3$ in (18) and the simulated tolerance limit G_3 in (19) have values $\tilde{G}_3 = 4.9207$ and $G_3 = 4.9058$. $\tilde{G}_3$ was obtained using 10,000 simulations, and the two tolerance limits are nearly the same.

Example 3. This example, from Ostle and Mensing (1984, p. 296), is on a study of the effect of storage conditions on the moisture content of white pine lumber. Five different storage conditions ($k = 5$) were studied with a varying number of sample boards stored under each condition. The example was also considered by Bhaumik and Kulkarni (1991), who gave the data in table II of their article. Bhaumik and Kulkarni assumed the one-way random model for the purpose of estimating a (.90, .95) upper tolerance limit for moisture content. The replicates are $n_1 = 5, n_2 = 3, n_3 = 2, n_4 = 3$, and $n_5 = 1$. The summary statistics are $\bar{\bar{x}} = 7.62$, $ss_{\bar{x}} = 3.80$, and $ss_e = 7.17$. The tolerance limits $\tilde{G}_2$ and G_2, given in (14) and (15), have values $\tilde{G}_2 = 11.04$ and $G_2 = 11.12$. $\tilde{G}_2$ was obtained using 10,000 simulations, and we note that the two tolerance limits are nearly the same.

For the same example, suppose that we are interested in a (.90, .95) upper tolerance limit for the distribution of the "true" moisture content, that is, the distribution of $\mu + \tau_i$. The approximation G_4 in (21) has the value 10.9404. We also simulated the 95th percentile of the random variable given in the second expression in (21). Based on 10,000 simulations, the value of the tolerance limit turned out to be 10.9413, very close to the foregoing value of G_4.

8. CONCLUDING REMARKS

We have succeeded in deriving one-sided tolerance limits for the observable random variable and the unobservable random effect in a one-way random model with balanced as well as unbalanced data. We used the concept of generalized confidence intervals for this purpose and developed some approximations for the tolerance limits.We reported simulation results on the actual confidence level of the tolerance intervals and the expected value of the tolerance limits. The generalized confidence interval idea turned out to be fruitful in all of the scenarios that we considered.

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