Closed-Form Approximate Tolerance Intervals for Some General Linear Models and Comparison Studies

K. Krishnamoorthy[1] AND **Xiaodong Lian**

Department of Mathematics, University of Louisiana at Lafayette, Lafayette, LA 70504-1010, USA

Abstract

The problems of constructing tolerance intervals in random effects model and in a mixed linear model are considered. The methods based on the generalized variable (GV) approach and the one based on the modified large sample (MLS) procedure are evaluated with respect to coverage probabilities and expected width in various setups using Monte Carlo simulation. Our comparison studies indicate that the tolerance intervals based on the MLS procedure are comparable to or better than those based on the GV approach. As the MLS tolerance intervals are in closed-form, they are easier to compute than those based on the GV approach. Tolerance intervals for a two-way nested model are also derived using the MLS method, and their merits are evaluated using simulation. The procedures are illustrated using a practical example.

Keywords: Balanced data; Generalized pivotal quantity; One-way random model; Satterthwaite approximation; Unbalanced data

[1]Corresponding Author. Tel.: +1-337-482-5283; fax: +1-337-482-5346.
E-mail address: krishna@louisiana.edu (K. Krishnamoorthy)

1 Introduction

A $(p, 1-\alpha)$ tolerance interval (TI) based on a sample is constructed so that it would include at least a proportion p of the sampled population with confidence $1-\alpha$. Such a TI is usually referred to as p-content $-$ $(1-\alpha)$ coverage TI or simply $(p, 1-\alpha)$ TI. A $(p, 1-\alpha)$ upper tolerance limit (TL) is simply a $1-\alpha$ upper confidence limit for the $100p$ percentile of the population, and a $(p, 1-\alpha)$ lower TL is a $1-\alpha$ lower confidence limit for the $100(1-p)$ percentile of the population. Tolerance intervals for many commonly used probability distributions are well addressed in the literature. A two-sided TI can be used to find a conservative estimate of the proportion of the population that falls between two specified values, and one-sided tolerance limits are useful to judge the proportion of the population that falls above or below a threshold value. For example, TIs are commonly used in situations requiring long-term prediction about numerous future observations from a process assumed to be in a state of statistical control (Wolfinger, 1998). See the books by Guttman (1970) and Krishnamoorthy and Mathew (2009) for numerous applications with examples, and references.

Construction of one-sided tolerance limits in one-way random model with balanced or unbalance data has been well addressed in the literature; for example, see Mee and Owen (1983), Mee (1984), Bhaumik and Kulkarni (1991), Vangel (1992), Krishnamoorthy and Mathew (2004) and Liao, Lin and Iyer (2005). It should be noted that all available methods of obtaining one-sided TLs are approximate, and comparison of some approximate methods can be found in Krishnamoorthy and Mathew (2004). Applications of one-sided tolerance limits in random effects model can be found in these articles cited above, Lyles et al. (1997), and Krishnamoorthy and Mathew (2009, Chapters 4 and 5).

In this article, we are mainly concerned two-sided TIs in random effects model. Mee (1984) and Beckman and Tietjen (1989) proposed some approximate methods for constructing two-sided TIs in random models with balanced data. These methods seem to be conservative unless some conditions are satisfied. Liao, Lin and Iyer (2005) have proposed a method for constructing TIs in one-way random model based on the *generalized variable* (GV) approach. The proposed approximate method reduces the problem of constructing a two-sided TI to the problem of setting upper confidence limit for a linear combination of variance components. Liao, Lin and Iyer's approach is very general and applicable to random models with balanced or unbalanced data, some mixed linear models, and two-way nested models. However, it is computationally involved, especially when the data are unbalanced, so it is not simple to use. Following the method of Liao, Lin and Iyer (2005), Krishnamoorthy and Mathew (2009) proposed an alternative approach which uses the modified large sample (MLS) procedure by Graybill and Wang (1980) for finding an upper confidence limit for a linear combination of variance components. Krishnamoorthy and Mathew (2009) illustrated the MLS method for finding TIs in various models, and found that the results are similar to those based on the GV approach. An advantage of the MLS approach is that the TIs are in closed-forms and they are easier to compute than the ones based on the GV approach.

Although Krishnamoorthy and Mathew (2009) noted via illustrative examples that the MLS procedure for finding TIs in various linear models are comparable with those based on Liao, Lin and Iyer's (2005) method, its merits (coverage properties and expected widths of TIs) are yet to be evaluated. The main purpose of this article is to evaluate the coverage probabilities and expected widths of MLS tolerance intervals and compare them with those of the GV tolerance intervals in various setups. Towards this, we describe the GV method and the MLS method for obtaining TIs in a general setup in the following section. In Section 3, we consider one-way random effects model with balanced data. Methods for obtaining TIs for the distribution of the response variable and for the distribution of unobservable "true value" are described. The results of Section 3 are extended to the unbalanced case in Section 4. The MLS procedure and the generalized variable approach to find TIs in a two-way nested model are given in Section 5. The coverage probabilities and expected widths of TIs on the basis of both approaches are estimated using Monte Carlo simulation in Section 6. Our comparison studies indicate that both procedures are conservative and comparable, and the MLS procedure is less conservative than the GV method in most situations. An illustrative example is given in Section 7, and some concluding remarks are given in Section 8.

2 Approximate Tolerance Intervals in a General Setting

We shall describe Liao, Lin and Iyer's (2005) approximate method of constructing $(p, 1-\alpha)$ two-sided TI for a $Y \sim N(\theta,\ \sum_{i=1}^{q} c_i\sigma_i^2)$, where θ and σ_i^2's are unknown parameters, and c_i's are known constants. It is assumed that independent estimates, independently of Y, $\widehat{\theta}$ and $\widehat{\sigma}_i^2$, $i = 1, 2, ..., q$ are available with distributions

$$\widehat{\theta} \sim N\left(\theta,\ \sum_{i=1}^{q} d_i\sigma_i^2\right) \quad \text{and} \quad \frac{\widehat{\sigma}_i^2}{\sigma_i^2} \sim \frac{\chi^2_{f_i}}{f_i},$$

where the d_i's are also known constants.

Following the approximate approach of constructing univariate normal TI by Howe (1969), Liao, Lin and Iyer (2005) developed an approximate TI of the form

$$\widehat{\theta} \pm D, \tag{1}$$

where D is to be determined so that

$$P_{\widehat{\sigma}_1,...,\widehat{\sigma}_q}\left\{D^2 \geq z^2_{\frac{1+p}{2}}(\sigma_c^2 + \sigma_d^2)\right\} = 1-\alpha. \tag{2}$$

If we take D^2 as a $1-\alpha$ upper confidence limit for $z_{\frac{1+p}{2}}(\sigma_c^2 + \sigma_d^2)$, then $\widehat{\theta} \pm D$ is an approximate $(p, 1-\alpha)$ two-sided tolerance interval for the $N(\theta,\ \sum_{i=1}^{q} c_i\sigma_i^2)$ distribution. Thus, the problem of constructing tolerance intervals reduces to the problem of finding confidence limits for $\sigma_c^2 + \sigma_d^2 = \sum_{i=1}^{q}(c_i + d_i)\sigma_i^2 = \sum_{i=1}^{q} a_i\sigma_i^2$, where $a_i = c_i + d_i$, $i = 1, ..., q$ and some of the a_i's could be negative.

In the following, we shall see two approximate methods of finding confidence intervals for $\sum_{i=1}^{q} a_i\sigma_i^2$.

Confidence Intervals for $\sum_{i=1}^{q} a_i\sigma_i^2$

Let $\widehat{\sigma}_1^2, ..., \widehat{\sigma}_q^2$ be independent random variables with $\widehat{\sigma}_i^2 \sim \sigma_i^2\frac{\chi^2_{m_i}}{m_i}$, so that $\widehat{\sigma}_i^2$ is an unbiased estimate of σ_i^2, $i = 1, ..., q$.

Modified Large Sample Confidence Interval: The MLS confidence interval for $\sum_{i=1}^{q} a_i\sigma_i^2$ was first proposed by Graybill and Wang (1980). When all a_i's are positive, the $1-\alpha$ MLS upper confidence bound for $\sum_{i=1}^{q} a_i\sigma_i^2$ is given by

$$\sum_{i=1}^{q} a_i\widehat{\sigma}_i^2 + \sqrt{\sum_{i=1}^{q} a_i^2\widehat{\sigma}_i^4\left(\frac{m_i}{\chi^2_{m_i;\alpha}}-1\right)^2}. \tag{3}$$

It has been shown that the coverage probability of the above MLS confidence interval approaches $1-\alpha$ as $m_i \to \infty$ for all i.

If some of the a_i's are negative, the $1-\alpha$ MLS upper confidence bound for $\sum_{i=1}^{q} a_i\sigma_i^2$ is obtained as

$$\sum_{i=1}^{q} a_i\widehat{\sigma}_i^2 + \sqrt{\sum_{i=1}^{q} a_i^2\widehat{\sigma}_i^4\left(\frac{m_i}{u_i}-1\right)^2}, \text{ where } u_i = \begin{cases} \chi^2_{m_i;\alpha}, & a_i > 0, \\ \chi^2_{m_i;1-\alpha}, & a_i < 0. \end{cases} \tag{4}$$

Furthermore, the $1-\alpha$ MLS lower confidence bound for $\sum_{i=1}^{q} a_i\sigma_i^2$ is obtained as

$$\sum_{i=1}^{q} a_i\widehat{\sigma}_i^2 + \sqrt{\sum_{i=1}^{q} a_i^2\widehat{\sigma}_i^4\left(\frac{m_i}{l_i}-1\right)^2}, \text{ where } l_i = \begin{cases} \chi^2_{m_i;1-\alpha}, & a_i > 0, \\ \chi^2_{m_i;\alpha}, & a_i < 0. \end{cases} \tag{5}$$

The $1-\frac{\alpha}{2}$ lower limit and the $1-\frac{\alpha}{2}$ upper limit form $1-\alpha$ two-sided confidence interval.

For more details along with numerical results regarding the performance of the MLS confidence intervals, we refer to the book by Burdick and Graybill (1992).

Generalized Variable Approach: Let $\widehat{\sigma}_{i0}^2$ be an observed value of $\widehat{\sigma}_i^2$, $i = 1, ..., q$. The *generalized pivotal quantity* (GPQ) for constructing a confidence interval for σ_i^2 is given by $m_i\widehat{\sigma}_{i0}^2/\chi^2_{m_i}$, $i = 1, ..., q$. The GPQ for $\sum_{i=1}^{q} a_i\sigma_i^2$ is obtained by combining the individual GPQs, and is given by

$$G = \sum_{i=1}^{q} a_i \frac{\widehat{m}_i\sigma_{i0}^2}{\chi^2_{m_i}}. \tag{6}$$

The $1-\alpha$ quantile of G is a $1-\alpha$ generalized upper confidence bound for $\sum_{i=1}^{q} a_i\sigma_i^2$. Note that, for a given $(\widehat{\sigma}_{10}^2, ..., \sigma_{q0}^2)$, the distribution of G does not depend on any parameters, and so its percentiles can be estimated using Monte Carlo simulation.

Remark 1. One could also use the Satterthwaite (1946) approximation to find a confidence interval for $\sum_{i=1}^{q} a_i\sigma_i^2$ with positive a_i's. However, our preliminary simulation studies indicated that the TIs based on the Satterthwaite approximation are in general not satisfactory and inferior to those based on the MLS method. For this reason, the Satterthwaite approximation is not considered in this article.

3 One-Way Random Effects Model: Balanced Case

Let Y_{ij} denote the jth observation corresponding to the ith level, assumed to follow the one-way random model

$$Y_{ij} = \mu + \tau_i + e_{ij}, \; j = 1, 2, ..., n, \; i = 1, 2, ..., a, \tag{7}$$

where μ is an unknown general mean, τ_i's represent random effects, and e_{ij}'s represent error terms. It is assumed that τ_i's and e_{ij}'s are all mutually independent having the distributions $\tau_i \sim N(0, \sigma_\tau^2)$ and $e_{ij} \sim N(0, \sigma_e^2)$. Thus, the observable random variable $Y_{ij} \sim N(\mu, \sigma_\tau^2 + \sigma_e^2)$, and the unobservable "true value" associated with the ith level is simply $\mu + \tau_i$, having the distribution $\mu + \tau_i \sim N(\mu, \sigma_\tau^2)$. We shall consider the problems of constructing two-sided tolerance intervals for the distribution $N(\mu, \sigma_\tau^2 + \sigma_e^2)$, and for the distribution $N(\mu, \sigma_\tau^2)$ based on balanced data. The tolerance intervals that we shall construct will be functions of $\bar{Y}_{..}$, SS_τ and SS_e, which are defined as follows. Let

$$\bar{Y}_{i.} = \frac{1}{n}\sum_{j=1}^{n} Y_{ij}, \; \bar{Y}_{..} = \frac{1}{an}\sum_{i=1}^{a}\sum_{j=1}^{n} Y_{ij}, \; SS_\tau = n\sum_{i=1}^{a}(\bar{Y}_{i.} - \bar{Y}_{..})^2 \text{ and } SS_e = \sum_{i=1}^{a}\sum_{j=1}^{n}(Y_{ij} - \bar{Y}_{i.})^2. \tag{8}$$

We note that $\bar{Y}_{..}$, SS_τ and SS_e are independent with

$$Z = \sqrt{an}\frac{(\bar{Y}_{..} - \mu)}{\sqrt{n\sigma_\tau^2 + \sigma_e^2}} \sim N(0, 1), \; U_\tau^2 = \frac{SS_\tau}{n\sigma_\tau^2 + \sigma_e^2} \sim \chi^2_{a-1}, \text{and } U_e^2 = \frac{SS_e}{\sigma_e^2} \sim \chi^2_{a(n-1)}. \tag{9}$$

3.1 Tolerance Intervals for $N(\mu, \sigma_\tau^2 + \sigma_e^2)$

To construct a $(p, 1-\alpha)$ tolerance interval for a $N(\mu, \sigma_\tau^2 + \sigma_e^2)$ distribution using the general procedure in the preceding section, let $\sigma_1^2 = n\sigma_\tau^2 + \sigma_e^2$ and $\sigma_2^2 = \sigma_e^2$, so that $\sigma_\tau^2 = (\sigma_1^2 - \sigma_2^2)/n$. Let $\widehat{\sigma}_1^2 = SS_\tau/(a-1)$ and $\widehat{\sigma}_2^2 = SS_e/(a(n-1))$. Note that

$$Y_{ij} \sim N\left(\mu, \frac{\sigma_1^2}{n} + \left(1 - \frac{1}{n}\right)\sigma_2^2\right), \; \bar{Y}_{..} \sim N\left(\mu, \frac{\sigma_1^2}{an}\right), \; \frac{\widehat{\sigma}_1^2}{\sigma_1^2} \sim \frac{\chi^2_{a-1}}{a-1}, \text{ and } \frac{\widehat{\sigma}_2^2}{\sigma_2^2} \sim \frac{\chi^2_{a(n-1)}}{a(n-1)},$$

and $\bar{Y}_{..}$, $\widehat{\sigma}_1^2$ and $\widehat{\sigma}_2^2$ are all mutually independent. Thus, to apply tolerance interval procedure given in the preceding section, we note that $c_1 = 1/n$, $c_2 = 1 - 1/n$, $d_1 = 1/(an)$ and $d_2 = 0$. Thus, construction of a $(p, 1-\alpha)$ TI simplifies to construction of a $1-\alpha$ upper confidence limit to $a_1\sigma_1^2 + a_2\sigma_2^2$, where $a_1 = c_1 + d_1 = (1 + 1/a)/n$ and $a_2 = c_2 + d_2 = (1 - 1/n)$.

MLS Tolerance Intervals

The MLS upper confidence limit for $a_1\sigma_1^2 + a_2\sigma_2^2$ in (3) can be expressed as

$$U^{(1)}_{\text{mls};1-\alpha} = (a_1\widehat{\sigma}_1^2 + a_2\widehat{\sigma}_2^2) + \left(a_1^2\widehat{\sigma}_1^4\left(\frac{a-1}{\chi^2_{a-1;\alpha}} - 1\right)^2 + a_2^2\widehat{\sigma}_2^4\left(\frac{a(n-1)}{\chi^2_{a(n-1);\alpha}} - 1\right)^2\right)^{\frac{1}{2}}, \tag{10}$$

where $a_1 = (1 + 1/a)/n$ and $a_2 = (1 - 1/n)$. The MLS tolerance interval on the basis of the above

CI is given by

$$\bar{Y}_{..} \pm z_{\frac{1+p}{2}} \sqrt{U^{(1)}_{\text{mls};1-\alpha}}. \tag{11}$$

Generalized Tolerance Intervals

A GPQ for $a_1\sigma_1^2 + a_2\sigma_2^2$ is given by $G_1 = a_1\frac{(a-1)\sigma_{10}^2}{\chi^2_{a-1}} + a_2\frac{a(n-1)\sigma_{20}^2}{\chi^2_{a(n-1)}}$, where $\widehat{\sigma}_{i0}^2$ is an observed value of $\widehat{\sigma}_i^2$, $i = 1, 2$, $a_1 = (1 + 1/a)/n$ and $a_2 = (1 - 1/n)$. Let $G_{1,1-\alpha}$ denote the $1-\alpha$ quantile of G_1. The generalized tolerance interval is given by

$$\bar{Y}_{..} \pm z_{\frac{1+p}{2}} \sqrt{G_{1,1-\alpha}}. \tag{12}$$

3.2 Tolerance Intervals for $N(\mu, \sigma_\tau^2)$

To compute tolerance interval for a $N(\mu, \sigma_\tau^2)$ distribution, we first note from Section 3.1 that $\sigma_\tau^2 = \frac{\sigma_1^2}{n} - \frac{\sigma_2^2}{n}$, so $c_1 = 1/n$ and $c_2 = -1/n$. Furthermore $\widehat{\mu} = \bar{Y}_{..} \sim N\left(\mu, \frac{\sigma_1^2}{an}\right)$, and so $d_1 = 1/(an)$ and $d_2 = 0$. Thus, $a_1 = c_1 + d_1 = (1 + 1/a)/n$ and $a_2 = c_2 + d_2 = -1/n$. The TI is $\bar{Y}_{..} \pm D$, where D is a $1-\alpha$ upper confidence limit for $z_{\frac{1+p}{2}}\sqrt{a_1\sigma_1^2 + a_2\sigma_2^2}$.

MLS Tolerance Intervals

An approximate $1-\alpha$ upper confidence limit for $a_1\sigma_1^2 + a_2\sigma_2^2$, where $a_1 = (1 + 1/a)/n$ and $a_2 = -1/n$. Noting that a_2 is negative, an upper confidence limit for $a_1\sigma_1^2 + a_2\sigma_2^2$ in (4) can be expressed as

$$U^{(2)}_{\text{mls};1-\alpha} = (a_1\widehat{\sigma}_1^2 + a_2\widehat{\sigma}_2^2) + \left(a_1^2\widehat{\sigma}_1^4\left(\frac{a-1}{\chi^2_{a-1;\alpha}} - 1\right)^2 + a_2^2\widehat{\sigma}_2^4\left(\frac{a(n-1)}{\chi^2_{a(n-1);1-\alpha}} - 1\right)^2\right)^{\frac{1}{2}}. \tag{13}$$

The $(p, 1-\alpha)$ tolerance interval for the $N(\mu, \sigma_\tau^2)$ distribution on the basis of the above confidence limit is given by

$$\bar{Y}_{..} \pm z_{\frac{1+p}{2}} \sqrt{U^{(2)}_{\text{mls};1-\alpha}}. \tag{14}$$

Generalized Tolerance Intervals

A $1-\alpha$ generalized upper confidence limit for $a_1\sigma_1^2 + a_2\sigma_2^2$ is the $1-\alpha$ quantile of $G_2 = \left(a_1\frac{(a-1)ss_\tau}{\chi^2_{a-1}} + a_2\frac{a(n-1)ss_e}{\chi^2_{a(n-1)}}\right)_+$, where $x_+ = \max\{x, 0\}$, $a_1 = (1+1/a)/n$ and $a_2 = -1/n$. Let $G_{2,1-\alpha}$ denote the $1-\alpha$ quantile of G_2. Then, the $(p, 1-\alpha)$ generalized tolerance interval for the $N(\mu, \sigma_\tau^2)$ distribution is given by

$$\bar{Y}_{..} \pm z_{\frac{1+p}{2}} \sqrt{G_{2,1-\alpha}}. \tag{15}$$

4 One-Way Random Model: Unbalanced Case

Suppose now we have n_i observations corresponding to the ith level of the factor, where the n_i's are not necessarily equal. Thus we have

$$Y_{ij} = \mu + \tau_i + e_{ij},\ j = 1, 2, ..., n_i,\ i = 1, 2, ..., a, \tag{16}$$

where the parameters μ and τ_i's are defined as in the balanced model (7), and e_{ij}'s represent error terms. Define

$$\bar{Y}_{i.} = \sum_{j=1}^{n_i} Y_{ij}/n_i, \text{ and } SS_e = \sum_{i=1}^{a}\sum_{j=1}^{n_i}(Y_{ij} - \bar{Y}_{i.})^2,$$

so that

$$\frac{SS_e}{\sigma_e^2} \sim \chi^2_{N-a}, \quad \text{where } N = \sum_{i=1}^{a} n_i. \tag{17}$$

4.1 MLS Tolerance Intervals

To derive the MLS tolerance intervals, we shall use the approximate pivotal quantities considered in Bhaumik and Kulkarni (1991), and Krishnamoorthy and Mathew (2004). Define

$$\tilde{n} = \frac{1}{a}\sum_{i=1}^{a} n_i^{-1},\ \bar{Y}_{i.} = \frac{1}{n_i}\sum_{j=1}^{n_i} Y_{ij},\ \bar{\bar{Y}} = \frac{1}{a}\sum_{i=1}^{a}\bar{Y}_{i.}, \text{ and } SS_{\bar{y}} = \sum_{i=1}^{a}(\bar{Y}_{i.} - \bar{\bar{Y}})^2. \tag{18}$$

Then

$$\bar{Y}_{i.} \sim N(\mu, \sigma_\tau^2 + \sigma_e^2/n_i) \text{ and } \bar{\bar{Y}} \sim N\left(\mu, \frac{\sigma_\tau^2 + \tilde{n}\sigma_e^2}{a}\right).$$

It can be verified that $E(SS_{\bar{y}}) = (a-1)(\sigma_\tau^2 + \tilde{n}\sigma_e^2)$. Furthermore, Thomas and Hultquist (1978) showed that

$$\frac{SS_{\bar{y}}}{\sigma_\tau^2 + \tilde{n}\sigma_e^2} \sim \chi^2_{a-1}, \quad \text{approximately.} \tag{19}$$

It is known that $\bar{\bar{Y}}$ and SS_e are independently distributed, and $SS_{\bar{y}}$ and SS_e are independently distributed. However, $\bar{\bar{Y}}$ and $SS_{\bar{y}}$ are not independent (except in the balanced case). As noted in Krishnamoorthy and Mathew (2004), this dependence dose not pose any serious problems to obtain a satisfactory tolerance intervals.

To construct a tolerance interval for the $N(\mu, \sigma_\tau^2 + \sigma_e^2)$ distribution, let $\sigma_1^2 = \sigma_\tau^2 + \tilde{n}\sigma_e^2$ and $\sigma_2^2 = \sigma_e^2$, so that $\sigma_\tau^2 = \sigma_1^2 - \tilde{n}\sigma_2^2$. In these notations, we see that $Y_{ij} \sim N(\mu, \sigma_1^2 + (1-\tilde{n})\sigma_2^2)$ and $\widehat{\mu} = \bar{\bar{Y}} \sim N(\mu, \sigma_1^2/a)$. Thus, $c_1 = 1, c_2 = 1 - \tilde{n}$, $d_1 = 1/a$ and $d_2 = 0$; $a_1 = c_1 + d_1 = (1 + 1/a)$ and $a_2 = c_2 + d_2 = 1 - \tilde{n}$. Using the procedures in Section 2, we can obtain a $(p, 1-\alpha)$ TI based on a $1-\alpha$ upper confidence limit of $a_1\sigma_1^2 + a_2\sigma_2^2$. To obtain a $1-\alpha$ upper confidence limit for $a_1\sigma_1^2 + a_2\sigma_2^2$, note that $\widehat{\sigma}_1^2 = SS_{\bar{y}}/(a-1) \sim \sigma_1^2\chi^2_{a-1}/(a-1)$ approximately, and $\widehat{\sigma}_2^2 = SS_e/(N-a) \sim \sigma_2^2\chi^2_{N-a}/(N-a)$. Furthermore, $\widehat{\sigma}_1^2$ and $\widehat{\sigma}_2^2$ are independent with $E(\widehat{\sigma}_1^2) = \sigma_1^2$ and $E(\widehat{\sigma}_2^2) = \sigma_2^2$. The MLS upper confidence limit for $a_1\sigma_1^2 + a_2\sigma_2^2$ in (3) can be expressed as

$$U^{(3)}_{mls,1-\alpha} = \left\{ (a_1\widehat{\sigma}_1^2 + a_2\widehat{\sigma}_2^2) + \left(a_1^2\widehat{\sigma}_1^4 \left(\frac{a-1}{\chi^2_{a-1;\alpha}} - 1 \right)^2 + a_2^2\widehat{\sigma}_2^4 \left(\frac{N-a}{\chi^2_{N-a;\alpha}} - 1 \right)^2 \right)^{\frac{1}{2}} \right\}, \tag{20}$$

where $a_1 = 1+1/a$ and $a_2 = 1-\tilde{n}$. The MLS tolerance interval on the basis of the above confidence limit is given by

$$\bar{\bar{Y}} \pm z_{\frac{1+p}{2}} \sqrt{U^{(3)}_{mls,1-\alpha}}, \tag{21}$$

where $\bar{\bar{Y}}$ is as defined in (18).

4.2 Generalized Tolerance Interval

We shall now describe the generalized tolerance interval due to Liao, Lin and Iyer's (2005). Note that

$$\bar{Y}_{i.} = \frac{1}{n_i}\sum_{j=1}^{n_i} Y_{ij} \sim N(\mu, v_i), \quad \text{where} \quad v_i = \sigma_\tau^2 + \frac{\sigma_e^2}{n_i}. \tag{22}$$

Write

$$\bar{\boldsymbol{Y}} = (\bar{Y}_{1.}, \bar{Y}_{2.}, ..., \bar{Y}_{a.})' \ \text{ so that } \quad N(\mu\mathbf{1}_a, \mathbf{V}), \tag{23}$$

where $\mathbf{1}_a$ is an $a \times 1$ vector of ones and $\mathbf{V} = \text{diag}(v_1, v_2, ..., v_a)$. If the variance components σ_τ^2 and σ_e^2 are known, so that the v_i's are known, the uniformly minimum variance unbiased estimator of μ, say $\bar{Y}_v$, is given by

$$\bar{Y}_v = \frac{\sum_{i=1}^a v_i^{-1}\bar{Y}_{i.}}{\sum_{i=1}^a v_i^{-1}} \sim N\left(\mu, \left(\sum_{i=1}^a v_i^{-1}\right)^{-1}\right). \tag{24}$$

The residual sum of squares SS_0 under the model (16) is given by

$$SS_0 = \sum_{i=1}^a v_i^{-1}\bar{Y}_{i.}^2 - \frac{1}{\sum_{i=1}^a v_i^{-1}}\left(\sum_{i=1}^a v_i^{-1}\bar{Y}_{i.}\right)^2. \tag{25}$$

It is known that $SS_0 \sim \chi^2_{a-1}$ independently of SS_e. To derive an exact GPQ for σ_τ^2, let ($\bar{y}_{1.}$, $\bar{y}_{2.}$, ... $\bar{y}_{a.}$), and ss_e denote the observed values of ($\bar{Y}_{1.}$, $\bar{Y}_{2.}$, ... $\bar{Y}_{a.}$), and SS_e, respectively. Let $G_{\sigma_e^2} = ss_e/\chi^2_{N-a}$. An exact GPQ $G_{\sigma_\tau^2}$ is obtained as the solution of the equation

$$U_0^2 = \sum_{i=1}^a (G_{\sigma_\tau^2} + G_{\sigma_e^2}/n_i)^{-1}\bar{y}_{i.}^2 - \frac{1}{\sum_{i=1}^a (G_{\sigma_\tau^2} + G_{\sigma_e^2}/n_i)^{-1}}\left(\sum_{i=1}^a (G_{\sigma_\tau^2} + G_{\sigma_e^2}/n_i)^{-1}\bar{y}_{i.}\right)^2, \tag{26}$$

where $U_0^2 \sim \chi^2_{a-1}$. Liao, Lin and Iyer (2005) showed that the right hand side of (26) is a monotone decreasing function of $G_{\sigma_\tau^2}$, with limiting value 0 as $G_{\sigma_\tau^2} \to \infty$. In view of this monotonicity property, there exists a unique $G_{\sigma_\tau^2}$ that solves (26), provided $G_{\sigma_\tau^2}$ is not restricted to be nonnegative.

In order to use the tolerance interval procedure discussed earlier, we need to find a $1-\alpha$ upper confidence limit for $\sigma_c^2 + \sigma_d^2 = \sigma_\tau^2 + \sigma_e^2 + \left(\sum_{i=1}^a \nu_i^{-1}\right)^{-1}$. The GPQ for this function of parameters is given by

$$G_3 = \left[G_{\sigma_\tau^2} + G_{\sigma_e^2} + \left(\sum_{i=1}^a (G_{\nu_i})^{-1} \right)^{-1} \right]_+, \quad \text{with } G_{\nu_i} = G_{\sigma_\tau^2} + \frac{G_{\sigma_e^2}}{n_i}. \tag{27}$$

Finally, we note that $\bar{Y}_v$ depends on unknown parameters, and hence it cannot be used as an estimate of μ. Liao, Lin and Iyer (2005) suggested to use the mean $\tilde{G}_\mu$ (or the median) of the GPQ

$$G_\mu = \frac{\sum_{i=1}^a \frac{1}{G_{v_i}} \bar{y}_{i.}}{\sum_{i=1}^a \frac{1}{G_{v_i}}} - Z \sqrt{\left[\left(\sum_{i=1}^a G_{v_i}^{-1} \right)^{-1} \right]_+} \tag{28}$$

for μ. Note that $\tilde{G}_\mu$ can be easily estimated using Monte Carlo simulation. If $G_{3,1-\alpha}$ is the $1-\alpha$ quantile of G_3 in (27), then the generalized TI is given by

$$\tilde{G}_\mu \pm z_{\frac{1+p}{2}} \sqrt{G_{3,1-\alpha}}. \tag{29}$$

4.3 Tolerance Intervals for $N(\mu, \sigma_\tau^2)$

As we want to find TIs for a $N(\mu, \sigma_\tau^2)$, and $\sigma_\tau^2 = \sigma_1^2 - \tilde{n}\sigma_2^2$, we have $c_1 = 1$ and $c_2 = -\tilde{n}$. Recall that $\bar{\bar{Y}} \sim N(\mu, \sigma_1^2/a)$, so $d_1 = 1/a$ and $d_2 = 0$. Thus, $a_1 = c_1 + d_1 = (1 + 1/a)$ and $a_2 = c_2 + d_2 = -\tilde{n}$. Using the procedures in Section 2, we can obtain a $(p, 1-\alpha)$ TI based on a $1-\alpha$ upper confidence limit of $a_1\sigma_1^2 + a_2\sigma_2^2$.

MLS Tolerance Intervals

Noting that a_2 is negative, the MLS upper confidence limit for $a_1\sigma_1^2 + a_2\sigma_2^2$ in (4) can be expressed as

$$U_{\text{mls};1-\alpha}^{(4)} = \left\{ (a_1\widehat{\sigma}_1^2 + a_2\widehat{\sigma}_2^2) + \left(a_1^2\widehat{\sigma}_1^4 \left(\frac{a-1}{\chi^2_{a-1;\alpha}} - 1 \right)^2 + a_2^2\widehat{\sigma}_2^4 \left(\frac{N-a}{\chi^2_{N-a;1-\alpha}} - 1 \right)^2 \right)^{\frac{1}{2}} \right\}, \tag{30}$$

where $a_1 = 1 + 1/a$ and $a_2 = -\tilde{n}$. The MLS tolerance interval on the basis of the above confidence limit is given by

$$\bar{\bar{Y}} \pm z_{\frac{1+p}{2}} \sqrt{U_{\text{mls};1-\alpha}^{(4)}}. \tag{31}$$

Generalized Tolerance Intervals

Let $\tilde{G_\mu}$ be as defined in (29), and let $G_{\sigma_\tau^2}$ be the GPQ defined by (26). Let $G_{4,1-\alpha}$ be the $1-\alpha$ quantile of $\left(G_{\sigma_\tau^2} + \left(\sum_{i=1}^a G_{\nu_i}^{-1}\right)^{-1}\right)_+$. The generalized TI for the $N(\mu, \sigma_\tau^2)$ distribution is given by

$$\tilde{G_\mu} \pm z_{\frac{1+p}{2}} \sqrt{G_{4,1-\alpha}}. \tag{32}$$

Remark 2. Note that the GPQ for σ_τ^2 in the unbalanced case is obtained as the root of the non-linear equation (26), which has to be solved a large number of times to obtain a generalized TI. Instead, one could use the approximate GPQ for σ_τ^2 given in Krishnamoorthy and Mathew (2004). This approximate GPQ can be obtained from the approximate distributional result for $SS_{\bar{y}}$ in (19), and is given by

$$G^*_{\sigma_\tau^2} = \frac{ss_{\bar{y}}}{\chi^2_{a-1}} - \tilde{n}\frac{ss_e}{\chi^2_{N-a}}. \tag{33}$$

This gives an approximate GPQ $G^*_{\nu_i} = G^*_{\sigma_\tau^2} + G_{\sigma_e^2}/n_i$ for ν_i defined in (22). Liao, Lin and Iyer (2005) compared the "exact" generalized one-sided tolerance limits based on the GPQ determined by the equation (26), and the ones based on the approximate GPQ $G^*_{\sigma_\tau^2}$ via Monte Carlo simulation. Their simulation studies indicated that these two one-sided tolerance limits are comparable. We also compared these two methods for the two-sided case, and found they are similar with respect to coverage probabilities and expected widths. For this reason and to save space, these results are not reported here but they can be found in Lian (2011).

5 A Two-Way Nested Model

Suppose there are two factors A and B with the levels of B nested within the levels of A. Suppose A has a levels, and b levels of B are nested within each level of A, with n observations obtained on each level combination. Let Y_{ijl} denote the lth observation corresponding to the jth level of B nested within the ith level of A. Thus, we have

$$\begin{aligned} Y_{ijl} &= \mu + \tau_i + \beta_{j(i)} + e_{ijl}, \\ i &= 1, ..., a; j = 1, ..., b; l = 1, ..., n, \end{aligned} \tag{34}$$

where μ is an overall mean effect, τ_i is the main effect due to the ith level of A, $\beta_{j(i)}$ is the effect due to the jth level of B nested within the ith level of A, and e_{ijl}'s are the error terms with $e_{ijl} \sim N(0, \sigma_e^2)$. If the levels of A and B are randomly selected, then we have a random effects model, and in addition to the above distributional assumptions, we also assume $\tau_i \sim N(0, \sigma_\tau^2)$ and $\beta_{j(i)} \sim N(0, \sigma_\beta^2)$, and τ_i, $\beta_{j(i)}$ and e_{ijl} are all independent. Define

$$\bar{Y}_{ij.} = \frac{1}{n}\sum_{l=1}^n Y_{ijl},\ \bar{Y}_{i..} = \frac{1}{bn}\sum_{j=1}^b\sum_{l=1}^n Y_{ijl},\ \bar{Y}_{...} = \frac{1}{abn}\sum_{i=1}^a\sum_{j=1}^b\sum_{l=1}^n Y_{ijl},$$

and let

$$SS_\tau = bn\sum_{i=1}^{a}(\bar{Y}_{i..} - \bar{Y}_{...})^2, \quad SS_\beta = n\sum_{i=1}^{a}\sum_{j=1}^{b}(\bar{Y}_{ij.} - \bar{Y}_{i..})^2, \text{ and } SS_e = \sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{l=1}^{n}(Y_{ijl} - \bar{Y}_{ij.})^2. \quad (35)$$

Under the distributional assumptions, we have

$$U_\beta^2 = \frac{SS_\beta}{n\sigma_\beta^2 + \sigma_e^2} \sim \chi^2_{a(b-1)}, \; U_e^2 = \frac{SS_e}{\sigma_e^2} \sim \chi^2_{ab(n-1)}, \text{ and } U_\tau^2 = \frac{SS_\tau}{(bn\sigma_\tau^2 + n\sigma_\beta^2 + \sigma_e^2)} \sim \chi^2_{a-1}, \quad (36)$$

where the above chi-square random variables are independent. Now the only fixed parameter in the model is μ, and we have the estimator

$$\widehat{\mu} = \bar{Y}_{...} \sim N\left(\mu, \frac{bn\sigma_\tau^2 + n\sigma_\beta^2 + \sigma_e^2}{abn}\right), \quad (37)$$

independent of the above chi-square random variables. A set of sufficient statistics for the random effects model (34) consists of $\bar{Y}_{...}$ along with SS_τ, SS_β and SS_e.

We shall now consider the problem of constructing a tolerance interval for the distribution $N(\mu, \sigma_\tau^2 + \sigma_\beta^2 + \sigma_e^2)$. Let $\sigma_1^2 = bn\sigma_\tau^2 + n\sigma_\beta^2 + \sigma_e^2$, $\sigma_2^2 = n\sigma_\beta^2 + \sigma_e^2$ and $\sigma_3^2 = \sigma_e^2$ so that $\sigma_\tau^2 = (\sigma_1^2 - \sigma_2^2)/(bn)$, $\sigma_\beta^2 = (\sigma_2^2 - \sigma_3^2)/n$ and $\sigma_e^2 = \sigma_3^2$. Thus, we have

$$Y_{ijl} \sim N\left(\mu, \sum_{i=1}^{3} c_i\sigma_i^2\right) \text{ and } \bar{Y}_{...} \sim N\left(\mu, \sum_{i=1}^{3} d_i\sigma_i^2\right)$$

with $c_1 = (bn)^{-1}$, $c_2 = (1 - 1/b)/n$, $c_3 = (1 - 1/n)$, $d_1 = (abn)^{-1}$, and $d_2 = d_3 = 0$.

MLS Tolerance Intervals: Following (3), the $(p, 1-\alpha)$ TI can be expressed as

$$\bar{Y}_{...} \pm z_{\frac{1+p}{2}} \left\{\sum_{i=1}^{3} a_i\widehat{\sigma}_i^2 + \sqrt{\sum_{i=1}^{3} a_i^2\widehat{\sigma}_i^4\left(\frac{m_i}{\chi^2_{m_i;\alpha}} - 1\right)^2}\right\}^{\frac{1}{2}}, \quad (38)$$

where $a_1 = c_1 + d_1 = (1 + 1/a)/(bn)$, $a_2 = c_2 + d_2 = (1 - 1/b)/n$, $a_3 = c_3 = (1 - 1/n)$, $m_1 = a - 1$, $m_2 = a(b-1)$ and $m_3 = ab(n-1)$. Furthermore, it follows from the distributional results that $\widehat{\sigma}_1^2 = SS_\tau/m_1$, $\widehat{\sigma}_2^2 = SS_\beta/m_2$ and $\widehat{\sigma}_3^2 = SS_e/m_3$ are unbiased estimators of σ_1^2, σ_2^2 and σ_3^2, respectively.

Generalize Tolerance Intervals: A GPQ for $a_1\sigma_1^2 + a_2\sigma_2^2 + a_3\sigma_3^2$ is given by $G_5 = a_1\frac{\hat{\sigma}_{10}^2(a-1)}{\chi^2_{a-1}} + a_2\frac{\hat{\sigma}_{20}^2(a(b-1))}{\chi^2_{a(b-1)}} + a_3\frac{\hat{\sigma}_{30}^2(ab(n-1))}{\chi^2_{ab(n-1)}}$, where $\hat{\sigma}_{i0}^2$ is an observed value of $\hat{\sigma}_i^2$, $i = 1, 2, 3$. Let $G_{5,1-\alpha}$ be the $1-\alpha$ quantile of G_5. The generalized tolerance interval for $a_1\sigma_1^2 + a_2\sigma_2^2 + a_3\sigma_3^2$ is given by

$$\bar{Y}_{...} \pm z_{\frac{1+p}{2}}\sqrt{G_{5,1-\alpha}} \quad (39)$$

where $a_1 = \frac{1}{bn} + \frac{1}{abn}, a_2 = \frac{1}{n}(1 - \frac{1}{b},), \text{ and } a_3 = 1 - \frac{1}{n}$.

6 Monte Carlo Studies

As both procedures for constructing TIs are approximate, we need to evaluate their coverage properties to judge their accuracy. We estimated the coverage probabilities of (.90, .95) TIs in various setups using Monte Carlo simulation. The simulation study for the generalized variable was carried out as follows. We first generate 2,500 samples according to the assumed model. Fora each sample, we used the generalized variable approach with 5,000 simulation runs to find the generalized TI. Proportion of these 2,500 TIs with content level at least p is a Monte Carlo estimate of the coverage probability. The maximum error of the Monte Carlo estimates is approximately determined as $2 \times \sqrt{.5 \times .5/2500} = .02$. As the MLS TIs are in closed-form, we use 10,000 simulation runs to estimate their coverage probabilities, and so the maximum error of the estimates is approximately $2 \times \sqrt{.5 \times .5/10000} = .01$.

As the procedures are location-scale invariant, the coverage probabilities depend on the parameters only via $\rho = \sigma_\tau^2/(\sigma_\tau^2 + \sigma_e^2)$, the intra-class correlation coefficient. We first estimated the coverage probabilities of TIs for $N(\mu, \sigma_\tau^2 + \sigma_e^2)$ distribution with balanced data. The estimated coverage probabilities for ρ ranging from .001 to .999 and some combinations of a and n are given in Table 1. Examination of estimated coverage probabilities clearly indicate that, in general, the MLS method and the GV approach perform similarly. Specifically, both methods appear to be conservative for small to moderate values of ρ, and they seem to be accurate for large values of ρ. Overall, the MLS TIs seem to be slightly less conservative than the generalized TIs.

Regarding the TIs for a $N(\mu, \sigma_\tau^2)$, we observe from the estimated coverage probabilities in Table 2 that both procedures exhibit very satisfactory performance for all values of ρ and (a, n) considered. In particular, the coverage probabilities of both TIs are close to the nominal level 0.95. We also estimated coverage probabilities of (.90, .99) and (.95, .90) TIs. These coverage probabilities are not reported here because they were very similar to those of (.90, .95) tolerance intervals reported in Tables 1 and 2.

The estimated coverage probabilities of TIs for $N(\mu, \sigma_\tau^2 + \sigma_e^2)$ distribution with unbalanced data are reported in Table 3. We used the generalized TIs in (29) with the median $\tilde{G}_\mu$ for coverage studies because the ones with the mean are too liberal in some cases. In general, we found that the generalized TIs with median $\tilde{G}_\mu$ are better than the ones with the mean. We see from Table 3 that the coverage probabilities of the MLS TIs are close to or larger than the nominal level in most cases, whereas the coverage probabilities of the generalized TIs are smaller than the nominal level in some cases, especially for small values of ρ. In general, the MLS TIs are slightly better than the generalized TIs. Coverage probabilities of (.90, .95) TIs based on unbalanced data for the $N(\mu, \sigma_\tau^2)$ distribution are given in Table 4. We once again observe that the MLS TIs exhibit better performance than the generalized TIs, in particular, for small values of ρ and a is moderately large.

To judge the performance of the TIs for the $N(\mu, \sigma_\tau^2 + \sigma_\beta^2 + \sigma_e^2)$ distribution, we estimated the coverage probabilities for some sample size and parameter configurations. As both TIs are location-scale invariant, without loss of generality, we can assume $\sigma_e^2 = 1$, $0 < \sigma_\beta^2 < 1$ and $0 < \sigma_\tau^2 < 1$. It is clear from the estimated coverage probabilities in Table 6 that both TIs are conservative for smaller values of $(\sigma_\tau^2, \sigma_\beta^2)$, and show improved performance for large values of $(\sigma_\tau^2, \sigma_\beta^2)$. In general, both MLS and generalized TIs are conservative.

In most cases both TIs perform similar with respect to coverage probabilities, and so their expected widths may be comparable. To check this, we estimated the expected widths of (.90, .95) TIs for the $N(\mu, \sigma_\tau^2)$ distribution with unbalanced data and reported them in Table 5. The expected widths are estimated for the cases of $a = 4$ and the values of n_i's as in Table 4, where both TIs are comparable with respect to coverage probabilities. We see from Table 5, the expected widths of the MLS and generalized TIs are practically the same.

Overall, our simulation results in Tables 1 – 6 clearly indicate that the MLS TIs are comparable to or better than generalized TIs.

Table 1: Coverage probabilities of (.90, .95) tolerance intervals for $N(\mu, \sigma_\tau^2 + \sigma_e^2)$
(a) MLS procedure; (b) Generalized variable approach

	(a, n)											
	(5,3)		(5,2)		(10,2)		(20,2)		(10,6)		(15,5)	
ρ	(a)	(b)	(a)	(b)	(a)	(b)	(a)	(b)	(a)	(b)	(a)	(b
.001	.983	.982	.985	.985	.978	.989	.973	.979	.971	.975	.964	.970
.010	.983	.986	.986	.994	.978	.977	.966	.963	.970	.971	.962	.964
.050	.980	.987	.986	.985	.977	.984	.968	.972	.974	.971	.960	.972
.100	.983	.978	.985	.992	.977	.979	.967	.972	.971	.977	.960	.969
.200	.979	.985	.979	.985	.972	.977	.968	.962	.967	.970	.960	.973
.300	.975	.974	.978	.976	.971	.977	.966	.964	.964	.962	.960	.970
.400	.971	.989	.979	.980	.966	.966	.965	.966	.955	.971	.959	.966
.500	.965	.978	.975	.981	.965	.969	.959	.961	.954	.951	.952	.962
.600	.962	.963	.972	.973	.961	.973	.956	.957	.950	.956	.952	.948
.700	.956	.967	.962	.972	.959	.963	.950	.953	.952	.961	.951	.950
.800	.953	.959	.956	.952	.951	.947	.948	.953	.946	.935	.948	.944
.900	.948	.939	.950	.958	.951	.940	.949	.944	.952	.945	.947	.941
.990	.948	.940	.949	.943	.945	.965	.945	.949	.949	.937	.948	.948
.999	.949	.940	.953	.948	.947	.936	.947	.942	.952	.944	.950	.942

Table 2: Coverage probabilities of (.90, .95) tolerance intervals for $N(\mu, \sigma_\tau^2)$
(a) MLS procedure; (b) Generalized variable approach

	(a, n)									
	(5,3)		(5,2)		(10,2)		(20,2)		(10,6)	
ρ	(a)	(b)	(a)	(b)	(a)	(b)	(a)	(b)	(a)	(b)
.001	0.956	.953	.956	.955	.952	.958	.954	.949	.959	.954
.010	0.956	.954	.952	.952	.952	.955	.951	.948	.954	.948
.050	0.954	.944	.952	.949	.947	.943	.952	.951	.950	.942
.100	0.954	.947	.959	.948	.947	.951	.949	.945	.952	.940
.200	0.953	.945	.958	.948	.951	.948	.947	.934	.950	.948
.300	0.951	.947	.953	.951	.942	.945	.953	.948	.946	.941
.400	0.949	.945	.951	.934	.940	.939	.952	.946	.947	.942
.500	0.947	.958	.946	.948	.942	.936	.950	.940	.948	.946
.600	0.956	.942	.949	.946	.941	.936	.945	.946	.948	.953
.700	0.952	.946	.946	.948	.936	.947	.948	.934	.948	.955
.800	0.947	.952	.952	.946	.953	.948	.950	.942	.951	.954
.900	0.948	.949	.949	.936	.945	.942	.948	.944	.951	.949
.990	0.951	.942	.947	.942	.952	.943	.949	.947	.948	.956
.999	0.950	.950	.946	.950	.947	.954	.951	.942	.947	.948

Table 3: Coverage probabilities of (.90, .95) tolerance intervals for $N(\mu, \sigma_\tau^2 + \sigma_e^2)$; unbalanced case
(a) MLS TIs; (b) Generalized TIs

	(5,4,3,8)		(5,12,20,12)		(2,7,12,30)		(2,3,2,4)		D_1		D_2		D_3		D_4	
ρ	(a)	(b)	(a)	(b)	(a)	(b)	(a)	(b)	(a)	(b)	(a)	(b)	(a)	(b)	(a)	(b)
.001	.959	.987	.954	.986	.943	.986	.963	.988	.957	.862	.958	.950	.941	.894	.962	.928
.010	.957	.984	.952	.981	.942	.981	.960	.985	.959	.866	.954	.952	.933	.884	.962	.924
.050	.954	.981	.951	.986	.945	.986	.957	.986	.960	.875	.952	.957	.938	.893	.963	.945
.100	.952	.983	.950	.981	.947	.981	.956	.983	.959	.909	.949	.949	.942	.898	.964	.948
.200	.952	.986	.949	.977	.949	.977	.953	.978	.958	.926	.950	.958	.946	.916	.960	.951
.300	.951	.972	.951	.968	.951	.968	.952	.980	.954	.942	.950	.946	.949	.928	.956	.942
.400	.950	.977	.951	.953	.950	.953	.952	.978	.951	.937	.948	.953	.947	.932	.952	.941
.500	.949	.968	.949	.954	.950	.954	.949	.978	.950	.934	.948	.948	.947	.934	.950	.936
.600	.951	.961	.950	.944	.950	.944	.951	.967	.948	.928	.949	.937	.947	.933	.949	.943
.700	.950	.956	.950	.954	.950	.954	.952	.963	.948	.939	.948	.945	.948	.936	.948	.942
.800	.950	.955	.951	.943	.949	.943	.950	.959	.947	.936	.948	.935	.949	.933	.949	.937
.900	.949	.951	.951	.952	.950	.952	.950	.955	.948	.933	.948	.934	.947	.931	.947	.936
.990	.950	.949	.949	.951	.951	.951	.951	.951	.948	.933	.947	.935	.948	.930	.947	.934
.999	.951	.955	.950	.951	.949	.951	.950	.952	.948	.929	.949	.926	.949	.940	.947	.932

Designs $(n_1, ..., n_a)$: D1 = (3,15,30,14,2,3,13,22,8,6,9,11); D2 = (3,4,3,4,2,3,3,2,2,2,2,2)
D3 = (2,2,2,2,10,10,10,10,40,40,40,40); D4 = (4,4,4,4,12,12,12,12, 20,20,20,20)

Table 4: Coverage probabilities of (.90, .95) tolerance intervals for $N(\mu, \sigma_\tau^2)$; unbalanced case
(a) MLS procedure; (b) Generalized variable approach

	(5,4,3,8)		(5,12,20,12)		(2,7,12,30)		(20,30,20,40)		(2,3,2,4)		D1		D2	
ρ	(a)	(b)	(a)	(b)	(a)	(b)	(a)	(b)	(a)	(b)	(a)	(b)	(a)	(b)
.001	.957	.966	.950	.950	.942	.938	.956	.961	.958	.959	.958	.908	.955	.894
.010	.956	.968	.950	.946	.943	.933	.953	.952	.955	.954	.954	.919	.948	.893
.050	.954	.956	.951	.946	.946	.952	.949	.951	.959	.956	.953	.930	.950	.917
.100	.952	.960	.952	.946	.948	.951	.952	.961	.955	.939	.952	.940	.949	.930
.200	.951	.954	.947	.953	.949	.947	.949	.945	.955	.944	.950	.938	.948	.929
.300	.949	.955	.947	.944	.950	.961	.951	.954	.952	.947	.953	.939	.952	.937
.400	.951	.951	.950	.948	.953	.949	.950	.957	.955	.949	.944	.942	.947	.942
.500	.948	.952	.950	.951	.951	.956	.952	.957	.951	.948	.940	.942	.948	.948
.600	.950	.946	.953	.950	.952	.946	.948	.955	.953	.945	.953	.946	.947	.949
.700	.952	.950	.951	.948	.951	.948	.952	.944	.950	.948	.945	.946	.949	.950
.800	.950	.950	.947	.952	.946	.954	.953	.943	.950	.947	.950	.940	.944	.949
.900	.952	.952	.950	.958	.950	.951	.951	.952	.950	.949	.942	.953	.945	.944
.990	.952	.946	.951	.951	.950	.953	.953	.952	.951	.951	.955	.952	.948	.950
.999	.947	.956	.951	.950	.952	.945	.948	.951	.949	.960	.946	.951	.948	.950

Designs $(n_1, ..., n_a)$: D1 = (3,15,30,14,2,3,13,22,8,6,9,11); D2 = (2,2,2,2,10,10,10,10,40,40,40,40)

Table 5: Expected widths of (.90, .95) tolerance intervals for $N(\mu, \sigma_\tau^2)$; unbalanced case
(a) MLS procedure; (b) Generalized variable approach

	(5,4,3,8)		(5,12,20,12)		(2,7,12,30)		(20,30,20,40)		(2,3,2,4)	
ρ	(a)	(b)	(a)	(b)	(a)	(b)	(a)	(b)	(a)	(b)
.001	4.36	4.38	2.93	2.94	3.80	3.86	1.84	1.79	5.82	5.55
.010	4.44	4.33	3.08	3.05	3.96	3.93	2.06	2.04	5.81	5.71
.050	4.93	4.90	3.69	3.66	4.51	4.44	2.90	2.86	6.18	6.06
.100	5.47	5.34	4.43	4.42	5.12	5.15	3.74	3.73	6.60	6.55
.200	6.60	6.50	5.76	5.68	6.30	6.32	5.24	5.24	7.60	7.45
.300	7.77	7.83	7.15	7.13	7.57	7.57	6.74	6.64	8.65	8.57
.400	9.21	9.11	8.61	8.38	9.03	9.02	8.22	8.28	9.89	9.83
.500	10.8	10.7	10.4	10.3	10.7	10.7	10.1	10.1	11.4	11.5
.600	12.9	12.9	12.4	12.4	12.7	12.7	12.2	12.2	13.4	13.4
.700	15.8	15.8	15.4	15.4	15.5	15.3	15.2	15.2	16.3	16.1
.800	20.3	20.3	19.9	19.9	20.3	20.2	19.8	20.0	20.5	20.6
.900	29.7	30.2	29.8	29.8	29.9	29.8	29.7	29.6	30.3	30.6
.990	98.4	97.5	98.7	98.7	98.6	98.3	98.6	98.2	99.1	98.4
.999	314	315	313	313	313	313	313	314	312	312

Table 6: Coverage probabilities of (.90, .95) TIs for $N(\mu, \sigma_\tau^2 + \sigma_\beta^2 + \sigma_e^2)$ in a two-way nested model
(a) MLS TIs; (b) Generalized TIs

		(a,b,n)															
		(5,5,4)		(5,5,10)		(10,5,3)		(10,10,5)		(15,3,5)		(15,3,15)		(10,10,4)		(15,15,3)	
σ_τ^2	σ_β^2	(a)	(b)	(a)	(b)	(a)	(b)	(a)	(b)	(a)	(b)	(a)	(b)	(a)	(b)	(a)	(b)
.01	.01	.983	.987	.979	.985	.972	.980	.965	.967	.961	.978	.977	.966	.966	.975	.961	.962
.01	.10	.983	.990	.982	.985	.975	.978	.968	.970	.969	.966	.977	.970	.967	.977	.962	.965
.01	.30	.984	.989	.983	.990	.974	.982	.969	.972	.971	.972	.973	.976	.969	.971	.963	.969
.01	.90	.983	.986	.981	.985	.974	.975	.969	.972	.971	.973	.973	.979	.969	.974	.960	.962
.10	.10	.980	.981	.977	.978	.974	.979	.969	.971	.967	.972	.973	.972	.969	.976	.967	.971
.10	.50	.982	.987	.979	.986	.974	.980	.971	.972	.970	.973	.978	.968	.972	.974	.967	.971
.10	.90	.982	.988	.980	.988	.973	.979	.972	.972	.970	.970	.980	.973	.971	.976	.966	.974
.30	.30	.973	.972	.967	.968	.969	.978	.961	.966	.960	.974	.962	.968	.961	.957	.957	.955
.30	.80	.977	.982	.975	.976	.972	.973	.966	.969	.968	.974	.966	.968	.968	.971	.962	.960
.30	.90	.979	.985	.976	.981	.971	.976	.967	.969	.967	.971	.969	.964	.967	.972	.964	.970
.50	.40	.967	.972	.962	.964	.963	.973	.954	.961	.957	.964	.954	.963	.957	.954	.954	.959
.50	.80	.972	.977	.969	.974	.966	.972	.960	.964	.963	.967	.959	.966	.960	.959	.957	.958
.50	.90	.973	.974	.971	.975	.967	.967	.960	.966	.964	.967	.964	.968	.962	.965	.958	.966
.60	.60	.967	.972	.964	.966	.963	.969	.956	.962	.959	.963	.963	.969	.955	.964	.953	.957
.60	.80	.969	.976	.966	.968	.965	.971	.958	.964	.961	.964	.963	.965	.959	.957	.955	.956
.60	.95	.971	.972	.968	.974	.966	.973	.959	.965	.963	.966	.967	.971	.960	.956	.958	.962
.70	.60	.965	.975	.960	.967	.961	.969	.954	.960	.958	.958	.964	.960	.955	.954	.953	.952
.70	.80	.968	.971	.964	.972	.962	.970	.955	.963	.960	.962	.960	.968	.957	.962	.954	.958
.80	.80	.966	.976	.962	.968	.962	.967	.954	.961	.958	.958	.955	.962	.955	.958	.952	.958
.80	.95	.968	.968	.964	.971	.962	.965	.955	.962	.960	.963	.960	.959	.956	.957	.954	.957
.90	.90	.965	.974	.961	.967	.959	.965	.954	.960	.959	.959	.957	.959	.954	.952	.952	.950
.95	.95	.963	.969	.961	.972	.960	.965	.953	.960	.959	.956	.954	.961	.955	.954	.953	.956
1	1	.953	.968	.962	.966	.960	.974	.953	.960	.959	.955	.957	.968	.954	.960	.952	.954

7 An Example

This example is taken from Ostle and Mensing (1975, p. 296) and is on a study of the effect of storage conditions on the moisture content of white pine lumber. Five different storage conditions ($k = 5$)

were studied with a varying number of sample boards stored under each condition, and the data are reproduced here in Table 7. The example is also considered by Bhaumik and Kulkarni (1991) and Krishnamoorthy and Mathew (2009, Chapter 5). These authors assumed the one-way random model for the purpose of computing a (.90, .95) upper tolerance limit for moisture content. We shall compute (.90, .95) TI for moisture contents. The replicates are: $n_1 = 5, n_2 = 3, n_3 = 2, n_4 = 3$

Table 7: Moisture contents of white plain lumber

	storage condition				
	1	2	3	4	5
	7.3	5.4	8.1	7.9	7.1
	8.3	7.4	6.4	9.5	
	7.6	7.1		10.0	
	8.4				
	8.3				
$\bar{y}_{i.}$	7.980	6.633	7.250	9.133	7.100

and $n_5 = 1$. The summary statistics are: $a = 5$, $N = \sum_{i=1}^{a} n_i = 14$, $\bar{\bar{y}} = 7.62$, $ss_{\bar{y}} = 3.80$, and $ss_e = 7.17$. To compute the (.90, .95) MLS tolerance interval, we note that $a_1 = (1 + 1/a) = 1.2$, $a_2 = 1 - \tilde{n} = 1 - .4733 = .5267$, $\chi^2_{4;.05} = 0.7107$ and $\chi^2_{9;.05} = 3.3251$. Further, $\widehat{\sigma}_1^2 = ss_{\bar{y}}/(a-1) = 3.80/4 = 0.95$ and $\widehat{\sigma}_2^2 = ss_e/(N-a) = 7.17/9 = 0.7967$. Using these quantities in (20), we get the 95% upper confidence limit for $\sqrt{a_1\sigma_1^2 + a_2\sigma_2^2}$ as 2.624. Noting that $z_{\frac{1+p}{2}} = z_{.95} = 1.645$, the (.90, .95) TI for the moisture contents is $7.62 \pm 1.645 \times 2.624 = (3.30, 11.94)$.

To construct the generalized TI, we computed the median $\tilde{G}_\mu = 7.70$ and the 95% generalized upper confidence limit for $\sqrt{\sigma_\tau^2 + \sigma_e^2 + \left(\sum_{i=1}^{a} \nu_i^{-1}\right)^{-1}} = 2.681$ using simulation consisting of 10,000 runs. Thus, the generalized TI is $7.70 \pm 1.645 \times 2.681 = (3.29, 12.11)$. Note that both TIs are practically the same.

Suppose we are interested in a (.90, .95) TI for the distribution of the 'true' moisture content, that is, the $N(\mu, \sigma_\tau^2)$ distribution. In this case, $a_1 = (1 + 1/a) = 1.2$ and $a_2 = -\tilde{n} = -.4733$. The critical value $\chi^2_{14,.95} = 16.919$. Substituting these quantities and the variance estimates in (30), we get the 95% MLS upper confidence limit for $\sqrt{a_1\sigma_1^2 + a_2\sigma_2^2}$ as 2.458. Thus, MLS TI is $7.62 \pm 1.645 \times 2.458 = (3.58, 11.66)$. To compute the generalized TI for the true moisture content, we estimated the 95% upper confidence limit for $\sqrt{\sigma_\tau^2 + \left(\sum_{i=1}^{a} \nu_i^{-1}\right)^{-1}}$ using simulation as 2.471. Using the value of $\tilde{G}_\mu$ given in the preceding paragraph, the required TI is $7.70 \pm 1.645 \times 2.471 = (3.64, 11.76)$. We once again observe that both methods produced practically the same tolerance interval.

8 Concluding Remarks

In this article, we outlined two procedures for constructing TIs in random effects model and in a two-way nested model, and compared them with respect to coverage probabilities. Our Monte Carlo simulation studies clearly indicate that the MLS TIs are comparable to or better than the corresponding generalized TIs with respect to coverage probabilities. Our limited study on expected width also indicates these TIs perform similar in terms of expected width wherever they have similar coverage properties. It is also clear from the description of the procedures that the MLS TIs are easier to compute than the generalized TIs. The various sums of squares required to compute the TIs can be obtained using standard software packages. Our overall conclusion is that both the MLS approach and the generalized variable approach are similar, with the MLS approach being easier to apply and performing slightly better. Finally, we note that the MLS TIs can be readily obtained in linear models where the problem of constructing TIs simplifies to the one of constructing confidence limits for a linear combination of variance components.

REFERENCES

Beckman, R. J. and Tietjen, G. L. (1989). Two-sided tolerance limits for balanced random-effects ANOVA models (Correction: 1998, 40, p. 269). *Technometrics*, 31, 185–197.

Bhaumik, D. K., and Kulkarni, P. M. (1991). One-sided tolerance limits for unbalanced one-way ANOVA random effects model. *Communications in Statistics-Theory and Methods*, 20, 1665-1675.

Burdick, R. K. and Graybill, F. A.(1992). *Confidence Intervals on Variance Components.* New York: Marcel-Dekker.

Graybill, F. A. and Wang, C. M. (1980). Confidence intervals on nonnegative linear combinations of variances. *Journal of the American Statistical Association*, 75, 869-873.

Guttman, I. (1970). *Statistical Tolerance Regions: Classical and Bayesian*, London: Charles Griffin and Company.

Howe, W. G. (1969). Two-sided tolerance limits for normal populations – Some improvements. *Journal of the American Statistical Association*, 64, 610–620.

Krishnamoorthy, K. and Mathew, T. (2004). One-sided tolerance limits in balanced and unbalanced one-way random models based on generalized confidence limits. *Technometrics*, 46, 44–52.

Krishnamoorthy, K. and Mathew, T. (2009). *Statistical Tolerance Regions: Theory, Applications and Computation*, Hoboken, NJ: Wiley.

Lian, X. (2011). Tolerance intervals in some linear models and variance components models. Ph.D. dissertation, Department of Mathematics, University of Louisiana at Lafayette.

Liao, C.-T., Lin, T. Y. and Iyer, H. K. (2005). One- and two-sided tolerance intervals for general balanced mixed models and unbalanced one-way random models. *Technometrics*, 47, 323–335.

Lyles, R. H., Kupper, L. L. and Rappaport, S. M. (1997). Assessing regulatory compliance of occupational exposures via the balanced one-way random effects ANOVA model. *Journal of Agricultural, Biological, and Environmental Statistics*, 2, 64–86.

Mee, R. W. (1984). β-expectation and β-content tolerance limits for balanced one-way ANOVA random model. *Technometrics*, 26, 251–254.

Mee, R. W. and Owen, D. B. (1983). Improved factors for one-sided tolerance limits for balanced one-way ANOVA random model. *Journal of the American Statistical Association*, 78, 901–905.

Ostle, B. and Mensing, R. W. (1975). *Statistics in Research: Basic Concepts and Techniques for Research Workers*, (3rd ed.). Ames, Iowa: Iowa State University Press.

Satterthwaite, F. E. (1946). An approximate distribution of estimates of variance components. *Biometrics Bulletin*, 2, 110-114.

Thomas, J. D. and Hultquist, R. A. (1978). Interval estimation for the unbalanced case of the one-way random effects model. *The Annals of Statistics*, 6, 582-587.

Vangel, M. G. (1992). New methods for one-sided tolerance limits for a one-way balanced random-effects ANOVA model. *Technometrics*, 34, 176–185.

Wolfinger, R. D. (1998). Tolerance intervals for variance component models using Bayesian simulation. *Journal of Quality Technology*, 30, 18–32.