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Inference on reliability in two-parameter exponential stress–strength model

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Abstract The problem of hypothesis testing and interval estimation of the reliability parameter in a stress–strength model involving two-parameter exponential distributions is considered. Test and interval estimation procedures based on the generalized variable approach are given. Statistical properties of the generalized variable approach and an asymptotic method are evaluated by Monte Carlo simulation. Simulation studies show that the proposed generalized variable approach is satisfactory for practical applications while the asymptotic approach is not satisfactory even for large samples. The results are illustrated using simulated data.

Keywords Coverage probability · Generalized confidence limit · Generalized p-value · Location-scale invariance · Pareto distribution · Power distribution · Size

1 Introduction

There has been continuous interest in the problem of estimating the probability that one random variable exceeds another, that is, $R = P(X > Y)$, where X and Y are independent random variables. The parameter R is referred to as the reliability parameter. This problem arises in the classical stress–strength reliability where one is interested in assessing the proportion of the times the

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random strength X of a component exceeds the random stress Y to which the component is subjected. If $X \leq Y$, then either the component fails or the system that uses the component may malfunction. This problem also arises in situations where X and Y represent lifetimes of two devices and one wants to estimate the probability that one fails before the other. Some practical examples can be found in Hall (1984) and Weerahandi and Johnson (1992). Hall provided an example of a system application where the breakdown voltage X of a capacitor must exceed the voltage output Y of a transverter (power supply) in order for a component to work properly. Weerahandi and Johnson (1992) presented a rocket–motor experiment data where X represents the chamber burst strength and Y represents the operating pressure. These authors proposed inferential procedures for $P(X > Y)$ assuming that X and Y are independent normal random variables. There are several papers that considered the stress–strength reliability problem, and for references see the recent article by Guo and Krishnamoorthy (2004) or the book by Kotz et al. (2003).

For the one-parameter (scale parameter) exponential case, Chao (1982) compared several procedures based on the maximum likelihood estimators (MLEs). Aminzadeh (1997) proposed an approximate method for setting confidence bounds for R when the stress and strength variables involve some covariates. Recently, Baklizi and El-Masri (2004) presented a shrinkage estimator of R when X and Y are independent two-parameter (scale-location) exponential random variables with common location parameter. Kunchur and Munoli (1993) considered the problem of estimating reliability for a multicomponent stress–strength model based on exponential distributions.

In this article, we want to develop inferential procedures about the reliability parameter $R = P(X > Y)$, where X and Y are independent two-parameter exponential random variables. A two-parameter exponential distribution has probability density function (pdf) given by

$$f(x;\mu,\theta) = \frac{1}{\theta}\mathrm{e}^{-(\mathrm{x}-\mu)/\theta}, \quad \mathrm{x} > \mu, \quad \mu \geq 0, \ \ \theta > 0, \tag{1}$$

where μ is the location parameter and θ is the scale parameter. In lifetime data analysis, μ is referred to as a threshold or "guarantee time" parameter, and θ is the mean time to failure.

As mentioned in Kotz et. al. (2003), the case of the two-parameter exponential distributions is of importance because it allows us to derive confidence limits for the reliability parameters involving Pareto distributions or power distributions by means of one-one transformations. In particular, if X follows a Pareto distribution with pdf $\lambda\sigma^{\lambda}/x^{\lambda+1}$, $x > \sigma$, then $Y = \ln(X)$ has the pdf in (1) with $\mu = \ln(\sigma)$ and $\theta = 1/\lambda$. If X follows a power distribution with pdf $\lambda x^{\lambda-1}/\sigma^{\lambda}$, $0 < x < \sigma$, then $Y = \ln(1/X)$ has the pdf in (1) with $\mu = \ln(1/\sigma)$ and $\theta = 1/\lambda$. Therefore, the inferential procedures about the reliability parameter that we will derive in the following sections are readily applicable to these distributions.

To formulate the present problem, let $X \sim$ exponential(μ_1,θ_1) independently of $Y \sim$ exponential (μ_2,θ_2). That is, the pdf of X is $f(x;\mu_1,\theta_1)$ and the pdf of Y

is $f(y;\mu_2,\theta_2)$, where f is given in (1). Then the stress–strength reliability can be expressed as follows.
If $\mu_1 > \mu_2$, then the reliability parameter R is given by

$$\begin{aligned} P(X>Y) &= P_Y P_{X|Y}(X>Y|\mu_2<Y<\mu_1) + P_Y P_{X|Y}(X>Y|Y>\mu_1) \\ &= \frac{1}{\theta_2}\int_{\mu_2}^{\mu_1} \mathrm{e}^{-\frac{(\mathrm{y}-\mu_2)}{\theta_2}}\mathrm{dy} + \frac{1}{\theta_2}\int_{\mu_1}^{\infty} \mathrm{e}^{-\frac{(\mathrm{y}-\mu_1)}{\theta_1}}\mathrm{e}^{-\frac{(\mathrm{y}-\mu_2)}{\theta_2}}\mathrm{dy} \\ &= 1 - \frac{\theta_2 \mathrm{e}^{\frac{(\mu_2-\mu_1)}{\theta_2}}}{\theta_1+\theta_2}. \end{aligned}$$

If $\mu_1 \le \mu_2$, then R is given by

$$\begin{aligned} P(X>Y) &= E_Y P_{X|Y}(X>Y|Y) \\ &= \frac{1}{\theta_2}\int_{\mu_1}^{\infty} \mathrm{e}^{-\frac{(\mathrm{y}-\mu_1)}{\theta_1}}\mathrm{e}^{-\frac{(\mathrm{y}-\mu_2)}{\theta_2}}\mathrm{dy} \\ &= \frac{\theta_1 \mathrm{e}^{\frac{(\mu_1-\mu_2)}{\theta_1}}}{\theta_1+\theta_2}. \end{aligned}$$

Thus, the reliability parameter R can be expressed as

$$R = \left(1 - \frac{\theta_2 \mathrm{e}^{(\mu_2-\mu_1)/\theta_2}}{\theta_1+\theta_2}\right) I(\mu_1>\mu_2) + \left(\frac{\theta_1 \mathrm{e}^{(\mu_1-\mu_2)/\theta_1}}{\theta_1+\theta_2}\right) I(\mu_1 \le \mu_2), \quad (2)$$

where $I(.)$ is the indicator function.

If $\mu_1 = \mu_2$, then the reliability parameter R simplifies to $\theta_1/(\theta_1+\theta_2)$, and exact confidence limits for R can be obtained using some conventional approaches (see Bhattacharyya and Johnson 1974). If $\mu_1 \neq \mu_2$, then the form for R, as shown in (2), is quite complex, and only large sample approach is available (see Kotz et al. 2003, Section 4.2.3). This large sample approach is based on the asymptotic normality of the MLE of R, and its accuracies are yet to be investigated. Small sample conventional approaches for the present problem may fail to yield any useful solution, and so we resorted to use the *generalized variable* approach. The concept of the generalized p-value for hypothesis testing was introduced by Tsui and Weerahandi (1989) and the idea was extended to interval estimation by Weerahandi (1993). Since then this *generalized variable* approach has been used to find solutions to many complex problems; among others, ANOVA with unequal error variances (Weerahandi 1995a), inferential methods for lognormal mean (Krishnamoorthy and Mathew 2003), tolerance limits for random effects model (Krishnamoorthy and Mathew 2004) and the multivariate Behrens-Fisher problem (Gamage et. al. 2004). For more details about this approach, see the book by Weerahandi (1995b).

This article is organized as follows. In the following section, we present some basic distributional results and the MLE of R. In section 3, we present the asymptotic results given in Kotz et. al. (2003). In section 4, we explain first the method of constructing generalized variables for the scale and location parameters of an exponential distribution. Using these generalized variables, in section 5, we construct a generalized variable for the reliability parameter R and outline the procedures for constructing confidence limits and hypothesis testing about R. We also show that the generalized variable approach produces exact confidence limits for R when $\mu_1 = \mu_2$. In section 6, Monte Carlo simulation studies are carried out to assess the validity of the generalized variable approach and large sample properties of the asymptotic approach. Our simulation studies indicate that the coverage probabilities of the generalized limits are in general slightly more than or close to the nominal level. The asymptotic approach has poor coverage properties even when the samples are large (as large as 100) . In section 7, we illustrate the inferential procedures using simulated data sets. Some concluding remarks are given in section 8.

2 Preliminaries

Let $Z_1, \ldots, Z_n$ be a sample of observations from an exponential distribution with pdf in (1). The maximum likelihood estimators of μ and θ are given by

$$\widehat{\mu} = Z_{(1)} \quad \text{and} \quad \widehat{\theta} = \frac{1}{n}\sum_{i=1}^{n}(Z_i - Z_{(1)}) = \bar{Z} - Z_{(1)}, \tag{3}$$

where $Z_{(1)}$ is the smallest of the Z_i's. It is known that $\widehat{\mu}$ and $\widehat{\theta}$ are independent with

$$\frac{2n(\widehat{\mu} - \mu)}{\theta} \sim \chi_2^2 \quad \text{and} \quad \frac{2n\widehat{\theta}}{\theta} \sim \chi_{2n-2}^2. \tag{4}$$

(see Lawless 1982, section 3.5)

Let X be an exponential random variable with pdf $f(x; \mu_1, \theta_1)$ and Y be an exponential random variable with pdf $f(y; \mu_2, \theta_2)$, where the pdfs are as defined in (1). Assume that X and Y are independent. Let $X_1, \ldots, X_{n_1}$ be a sample of observations on X and $Y_1, \ldots, Y_{n_2}$ be a sample of observations on Y. Furthermore, let $\widehat{\mu}_1$ and $\widehat{\theta}_1$ denote respectively the MLEs of μ_1 and θ_1 based on X observations, and let $\widehat{\mu}_2$ and $\widehat{\theta}_2$ denote respectively the MLEs of μ_2 and θ_2 based on Y observations (see (3) for the formulas). Specifically, the MLEs are

$$\widehat{\mu}_1 = X_{(1)},\ \widehat{\mu}_2 = Y_{(1)}, \widehat{\theta}_1 = \bar{X} - X_{(1)} \ \text{and} \ \widehat{\theta}_2 = \bar{Y} - Y_{(1)},$$

The MLE of the reliability parameter R can be obtained by replacing the parameters μ_1, μ_2, θ_1 and θ_2 in (2) by their MLEs. That is, the MLE of R is given by

$$\widehat{R} = \left(1 - \frac{\widehat{\theta}_2 e^{(\widehat{\mu}_2 - \widehat{\mu}_1)/\widehat{\theta}_2}}{\widehat{\theta}_1 + \widehat{\theta}_2}\right) I(\widehat{\mu}_1 > \widehat{\mu}_2) + \left(\frac{\widehat{\theta}_1 e^{(\widehat{\mu}_1 - \widehat{\mu}_2)/\widehat{\theta}_1}}{\widehat{\theta}_1 + \widehat{\theta}_2}\right) I(\widehat{\mu}_1 \le \widehat{\mu}_2). \quad (5)$$

3 An asymptotic approach

An asymptotic confidence interval for R is given in Kotz et. al. (2003, section 4.2.3). This confidence interval is based on an asymptotic distribution of the MLE of R. We shall now present an asymptotic mean squared error of $\widehat{R}$ given in Kotz et al. (2003). Let $\lambda = n_1/(n_1 + n_2)$ and define

$$C_j = \begin{cases} \frac{\widehat{\theta}_i}{(\widehat{\theta}_i + \widehat{\theta}_j)^2} \exp\left[-\frac{(\widehat{\mu}_j - \widehat{\mu}_i)}{\widehat{\theta}_i}\right], & \text{if } \widehat{\mu}_j > \widehat{\mu}_i, \\ \left[\frac{\widehat{\theta}_i}{(\widehat{\theta}_i + \widehat{\theta}_j)^2} + \frac{\widehat{\mu}_i - \widehat{\mu}_j}{\widehat{\theta}_j(\widehat{\theta}_i + \widehat{\theta}_j)}\right] \exp\left[-\frac{(\widehat{\mu}_i - \widehat{\mu}_j)}{\widehat{\theta}_j}\right], & \text{if } \widehat{\mu}_j \le \widehat{\mu}_i, \end{cases}$$

where $i = 2$ if $j = 1$ and $i = 1$ if $j = 2$. Using these terms, an estimate of asymptotic MSE of $\widehat{R}$ is given by

$$\widehat{\sigma}^2_{\widehat{R}} = (\lambda^{-1} C_1^2 \widehat{\theta}_1^2 + (1 - \lambda)^{-1} C_2^2 \widehat{\theta}_2^2)/(n_1 + n_2).$$

Using this estimate, for large $n_1 + n_2$, we have

$$\frac{\sqrt{n_1 + n_2}(\widehat{R} - R)}{\widehat{\sigma}_{\widehat{R}}} \sim N(0, 1) \text{ approximately.}$$

A $1 - \alpha$ lower limit for R based on the above asymptotic distribution is given by

$$\widehat{R} - z_{1-\alpha} \frac{\widehat{\sigma}_{\widehat{R}}}{\sqrt{n_1 + n_2}}, \quad (6)$$

where z_p denotes the pth quantile of the standard normal distribution.

4 Generalized variables for μ and θ

The reliability parameter in (2) is a function of both μ's and θ's. So we develop first generalized variables for μ and θ for the one-sample case. Even though it is not our primary interest, knowing the results of the one-sample case will make it easier to understand the approach and results for the stress–strength reliability in section 5.

4.1 A generalized variable for μ

Let $\widehat{\mu}_0$ and $\widehat{\theta}_0$ be observed values of $\widehat{\mu}$ and $\widehat{\theta}$ respectively. Based on the above distributional results of the MLEs, a *generalized pivot variable* for μ can be constructed as follows.

$$
\begin{aligned}
G_\mu &= \widehat{\mu}_0 - \frac{2n(\widehat{\mu}-\mu)}{2n\theta}\frac{\theta}{\widehat{\theta}}\widehat{\theta}_0 \\
&= \widehat{\mu}_0 - \frac{2n(\widehat{\mu}-\mu)}{\theta}\frac{1}{(2n\widehat{\theta})/\theta}\widehat{\theta}_0 \\
&= \widehat{\mu}_0 - \frac{\chi^2_2}{\chi^2_{2n-2}}\widehat{\theta}_0. \qquad (7)
\end{aligned}
$$

To get the last step, we used the distributional results in (4). The *generalized test variable* for testing hypothesis about μ is given by

$$
G^t_\mu = G_\mu - \mu = \widehat{\mu}_0 - \frac{\chi^2_2}{\chi^2_{2(n-1)}}\widehat{\theta}_0 - \mu. \qquad (8)
$$

In general, a generalized pivot variable should satisfy two properties. More details can be found in Weerahandi (1995b).

(1) The value of G_μ at $(\widehat{\mu}, \widehat{\theta}) = (\widehat{\mu}_0, \widehat{\theta}_0)$ should be the parameter of interest. Here, using the step 1 of (7), we see that the value of G_μ at $(\widehat{\mu}, \widehat{\theta}) = (\widehat{\mu}_0, \widehat{\theta}_0)$ is μ.
(2) For given $\widehat{\mu}_0$ and $\widehat{\theta}_0$, the distribution of G_μ should be independent of any unknown parameters. This property also holds because we see from step 3 of (7) that the distribution of G_μ, when $\widehat{\mu}_0$ and $\widehat{\theta}_0$ are fixed, does not depend on any parameter.

A generalized test variable should satisfy the following three properties:

(1) The value of G^t_μ at $(\widehat{\mu}, \widehat{\theta}) = (\widehat{\mu}_0, \widehat{\theta}_0)$ should not depend on any parameter. Here, using the step 1 of (7) and (8), we see that the value of G^t_μ at $(\widehat{\mu}, \widehat{\theta}) = (\widehat{\mu}_0, \widehat{\theta}_0)$ is zero.
(2) For given $\widehat{\mu}_0$ and $\widehat{\theta}_0$, the distribution of G^t_μ should depend only on the parameter of interest. Using (8), it is easy to see that this property also holds.
(3) For given $\widehat{\mu}_0$ and $\widehat{\theta}_0$, the distribution of G^t_μ should be stochastically monotone with respect to the parameter of interest. From the definition of G^t_μ in (8), we see that the distribution of G^t_μ is stochastically decreasing with respect to μ.

Thus, we showed that G_μ is a bona fide generalized pivot variable for constructing confidence limits for μ, and G^t_μ is a valid generalized test variable for hypothesis testing about μ. For example, the 100αth percentile of G_μ, that is, $\widehat{\mu}_0 - \frac{2}{2n-2}F_{2,2n-2,1-\alpha}\widehat{\theta}_0$, where $F_{m,n,p}$ denotes the 100pth percentile of an F distribution with degrees of freedoms m and n, is a $1-\alpha$ lower confidence limit for μ. If one is interested in testing

$$H_0 : \mu \leq \mu_0 \ \text{ vs. } \ H_a : \mu > \mu_0,$$

then, noting that G_μ^t is stochastically decreasing in μ, the generalized p-value is given by

$$P\left(\sup_{H_0} G_\mu^t < 0\right) = P(G_\mu^t < 0|\mu = \mu_0) = P(G_\mu < \mu_0).$$

Using (7), and after some simplification, we see that the above p-value can be expressed as

$$P\left(\frac{2}{2n-2}F_{2,2n-2} > \frac{\widehat{\mu}_0 - \mu_0}{\widehat{\theta}_0}\right).$$

The test or interval estimation of μ based on our generalized variable approach are the same as the usual exact ones (see Lawless 1982, p. 128).

4.2 A generalized variable for θ

A generalized variable for θ is given by

$$G_\theta = \frac{\theta}{2n\widehat{\theta}} 2n\widehat{\theta}_0 = \frac{2n\widehat{\theta}_0}{\chi^2_{2n-2}}, \tag{9}$$

and the generalized test variable based on G_θ is given by

$$G_\theta^t = \frac{2n\widehat{\theta}_0}{\chi^2_{2n-2}} - \theta.$$

It is easy to see that the generalized pivot variable and the generalized test variable satisfy the properties given earlier. Furthermore, it is easy to see that the confidence interval and hypothesis testing [based on (9)] about θ are the same as the usual exact ones (see Lawless 1982, p. 128).

Notice that under the transformation $X \to aX + b$, where $a > 0$, $G_\mu \to aG_\mu + b$ and $G_\theta \to aG_\theta$. Therefore, inferential procedures based on the generalized variables are scale-location invariant.

5 Generalized confidence limits for R

The generalized variable for R can be obtained by replacing the parameters by their generalized variables. The reliability parameter R simplifies to $\theta_1/(\theta_1+\theta_2) = 1/(1+\theta_2/\theta_1)$ when $\mu_1 = \mu_2$. In this case, it is enough to find confidence limit for θ_2/θ_1. The generalized variable for θ_2/θ_1 is given by $G_{\theta_2}/G_{\theta_1}$, and after some algebraic manipulation, it is easy to see that the generalized limits are equal to the exact limits [see Bhattacharyya and Johnson (1974, section 5)] for the reliability parameter.

If the location parameters are unknown and arbitrary, then a generalized pivot variable for R can be obtained by replacing the parameters in (2) by their generalized variables. Denoting the resulting generalized variable by G_R, we have

$$G_R = \left(1 - \frac{G_{\theta_2} e^{\frac{(G_{\mu_2} - G_{\mu_1})}{G_{\theta_2}}}}{G_{\theta_1} + G_{\theta_2}}\right) I(G_{\mu_1} > G_{\mu_2}) + \left(\frac{G_{\theta_1} e^{\frac{(G_{\mu_1} - G_{\mu_2})}{G_{\theta_1}}}}{G_{\theta_1} + G_{\theta_2}}\right) I(G_{\mu_1} \le G_{\mu_2}), \tag{10}$$

where

$$G_{\mu_i} = \widehat{\mu}_{i0} - \frac{Q_i}{W_i}\widehat{\theta}_{i0}, \quad G_{\theta_i} = \frac{2n_i\widehat{\theta}_{i0}}{W_i}, \quad i = 1, 2, \tag{11}$$

$(\widehat{\mu}_{i0}, \widehat{\theta}_{i0})$ is an observed value of $(\widehat{\mu}_i, \widehat{\theta}_i)$, $i = 1, 2$, and Q_1, Q_2, W_1 and W_2 are independent random variables with $Q_i \sim \chi^2_2$ and $W_i \sim \chi^2_{2n_i-2}$, $i = 1, 2$. The generalized test variable for R is given by

$$G_R^t = G_R - R.$$

It is easy to check that the generalized pivot variable G_R satisfies the two properties given in Section 3. In particular, for given $\widehat{\mu}_{10}$, $\widehat{\mu}_{20}$, $\widehat{\theta}_{10}$ and $\widehat{\theta}_{20}$, the distribution of G_R does not depend on any unknown parameters. So, Monte Carlo method given in Algorithm 1, can be used to find confidence limits for R or to find the generalized p-value for hypothesis testing about R.

Algorithm 1

For a given data set, compute the MLEs $\widehat{\mu}_{10}, \widehat{\theta}_{10}, \widehat{\mu}_{20}, \widehat{\theta}_{20}$ using the formulas in (3)

For $i = 1, m$

Generate $Q_1 \sim \chi^2_2$, $Q_2 \sim \chi^2_2$, $W_1 \sim \chi^2_{2n_1-2}$, $W_2 \sim \chi^2_{2n_2-2}$

Compute $G_{\mu_1}, G_{\mu_2}, G_{\theta_1}, G_{\theta_2}$ and G_R [see (10) and (11)]

(end loop)

The 100αth percentile of the generated G_R's is a $1-\alpha$ lower limit for the reliability parameter R. If we are interested in testing

$$H_0 : R \le R_0 \ \text{vs.}\ H_a : R > R_0,$$

where R_0 is a specified value, then the generalized p-value is the proportion of the G_R's that are less than R_0.

6 Monte Carlo studies

We note first that the exponential distributions, the parameter of interest R in (2), and the generalized pivot variable G_R are scale-location invariant. Therefore, without loss of generality, we can assume that $\mu_2 = 0$ and $\theta_2 = 1$ to compute the coverage probabilities. We estimated the coverage probabilities via Monte Carlo simulation. The simulation is carried out as follows. For a given $(n_1, \mu_1, \theta_1, n_2)$, we first generated 2,500 $(\widehat{\mu}_{10}, \widehat{\theta}_{10}, \widehat{\mu}_{20}, \widehat{\theta}_{20})$'s using the distributional results in (4). For each simulated set $(\widehat{\mu}_{10}, \widehat{\theta}_{10}, \widehat{\mu}_{20}, \widehat{\theta}_{20})$, we used Algorithm 1 with $m = 5,000$ to find the 95% lower limit for R. The proportion of the 2,500 lower limits that are below the value of R is a Monte Carlo estimate of the coverage probability. The coverage probabilities of the asymptotic limit in (6) were estimated using simulation consisting of 100,000 runs.

In Table 1, we present coverage probabilities of asymptotic limits and generalized confidence limits for samples $n_1 = n_2 = 50$ and $n_1 = n_2 = 100$. We chose large sample sizes because the asymptotic limits are valid only for large samples. In Table 2, we give the coverage probabilities of the generalized limits for small samples. Furthermore, to understand the closeness of the lower confidence limits to the value of the reliability parameter, we present estimates of the expectation of the lower limits and the value of R for each of the parameter and sample size configurations.

We observe the following from Monte Carlo estimates given in Tables 1 and 2.

1. We observe from Table 1 that, even for large samples, the coverage probabilities of the asymptotic approach are in general smaller than the nominal level 0.95, and they are close to the nominal level only when μ_1 and μ_2 are close to zero. For larger θ_1, the coverage probabilities are much lower than the nominal level. Even for samples as large as 100, the coverage probabilities of the asymptotic limits go as low as 0.77 when the nominal level is 0.95. So, the asymptotic approach should not be recommended for practical applications. Furthermore, we notice that the generalized variable limits not only have good coverage probabilities but also have expected values close to those of the asymptotic limits.
2. The coverage probabilities of the generalized confidence limits are in general either close to or slightly more than the nominal level 0.95. The coverage probabilities seldom exceed 0.97. Comparison of values for $n_1 = 10, n_2 = 12$

Table 1 Coverage probabilities (CP) and expected lengths (EL) of 95% lower confidence limits for R

		θ_1									
		1		1.5		2		2.5		10	
μ_1		R	CP(EL)	R	CP(EL)	R	CP(EL)	R	CP(EL)	R	CP(EL)
$\mu_2 = 0, \theta_2 = 1;\ n_1 = 50, n_2 = 50$											
0	a	0.50	0.95(0.41)	0.60	0.95(0.52)	0.67	0.95(0.59)	0.71	0.95(0.64)	0.91	0.97(0.86)
	b		0.94(0.42)		0.92(0.52)		0.90(0.60)		0.89(0.65)		0.69(0.90)
0.2	a	0.59	0.95(0.51)	0.67	0.95(0.59)	0.73	0.94(0.65)	0.77	0.95(0.69)	0.93	0.97(0.88)
	b		0.93(0.51)		0.91(0.60)		0.90(0.66)		0.89(0.71)		0.70(0.91)
0.4	a	0.66	0.95(0.58)	0.73	0.95(0.65)	0.78	0.95(0.70)	0.81	0.95(0.74)	0.94	0.96(0.90)
	b		0.92(0.59)		0.91(0.66)		0.89(0.72)		0.88(0.76)		0.72(0.93)
0.6	a	0.73	0.95(0.64)	0.78	0.95(0.70)	0.82	0.95(0.75)	0.84	0.96(0.78)	0.95	0.96(0.91)
	b		0.92(0.65)		0.90(0.72)		0.89(0.76)		0.88(0.80)		0.73(0.94)
0.8	a	0.78	0.95(0.70)	0.82	0.95(0.75)	0.85	0.96(0.78)	0.87	0.95(0.81)	0.96	0.97(0.93)
	b		0.91(0.71)		0.90(0.76)		0.89(0.80)		0.88(0.83)		0.73(0.95)
1	a	0.82	0.95(0.74)	0.85	0.94(0.79)	0.88	0.96(0.82)	0.89	0.95(0.84)	0.97	0.96(0.93)
	b		0.90(0.75)		0.89(0.80)		0.88(0.83)		0.87(0.85)		0.74(0.96)
1.3	a	0.86	0.95(0.79)	0.89	0.95(0.83)	0.91	0.95(0.85)	0.92	0.95(0.87)	0.98	0.96(0.95)
	b		0.89(0.81)		0.89(0.84)		0.88(0.87)		0.87(0.89)		0.75(0.97)
1.6	a	0.90	0.95(0.84)	0.92	0.94(0.86)	0.93	0.95(0.88)	0.94	0.95(0.90)	0.98	0.97(0.96)
	b		0.88(0.85)		0.88(0.88)		0.87(0.90)		0.86(0.91)		0.75(0.97)
1.9	a	0.93	0.95(0.87)	0.94	0.96(0.89)	0.95	0.95(0.91)	0.96	0.95(0.92)	0.99	0.95(0.97)
	b		0.88(0.89)		0.87(0.91)		0.86(0.92)		0.86(0.93)		0.76(0.98)
2.2	a	0.94	0.95(0.90)	0.96	0.95(0.92)	0.96	0.95(0.93)	0.97	0.95(0.94)	0.99	0.97(0.98)
	b		0.87(0.91)		0.86(0.93)		0.86(0.94)		0.85(0.95)		0.76(0.98)
$\mu_2 = 0, \theta_2 = 1;\ n_1 = 100, n_2 = 100$											
0	a	0.50	0.95(0.44)	0.60	0.95(0.54)	0.67	0.95(0.61)	0.71	0.95(0.66)	0.91	0.95(0.88)
	b		0.95(0.44)		0.93(0.55)		0.92(0.62)		0.91(0.67)		0.77(0.90)
0.2	a	0.59	0.94(0.53)	0.67	0.95(0.62)	0.73	0.96(0.67)	0.77	0.95(0.72)	0.93	0.96(0.90)
	b		0.94(0.53)		0.93(0.62)		0.92(0.68)		0.91(0.73)		0.78(0.91)
0.4	a	0.66	0.95(0.61)	0.73	0.95(0.68)	0.78	0.95(0.73)	0.81	0.96(0.76)	0.94	0.97(0.92)
	b		0.93(0.61)		0.92(0.68)		0.92(0.73)		0.91(0.77)		0.79(0.93)
0.6	a	0.73	0.95(0.67)	0.78	0.94(0.73)	0.82	0.94(0.77)	0.84	0.95(0.80)	0.95	0.96(0.93)
	b		0.93(0.67)		0.92(0.73)		0.91(0.78)		0.90(0.81)		0.80(0.94)
0.8	a	0.78	0.95(0.72)	0.82	0.95(0.77)	0.85	0.95(0.81)	0.87	0.95(0.83)	0.96	0.95(0.94)
	b		0.93(0.73)		0.92(0.78)		0.91(0.81)		0.90(0.84)		0.81(0.95)
1	a	0.82	0.95(0.76)	0.85	0.95(0.81)	0.88	0.95(0.84)	0.89	0.96(0.86)	0.97	0.95(0.95)
	b		0.92(0.77)		0.91(0.81)		0.91(0.84)		0.90(0.87)		0.81(0.96)
1.3	a	0.86	0.96(0.82)	0.89	0.95(0.85)	0.91	0.95(0.87)	0.92	0.95(0.89)	0.98	0.96(0.96)
	b		0.91(0.82)		0.91(0.86)		0.90(0.88)		0.90(0.90)		0.82(0.97)
1.6	a	0.90	0.95(0.86)	0.92	0.95(0.88)	0.93	0.95(0.90)	0.94	0.95(0.91)	0.98	0.96(0.97)
	b		0.91(0.86)		0.90(0.89)		0.90(0.91)		0.89(0.92)		0.82(0.98)
1.9	a	0.93	0.95(0.89)	0.94	0.95(0.91)	0.95	0.96(0.92)	0.96	0.95(0.93)	0.99	0.96(0.98)
	b		0.91(0.90)		0.90(0.92)		0.89(0.93)		0.89(0.94)		0.83(0.98)
2.2	a	0.94	0.95(0.91)	0.96	0.95(0.93)	0.96	0.95(0.94)	0.97	0.95(0.95)	0.99	0.96(0.98)
	b		0.90(0.92)		0.89(0.94)		0.89(0.95)		0.88(0.95)		0.83(0.99)

a Generalized limit, b asymptotic limit, R= Reliability parameter

Table 2 Coverage probabilities (CP) and expected lengths of 95% generalized lower confidence limits for R

	θ_1									
	1		1.5		2		2.5		10	
μ_1	R	CP(EL)	R	CP(EL)	R	CP(EL)	R	CP(EL)	R	CP(EL)
$\mu_2 = 0, \theta_2 = 1;\ n_1 = 12, n_2 = 10$										
0	0.50	0.96(0.33)	0.60	0.96(0.41)	0.67	0.96(0.48)	0.71	0.97(0.53)	0.91	0.97(0.75)
0.2	0.59	0.96(0.40)	0.67	0.96(0.48)	0.73	0.97(0.53)	0.77	0.97(0.58)	0.93	0.97(0.76)
0.4	0.66	0.96(0.48)	0.73	0.94(0.55)	0.78	0.97(0.59)	0.81	0.97(0.62)	0.94	0.98(0.78)
0.6	0.73	0.96(0.53)	0.78	0.95(0.59)	0.82	0.97(0.64)	0.84	0.97(0.66)	0.95	0.97(0.80)
0.8	0.78	0.95(0.59)	0.82	0.96(0.64)	0.85	0.95(0.67)	0.87	0.97(0.70)	0.96	0.97(0.81)
1	0.82	0.95(0.63)	0.85	0.96(0.67)	0.88	0.96(0.70)	0.89	0.97(0.73)	0.97	0.97(0.83)
1.3	0.86	0.95(0.68)	0.89	0.95(0.72)	0.91	0.95(0.75)	0.92	0.96(0.77)	0.98	0.97(0.85)
1.6	0.90	0.94(0.73)	0.92	0.95(0.77)	0.93	0.95(0.79)	0.94	0.96(0.81)	0.98	0.96(0.87)
1.9	0.93	0.95(0.77)	0.94	0.94(0.80)	0.95	0.95(0.82)	0.96	0.96(0.84)	0.99	0.97(0.88)
2.2	0.94	0.94(0.81)	0.96	0.95(0.82)	0.96	0.96(0.85)	0.97	0.96(0.86)	0.99	0.97(0.90)
2.5	0.96	0.95(0.83)	0.97	0.94(0.85)	0.97	0.95(0.87)	0.98	0.96(0.88)	0.99	0.96(0.92)
3.0	0.98	0.95(0.87)	0.98	0.95(0.88)	0.98	0.95(0.90)	0.99	0.95(0.91)	0.99	0.96(0.93)
$\mu_2 = 0, \theta_2 = 1; n_1 = 12, n_2 = 10$										
0.2	0.49	0.96(0.33)	0.69	0.95(0.50)	0.81	0.95(0.62)	0.89	0.95(0.71)	0.98	0.96(0.89)
0.4	0.57	0.96(0.39)	0.74	0.95(0.54)	0.84	0.95(0.65)	0.90	0.95(0.73)	0.99	0.95(0.90)
0.6	0.62	0.95(0.43)	0.77	0.95(0.57)	0.86	0.94(0.68)	0.92	0.94(0.75)	0.99	0.95(0.91)
0.8	0.66	0.95(0.48)	0.80	0.95(0.61)	0.88	0.95(0.70)	0.92	0.96(0.77)	0.99	0.95(0.91)
1.3	0.74	0.96(0.55)	0.84	0.96(0.66)	0.90	0.95(0.74)	0.94	0.96(0.80)	0.99	0.94(0.92)
1.6	0.77	0.96(0.58)	0.86	0.96(0.68)	0.91	0.95(0.76)	0.95	0.94(0.82)	0.99	0.95(0.93)
1.9	0.79	0.97(0.60)	0.87	0.96(0.70)	0.92	0.96(0.77)	0.95	0.96(0.82)	0.99	0.95(0.93)
2.2	0.81	0.97(0.63)	0.89	0.96(0.72)	0.93	0.97(0.79)	0.96	0.96(0.84)	0.99	0.97(0.94)
2.5	0.83	0.97(0.64)	0.89	0.96(0.73)	0.94	0.96(0.80)	0.96	0.96(0.85)	0.99	0.95(0.94)
3.0	0.85	0.97(0.67)	0.91	0.97(0.75)	0.94	0.95(0.81)	0.97	0.97(0.86)	0.99	0.94(0.95)
3.5	0.87	0.97(0.69)	0.92	0.97(0.76)	0.95	0.97(0.82)	0.97	0.96(0.87)	0.99	0.96(0.95)

(in Table 2) and those for $n_1 = n_2 = 50$ (in Table 1) suggests that the coverage probabilities approach nominal level as sample sizes increase. Thus, we see that the generalized inference is in general conservative, and its accuracy increases as the sample sizes increase.

3. Comparison between the estimates of the expectation of the lower limits and the values of R indicates that the lower limits are expected to be fairly close to R even though the generalized estimation procedures is slightly conservative. Furthermore, for fixed confidence level, the lower limits tend to increase as the sample sizes increase. For example, when $n_1 = 10, n_2 = 12$ and $\theta_1 = 1.5, \mu_1 = 1.0$, $R = 0.85$ and the lower limit is 0.67 (see Table 2); at the same parameter configuration, the lower limit is 0.79 when $n_1 = n_2 = 50$ and is 0.81 when $n_1 = n_2 = 100$ (see Table 1). Thus, the lower confidence

Table 3 Simulated data

X	4.21	4.88	5.17	5.64	6.31	7.42	7.89	8.14	8.27	9.92
	10.45	10.59	11.37	12.98	13.94	14.18	14.19	14.94	18.83	20.91
Y	1.07	1.09	1.16	1.17	1.65	1.98	2.12	2.13	2.54	3.18
	3.19	3.30	3.33	3.40	3.62	4.29	5.80	5.95	6.39	6.74

limit is expected to increase with increasing sample sizes, which is a desirable property.

4. The size properties of the generalized test can be understood from the above coverage properties. In particular, the sizes of the test should be close to or less than the nominal level, and they are expected to be close to the nominal level for large samples.

7 An illustrative example

To illustrate the generalized inferential procedures in section 5, we simulated data on $X \sim$ exponential(4, 5) and on $Y \sim$ exponential(1, 2). The value of the reliability parameter R is 0.936. The ordered data are given in Table 3.

The MLEs are computed as

$$\widehat{\mu}_{10} = 4.21, \quad \widehat{\theta}_{10} = 6.298, \quad \widehat{\mu}_{20} = 1.07, \quad \widehat{\theta}_{20} = 2.138 \quad \text{and} \quad \widehat{R} = 0.942.$$

Using Algorithm 1 with $m = 100,000$, we computed the p-value for testing

$$H_0 : R \leq 0.83 \text{ vs. } H_a : R > 0.83 \tag{12}$$

as 0.027, which suggests that the data provide evidence against H_0. We also computed the 95% lower confidence limit for R as 0.849. Thus, we see that the conclusions based on the p-value and the 95% lower limit are the same.

The asymptotic approach in section 3 produced 95% lower limit as 0.892, and p-value for testing the hypotheses in (12) as 0.0001.

8 Some concluding remarks

In this article, we developed inferential procedures using the novel concept of generalized p-value and generalized confidence limits. Even though the generalized variable approach can be used to obtain solutions to complex problems such as the present one, its statistical properties should be investigated numerically. Based on Monte Carlo studies we conclude that the existing asymptotic method is not accurate even for very large samples, and the generalized inferential procedures are conservative for some parameter configurations when the sample sizes are moderate. The accuracy of the generalized variable method increases as the sample sizes increase. Finally, we note that conservative procedures are safer to use in practical applications than the liberal ones. Therefore, our procedures can be used to assess the reliability in stress–strength model involving two-parameter exponential distributions, or, as noted in the introduction, the related Pareto and power distributions.

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