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Inferences on the difference and ratio of the means of two inverse Gaussian distributions

K. Krishnamoorthy[a,*], Lili Tian[b]

[a] Department of Mathematics, University of Louisiana at Lafayette, Lafayette, LA 70504, USA
[b] Department of Biostatistics, University at Buffalo, Buffalo, NY 14214, USA

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Abstract

Methods for interval estimation and hypothesis testing about the ratio of two independent inverse Gaussian (IG) means based on the concept of generalized variable approach are proposed. As assessed by simulation, the coverage probabilities of the proposed approach are found to be very close to the nominal level even for small samples. The proposed new approaches are conceptually simple and are easy to use. Similar procedures are developed for constructing confidence intervals and hypothesis testing about the difference between two independent IG means. Monte Carlo comparison studies show that the results based on the generalized variable approach are as good as those based on the modified likelihood ratio test. The methods are illustrated using two examples.

MSC: 03C98; 62P20

Keywords: Generalized p-values; Generalized confidence intervals; Goodness-of-fit test; Power

1. Introduction

It is well known that for non-negative right skewed data, the inverse Gaussian (IG) family offers a convenient modeling alternative to the normal family, especially because of the intriguing similarities between the inference methods for these two families. The density function of the two-parameter IG distribution, IG(μ, λ), is defined as

$$f(x, \mu, \lambda) = \left(\frac{\lambda}{2\pi x^3}\right)^{1/2} \exp\left\{-\frac{\lambda(x-\mu)^2}{2\mu^2 x}\right\}, \quad x > 0, \quad \mu, \lambda > 0, \tag{1.1}$$

where μ is the mean parameter and λ is the scale parameter. The shape parameter for the IG(μ, λ) is defined as λ/μ. The IG distribution can represent a highly skewed to nearly a normal distribution, and is useful to model lifetime distributions and wind energy distributions. Examples from many diverse fields such as cardiology, hydrology, demography and finance are given in Chapter 10 of Chhikara and Folks (1989). See this book and the articles by Seshadri (1993, 1999) for more details about the IG distribution and its applications.

* Corresponding author. Tel.: +1 337 482 5283; fax: +1 337 482 5346.
E-mail address: krishna@louisiana.edu (K. Krishnamoorthy).

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Consider two IG distributions, $\mathrm{IG}(\mu_1, \lambda_1)$ and $\mathrm{IG}(\mu_2, \lambda_2)$. Chhikara and Folks (1989) proposed an exact method of finding a confidence interval (CI) for the ratio μ_1/μ_2 when the shape parameters of these two populations are the same, that is, $\lambda_1/\mu_1 = \lambda_2/\mu_2$. However, under most circumstances, it is not practical to expect two IG populations to have the same shape parameter. Therefore, it is desirable to develop an interval estimation method when there is no restriction on the parameter space.

Tian and Wilding (2005) developed a likelihood based inferential procedure to find a CI for the ratio of two IG means. This modified likelihood ratio approach is not only complex but also computationally involved. Specifically, to develop the modified directed likelihood ratio test (MLRT), we note first that the constrained maximum likelihood estimator of the nuisance parameters at a given value of the parameter of interest (either ratio or the difference of the means) have no closed form; a system of equations has to be solved numerically to find the MLEs. Secondly, to compute the confidence limits of the parameter of interest, yet another set of complex equations for the directed LRT must be solved numerically. Furthermore, it should be noted that the MLRT based confidence limits are not available for the difference between two IG means.

The purpose of this paper is to develop a simple approach which can be used to obtain CIs for the ratio of two IG means or for the difference between two IG means. Toward this, we develop methods based on the concept of generalized variable. The generalized p-value was introduced by Tsui and Weerahandi (1989) and generalized CI by Weerahandi (1993). The concepts of generalized CI and generalized p-value have been widely applied to a variety of practical settings where standard methods are failed to produce satisfactory results. For example, see Weerahandi (1991, 1995b), Weerahandi and Johnson (1992), Zhou and Mathew (1994),Weerahandi and Berger (1999), Krishnamoorthy and Mathew (2003), Krishnamoorthy and Lu (2003), McNally et al. (2003) and Iyer et al. (2004). For a recipe of constructing generalized pivotal quantities (GPQs), see Iyer and Patterson (2002).

This article is organized as follows. In Section 2, the concepts of the generalized p-values and generalized CIs are outlined. In Section 3, the GPQs for the ratio of two IG means and for the difference between two IG means are developed. Interval estimation and hypothesis testing procedures about the ratio of means and about the difference between two means are developed based on the GPQs. In Section 4, simulation studies are carried out to evaluate the coverage probabilities of the generalized CIs and the likelihood based CIs due to Tian and Wilding (2005). Simulation studies indicate that the coverage probabilities of the generalized CIs are very close to the nominal confidence level. In Section 5, we illustrate the new procedures using two examples.

2. Generalized CI and generalized p-values

Let $\boldsymbol{X} = (X_1, X_2, \ldots, X_n)$ be a random sample from a distribution which depends on the parameters $(\theta, \boldsymbol{v})$, where θ is the parameter of interest and $\boldsymbol{v}$ is a vector of nuisance parameters. A GPQ $R(\boldsymbol{X}; \boldsymbol{x}, \theta, \boldsymbol{v})$, where $\boldsymbol{x}$ is an observed value of $\boldsymbol{X}$, for interval estimation defined in Weerahandi (1995a) has the following two properties:

1. For fixed $\boldsymbol{x}$, $R(\boldsymbol{X}; \boldsymbol{x}, \theta, \boldsymbol{v})$ has a distribution free of unknown parameters.
2. $R(\boldsymbol{x}; \boldsymbol{x}, \theta, \boldsymbol{v}) = \theta$.

Let R_α be the 100αth percentile of R. Then R_α is a $1-\alpha$ lower limit for θ and $(R_{\alpha/2}, R_{1-\alpha/2})$ is a $1-\alpha$ two-sided CI for θ.

Consider testing $\mathrm{H}_0 : \theta \leqslant \theta_0$ vs. $\mathrm{H}_1 : \theta > \theta_0$, where θ_0 is a specified quantity. A generalized test variable of the form $T(\boldsymbol{X}; \boldsymbol{x}, \theta, \boldsymbol{v})$, where $\boldsymbol{x}$ is an observed value of $\boldsymbol{X}$, is chosen to satisfy the following three conditions:

1. For fixed $\boldsymbol{x}$, the distribution of $T(\boldsymbol{X}; \boldsymbol{x}, \theta, \boldsymbol{v})$ is free of the vector of nuisance parameters $\boldsymbol{v}$.
2. The value of $T(\boldsymbol{X}; \boldsymbol{x}, \theta, \boldsymbol{v})$ at $\boldsymbol{X} = \boldsymbol{x}$ is free of any unknown parameters.
3. For fixed $\boldsymbol{x}$ and $\boldsymbol{v}$, and for all t, $P[T(\boldsymbol{X}; \boldsymbol{x}, \theta, \boldsymbol{v}) > t]$ is either an increasing or a decreasing function of θ.

A generalized extreme region is defined as $G = \{\boldsymbol{X} : T(\boldsymbol{X}; \boldsymbol{x}, \theta, \boldsymbol{v}) \geqslant T(\boldsymbol{x}; \boldsymbol{x}, \theta, \boldsymbol{v})\}$ (or $\{\boldsymbol{X} : T(\boldsymbol{X}; \boldsymbol{x}, \theta, \boldsymbol{v}) \leqslant T(\boldsymbol{x}; \boldsymbol{x}, \theta, \boldsymbol{v})\}$) if $T(\boldsymbol{X}; \boldsymbol{x}, \theta, \boldsymbol{v})$ is stochastically increasing (or decreasing) in θ. The generalized p-value is defined as $\sup_{\mathrm{H}_0} P(G|\mathrm{H}_0)$, where G is the extreme region defined above.

For further details on the concepts of generalized p-values and generalized CIs, we refer readers to the book by Weerahandi (1995a).

3. The generalized variable approach

We shall first develop a GPQ for the mean of a IG distribution. Even though, one-sample case is not the primary interest of this article, it is considered just to demonstrate that the generalized variable approach can produce exact results. Furthermore, details of the one-sample case is used to find the GPQ for the two-sample case.

3.1. One-sample inference

Let $X_1, X_2, \ldots, X_n$ be a sample from an IG(μ, λ) distribution with λ unknown. Let

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i \quad \text{and} \quad V = \frac{1}{n}\sum_{i=1}^{n}(1/X_i - 1/\bar{X})$$

and let $\bar{x}$ and v denote the observed values of $\bar{X}$ and V, respectively.

The UMP-unbiased test for

$$\mathrm{H}_0 : \mu = \mu_0 \quad \text{vs.} \quad \mathrm{H}_a : \mu \neq \mu_0 \tag{3.1}$$

rejects null hypothesis whenever $|w_0| > t_{n-1;1-\alpha/2}$, where w_0 is an observed value of W_0 defined by

$$W_0 = \frac{\sqrt{n-1}(\bar{X} - \mu_0)}{\mu_0\sqrt{\bar{X}V}} \tag{3.2}$$

and $t_{n-1;1-\alpha/2}$ is the $100(1-\alpha/2)$ percentage point of Student's t distribution with $n-1$ degrees of freedom. The distribution of W_0 is unimodal but asymmetric; however, $|W_0|$ is distributed as $|t_{n-1}|$, or equivalently, W_0^2 follows an F distribution with 1 and $n-1$ degrees of freedoms.

From the optimum tests presented above, one can obtain uniformly most accurate (UMA) or UMA-unbiased CIs for μ by inverting their acceptance regions. Let $(\bar{x}, v)$ be an observed value of $(\bar{X}, V)$ and $w_0 = \frac{\sqrt{n-1}(\bar{X}-\mu_0)}{\mu_0\sqrt{\bar{X}V}}$ be the corresponding observed value of W_0. A UMA-unbiased CI for μ can be obtained by solving the inequality $|w_0| \leqslant t_{n-1;1-\alpha/2}$ for μ_0, and is given by

$$\left(\frac{\bar{x}}{1 + t_{n-1;1-\alpha/2}\sqrt{\frac{v\bar{x}}{n-1}}}, \frac{\bar{x}}{\max\{0, 1 - t_{n-1;1-\alpha/2}\sqrt{\frac{v\bar{x}}{n-1}}\}}\right). \tag{3.3}$$

It should be noted that exact one-sided tests are available but they are not so simple as the two-sided test. Furthermore, one-sided CIs do not have closed form. For more details on one-sample inference, see Chhikara and Folks (1989, Chapter 6).

3.2. A GPQ for the mean of an IG distribution

The GPQ that we consider for interval estimation or two-sided hypothesis testing about μ is given by

$$\begin{aligned} G^* &= \frac{\bar{x}}{\max\left\{0, 1 + \frac{\sqrt{(n-1)}(\bar{X}/\mu-1)}{V\bar{X}}\sqrt{\frac{v\bar{x}}{n-1}}\right\}} \\ &= \frac{\bar{x}}{\max\left\{0, 1 + W\sqrt{\frac{v\bar{x}}{n-1}}\right\}}, \end{aligned} \tag{3.4}$$

where $W = \frac{\sqrt{(n-1)}(\bar{X}/\mu-1)}{\sqrt{V\bar{X}}}$. It can be easily verified that the value of G at $(\bar{X}, V) = (\bar{x}, v)$ is μ. As mentioned earlier, $|W|$ is distributed as $|t_{n-1}|$ but the distribution of W is asymmetric and depends on λ/μ. However, using the moment matching method, we can argue that W is approximately distributed as t_{n-1} if n is large and/or λ is large compared to μ. This approximation is noted in Chhikara and Folks (1989, p. 81), and can be established as follows. The second moments of these variables are the same because W^2 is distributed as t_{n-1}^2. The first moment of t_{n-1} is zero, and the first moment of W is also close to zero provided n is large and/or λ/μ is large (see Appendix). On the basis of this approximation, we conclude that, for a given $\bar{x}$ and v, $G^* \sim G$ approximately, where

$$G = \frac{\bar{x}}{\max\left\{0,\, 1 + t_{n-1}\sqrt{\frac{v\bar{x}}{n-1}}\right\}}. \tag{3.5}$$

The generalized CI for μ is given by $(G_{\alpha/2}, G_{1-\alpha/2})$, where G_p denotes the pth quantile of G in (3.5), which is the same as the exact CI in (3.3).

A $1-\alpha$ generalized CI is usually defined as the interval formed by the lower and upper $\alpha/2$ quantiles of a GPQ. However, note that any interval (L, U) that satisfies $P(L \leqslant GPQ \leqslant U) = 1-\alpha$ is a generalized CI. We shall now show that $(G_{\alpha/2}, G_{1-\alpha/2})$ is also a generalized CI based on the GPQ in (3.4). As $|W| \sim |t_{n-1}|$, for any given $c > 0$, we have

$$P(-c \leqslant t_{n-1} \leqslant c) = P(-c \leqslant W \leqslant c) \tag{3.6}$$

and this probability does not depend on any unknown parameters. After some algebraic manipulations, it can be easily checked that, for any given $(n, \bar{x}, v)$ and $c > 0$,

$$\begin{aligned} &P(-c \leqslant W \leqslant c) \\ &\quad = P\left(\frac{\bar{x}}{1 + c\sqrt{\frac{v\bar{x}}{n-1}}} \leqslant \frac{\bar{x}}{\max\left\{0,\, 1 + W\sqrt{\frac{v\bar{x}}{n-1}}\right\}} \leqslant \frac{\bar{x}}{\max\left\{0,\, 1 - c\sqrt{\frac{v\bar{x}}{n-1}}\right\}}\right) \\ &\quad = P\left(\frac{\bar{x}}{1 + c\sqrt{\frac{v\bar{x}}{n-1}}} \leqslant \frac{\bar{x}}{\max\left\{0,\, 1 + t_{n-1}\sqrt{\frac{v\bar{x}}{n-1}}\right\}} \leqslant \frac{\bar{x}}{\max\left\{0,\, 1 - c\sqrt{\frac{v\bar{x}}{n-1}}\right\}}\right). \end{aligned} \tag{3.7}$$

Note that the second equation follows from (3.6). If we take c to be $t_{n-1;1-\alpha/2}$, then the first equation in (3.7) implies that the probability that the GPQ in (3.4) lies in the exact CI (3.3) is $1-\alpha$, and this probability (with respect to the distribution of W) does not depend on any parameters. As the GPQ G^* lies in the exact interval with probability $1-\alpha$, the exact CI can be regarded as a generalized CI based on G^* or G.

Finally, we note that the interval $(G^*_{\alpha/2}, G^*_{1-\alpha/2})$, where G^*_p is the pth quantile of G^* in (3.4), is not necessarily equal to the exact CI, and also not necessarily a $1-\alpha$ generalized CI. For a given α, the endpoints of this interval depend on the unknown parameters, and so it is not useful to make inferences on μ. On the other hand, the GPQ G in (3.5) can be used to find an exact CI for μ or to test a two-sided hypothesis on μ.

3.3. Two-sample inference based on generalized variable method

Let $X_{i1}, \ldots, X_{in_i}$ be a random sample from $\mathrm{IG}(\mu_i, \lambda_i)$, $i = 1, 2$. Define

$$\bar{X}_i = \frac{1}{n_i}\sum_{j=1}^{n_i} X_{ij} \quad \text{and} \quad V_i = \frac{\sum_{j=1}^{n_i}(X_{ij}^{-1} - \bar{X}_i^{-1})}{n_i}, \quad i = 1, 2. \tag{3.8}$$

It is well known that $\bar{X}_1$, $\bar{X}_2$, V_1 and V_2 are mutually independent with

$$\bar{X}_i \sim \mathrm{IG}(\mu_i, n_i\lambda_i) \quad \text{and} \quad n_i\lambda_i V_i \sim \chi^2_{n_i-1}, \; i = 1, 2.$$

The approximate GPQ for μ_i based on (3.5) is given by

$$G_i = \frac{\bar{x}_i}{\max\left\{0,\, 1 + t_{n_i-1}\sqrt{\frac{\bar{x}_i v_i}{n_i-1}}\right\}}, \quad i = 1, 2, \tag{3.9}$$

where $\bar{x}_i$ and v_i denote the observed values of $\bar{X}_i$ and V_i, respectively.

The GPQ for μ_1/μ_2 is given by

$$R = G_1/G_2. \tag{3.10}$$

For testing $H_0 : \mu_1/\mu_2 = 1$ vs. $H_a : \mu_1/\mu_2 \neq 1$, the generalized p-value is given by $2\,\min\{P(R<1|H_0),\, P(R>1)\}$.

Similarly, the GPQ for $\mu_1 - \mu_2$ is given by

$$D = G_1 - G_2, \tag{3.11}$$

and the generalized p-value for testing $H_0 : \mu_1 = \mu_2$ vs. $H_a : \mu_1 \neq \mu_2$ is given by $2\,\min\{P(D<0|H_0),\, P(D>0|H_0)\}$.

For given two independent IG samples, the appropriate percentiles of R form a $1-\alpha$ confidence limits for μ_1/μ_2. Similarly, the percentiles of D can be used to construct CI for $\mu_1 - \mu_2$.

3.4. Computing algorithms

For given two independent samples from IG populations, let the first sample contain n_1 observations with statistics $\bar{x}_1$ and v_1, and the second sample contain n_2 observations with statistics $\bar{x}_2$ and v_2. The generalized CIs for μ_1/μ_2 and the generalized p-values for testing can be computed using the following steps:

1. Generate $T_1 \sim t_{n_1-1}$ and $T_2 \sim t_{n_2-1}$. Compute G_1 and G_2 according to (3.9). Calculate $R = G_1/G_2$.
2. Repeat Step 1 a total of m times and obtain m values of R.

Let R_p denote the 100pth percentile of R's in Step 2. Then, $[R(\alpha/2),\, R(1-\alpha/2)]$ is a Monte Carlo estimate of $1-\alpha$ CI for μ_1/μ_2. The generalized p-value for testing $\mu_1/\mu_2 = r$ vs. $\mu_1/\mu_2 \neq r$ is

$$2\,\min[P(R \geqslant r),\, P(R \leqslant r)]. \tag{3.12}$$

The probability $P(R \geqslant r)$ can be estimated by the proportion of the R's in Step 2 that are greater than or equal to r. Similarly, $P(R \leqslant r)$ can also be estimated.

Remark 3.1. Weerahandi (1993) expressed the generalized p-value for the Behrens–Fisher problem in terms of integrals (double integral), and it can be used to compute the generalized p-values more accurately. However, for the present problem, it is difficult to express the generalized p-value in terms of integrals, as the GPQ involves the term $\max\{0,\, 1 + t_{n_i-1}\sqrt{\frac{\bar{x}_i v_i}{n_i-1}}\}$ which poses some serious problems. In similar cases, Weerahandi (1995a, b) and Krishnamoorthy and Mathew (2004) noted that the results based on numerical integration and Monte Carlo simulation are practically the same. Furthermore, the numerical procedures enable us to compute the generalized p-values more accurately but the tests based on them are not necessarily exact.

4. Simulation studies

This section describes the results of simulation studies evaluating the properties of the proposed CI estimates of the ratio of two IG means in comparison with the likelihood based CIs proposed by Tian and Wilding (2005). In this simulation, we use a variety of scale parameter configurations and three different settings of small sample sizes: (i) $(n_1, n_2) = (5, 5)$, (ii) $(n_1, n_2) = (10, 5)$ and (iii) $(n_1, n_2) = (10, 10)$ in which $\mu_1/\mu_2 = 0.5$ and $\mu_1/\mu_2 = 0.2$. We select these settings with various configurations of scale parameters to investigate the performance of the proposed procedure under different skewnesses of the IG distributions. The coverage probabilities of the CIs based on the likelihood approach are taken from Tian and Wilding (2005). The coverage probabilities of the generalized CIs are estimated as follows: For each parameter setting, 2500 summary statistics $(\bar{x}_1, \bar{x}_2, v_1, v_2)$ are generated. Based on each summary statistic, the generalized CI for the ratio of the means is constructed using Monte Carlo simulation consisting of 5000 runs (steps presented in Section 3.4 with $m = 5000$). Proportion of the generalized CIs containing the ratio of the means is a Monte Carlo estimate of the coverage probability.

From the numerical results in Table 1, we observe that the coverage probabilities of the generalized CIs are very close to the nominal level 0.95 for all the sample size and parameter configurations considered. We also note that the MLR approach appears to be liberal (coverage probability smaller than the nominal level) for very small samples.

Table 1
Coverage probabilities of 95% CIs for the ratio of two IG means

(n_1, n_2)	$(\mu_1, \mu_2, \lambda_1, \lambda_2)$	MLR	GCI	$(\mu_1, \mu_2, \lambda_1, \lambda_2)$	MLR	GCI
(5,5)	(1,2,1,1)	0.91	0.95	(1,5,1,1)	0.93	0.94
	(1,2,2,1)	0.94	0.95	(1,5,2,1)	0.95	0.95
	(1,2,5,1)	0.95	0.95	(1,5,5,1)	0.95	0.95
(10,5)	(1,2,1,1)	0.95	0.95	(1,5,1,1)	0.95	0.95
	(1,2,2,1)	0.95	0.95	(1,5,2,1)	0.95	0.95
	(1,2,5,1)	0.95	0.95	(1,5,5,1)	0.95	0.95
(10,10)	(1,2,1,1)	0.95	0.95	(1,5,1,1)	0.95	0.95
	(1,2,2,1)	0.95	0.95	(1,5,2,1)	0.95	0.95
	(1,2,5,1)	0.95	0.95	(1,5,5,1)	0.95	0.95

Note: MLR=CI based on the modified likelihood ratio test with $\psi = 0.5$ (Tian and Wilding, 2005); GCI=generalized CI.

Table 2
Coverage probabilities of generalized CIs for the ratio of two IG means

(n_1, n_2)	$\left(\frac{\lambda_1}{\mu_1}, \frac{\lambda_2}{\mu_2}\right)$	Confidence level		
		0.90	0.95	0.99
(5,5)	(0.2,0.2)	0.86	0.89	0.94
	(0.2,0.5)	0.89	0.92	0.95
	(0.4,0.5)	0.90	0.94	0.96
	(1, 2)	0.90	0.96	0.99
	(1, 4)	0.91	0.95	0.99
(10,10)	(0.2,0.2)	0.89	0.93	0.96
	(0.2,0.5)	0.90	0.95	0.98
	(0.4,0.5)	0.90	0.95	0.99
	(1, 2)	0.90	0.96	0.99
	(1, 4)	0.91	0.95	0.99
(20,20)	(0.2,0.2)	0.90	0.95	0.99
	(0.2,0.5)	0.90	0.95	0.99
	(0.4,0.5)	0.90	0.95	0.99
	(1, 2)	0.90	0.95	0.99
	(1, 4)	0.90	0.95	0.99
(30,30)	(0.1,0.1)	0.90	0.94	0.99
	(0.1, 1)	0.90	0.95	0.99
	(0.4, 2)	0.90	0.95	0.99
	(0.1, 9)	0.90	0.95	0.99
	(2, 3)	0.90	0.95	0.99

The coverage probabilities of the CIs for $\mu_1 - \mu_2$ for the sample size and parameter configurations given in Table 1 coincided with those of the generalized CIs for the ratio, and so they are not reported in Table 1.

Recall that the GPQs are only approximately distributed as t random variables provided the sample sizes are large and/or λ_i/μ_i's are large. We observe from Table 1 that even when λ/μ is small and n is small the approximation seems to provide accurate CIs. To judge the sample sizes for which the generalized variable procedures will work satisfactorily, we estimated the coverage probabilities as a function of the ratios λ_1/μ_1 and λ_2/μ_2 and they are reported in Table 2. We see from Table 2 that the estimated coverage probabilities are close to the nominal levels even when the ratios are small (as small as 0.2) provided the sample sizes are moderate. We also note that the coverage probabilities are not appreciably different from the nominal levels even when sample sizes are as small as 10. Finally, our simulation studies indicated (not reported here) that the generalized variable procedure seems to be satisfactory even for small samples (as small as 3) provided both ratios λ_1/μ_1 and λ_2/μ_2 are around 1 or larger. If both sample sizes are 15 or larger, then the generalized variable procedure provides accurate results even for very small values of λ_1/μ_1 and λ_2/μ_2.

5. Examples

We shall now illustrate the methods by computing generalized CIs and generalized p-values for two examples. The generalized CIs and the p-values are computed using the algorithm in Section 3.4 with $m = 100,000$.

Example 1. The data represent the shelf life in days of two products, and are reported in Gacula and Kubaba (1975).

Product M:	24 24 26 32 32 33 33 33 35 41 42 43 47 48 48 48
	50 52 54 55 57 57 57 57 61
Product K:	21 23 25 38 43 43 52 56 61 63 67 69 70 75 85 107

Each sample fits an IG distribution very well (see Chhikara and Folks 1989, p. 91). The summary statistics are: $n_1 = 25$, $n_2 = 16$, $\bar{x}_1 = 43.56$, $\bar{x}_2 = 56.13$, $v_1 = 0.0019035$ and $v_2 = 0.0041468$. The maximum likelihood estimates (MLEs) of λ_1 and λ_2 are $v_1^{-1} = 518.05$ and $v_2^{-1} = 248.15$, respectively. Comparison of these MLEs with the sample means indicates that the ratios λ_1/μ_1 and λ_2/μ_2 are likely to be large, and so the generalized variable approach is expected to produce accurate results.

The 95% CI given in Chhikara and Folks (1989, p. 92) is (0.64, 0.91), and the likelihood based CI due to Tian and Wilding (2005) is (0.56,0.99). The 95% generalized CI for the ratio μ_1/μ_2 is (0.56, 1.02). Even though the generalized CI is slightly wider than the likelihood based CI, they are practically the same, and most practitioners interpret the result as barely significant. The 95% generalized CI for $(\mu_1 - \mu_2)$ is $(-33.25, 0.73)$. The generalized p-value for two-sided hypothesis testing about the ratio

$$\mathrm{H}_0 : \frac{\mu_1}{\mu_2} = 1 \quad \text{vs.} \quad \mathrm{H}_a : \frac{\mu_1}{\mu_2} \neq 1,$$

or about the difference between the means are the same, and is 0.064. We used Monte Carlo simulation consisting of 100,000 runs to compute the generalized CIs and generalized p-values.

Example 2. The data sets were obtained from a study of lymphocyte abnormalities in patients in remission from Hodgkin's disease (Shapiro et al., 1986). The data represent the numbers of T_4 cells per mm^3 in the patients' blood. We like to compare the means of T_4 cells in the Hodgkin's disease group and non-Hodgkin's disease group.

Hodgkin's disease:	396	567	1212	171	554	1104	257	435	295	397
	288	1004	431	795	1621	1378	902	958	1283	2415
Non-Hodgkin's disease:	375	375	752	208	151	116	736	192	315	1252
	675	700	440	771	688	426	410	979	377	503

Chhikara and Folks (1989)showed that the data (divided by 1000) fit an IG distribution. After dividing the data by 1000, we computed $\bar{x}_1 = 0.8232$, $\bar{x}_2 = 0.5221$, $v_1 = 0.7105$ and $v_2 = 0.8663$. The generalized CI for the ratio μ_1/μ_2 is (0.37, 1.04) (the likelihood based CI due to Tian and Wilding, 2005 is (0.37, 1.03)), and the generalized CI for $(\mu_1 - \mu_2)$ is $(-0.79, 0.03)$. The generalized p-value for two-sided hypothesis testing about the ratio or about the difference between the means are the same, and is 0.069.

We observe from the above two examples that the generalized variable approach produced CIs for the ratio of the means which are practically equal to those of the likelihood based approach.

Appendix A.

We here show that $E(W)$ is close to zero as n is sufficiently large and/or λ/μ is large. Let $Z = \sqrt{\lambda n}(\bar{X}/\mu - 1)/\sqrt{\bar{X}}$ and $Q = \sqrt{\lambda n V}$. Then

$$W = \frac{\sqrt{\lambda n}(\bar{X}/\mu - 1)/\sqrt{\bar{X}}}{\sqrt{\lambda n V}} = \frac{Z}{\sqrt{Q}}, \tag{A.1}$$

where $Q = n\lambda V$ follows a χ^2_{n-1} distribution independently of $Z = \sqrt{\lambda n}(\bar{X}/\mu - 1)/\sqrt{\bar{X}}$. The pdf of Z (see Chhikara and Folks, 1989, Theorem 2.1) is given by

$$\left(1 - \frac{z}{\sqrt{4\lambda n/\mu + z^2}}\right)\phi(z), \quad -\infty < z < \infty,$$

where ϕ is the standard normal pdf. As the mean of a standard normal random variable is zero, we have

$$E(Z) = -E\left(\frac{Z^2}{\sqrt{4\lambda n/\mu + Z^2}}\right),$$

where Z^2 is a χ^2_1 random variable. Again, using the identity that, for a real valued function f,

$$E\left[\left(\chi^2_m\right)^k f\left(\chi^2_m\right)\right] = \frac{2^k\Gamma(m/2+k)}{\Gamma(m/2)} E\left[f\left(\chi^2_{m+2k}\right)\right],$$

we see that

$$E(Z) = -E\left(\frac{1}{4\lambda n/\mu + \chi^2_3}\right) < -\left(\frac{1}{4\lambda n/\mu + E(\chi^2_3)}\right) = -\left(\frac{1}{4\lambda n/\mu + 3}\right). \tag{A.2}$$

As $Q \sim \chi^2_{n-1}$, we have $E(1/\sqrt{Q}) = 2^{-1/2}\Gamma(n/2-1)/\Gamma(n/2-1/2)$, which approaches zero as $n \to \infty$. Thus, it follows from (A.1) and (A.2) that $E(W) = E(Z)E(1/\sqrt{Q})$ is close to zero when n is large and/or λ/μ is large.

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