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Tolerance intervals for symmetric location-scale families based on uncensored or censored samples

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ABSTRACT

Exact methods for constructing two-sided tolerance intervals (TIs) and tolerance intervals that control percentages in both tails for a location-scale family of distributions are proposed. The proposed methods are illustrated by constructing TIs for a normal, logistic, and Laplace (double exponential) distributions based on type II singly censored samples. Factors for constructing one-sided and two-sided TIs for a logistic distribution are tabulated for the case of uncensored samples. Factors for constructing TIs based on censored samples for all three distributions are also tabulated. The factors for all cases are estimated by Monte Carlo simulation. An adjustment to the tolerance factors based on type II censored samples is proposed so that they can be used to find approximate TIs based on type I censored samples. Coverage studies of the approximate TIs based on type I censored samples indicate that the approximation is satisfactory as long as the proportion of censored observations is no more than 0.70. The methods are illustrated using some practical examples.

1. Introduction

An interval estimate based on a random sample that includes at least a proportion p of the sampled population with confidence level $1-\alpha$ is referred to as a p content—$(1-\alpha)$ coverage tolerance interval (TI) or simply $(p,1-\alpha)$ TI. Another type of TI is constructed so that not only it should include at least a proportion p of the population, but should also meet the requirement that no more than a proportion $(1-p)/2$ of the population is less than the lower endpoint and no more than a proportion $(1-p)/2$ of the population is greater than the upper endpoint. Krishnamoorthy and Mathew (2009, Section 2.3.2) referred to the TI that controls the percentages in both tails as the $(p,1-\alpha)$ "equal-tailed" TI. A $(p,1-\alpha)$ one-sided lower tolerance limit (TL) is constructed so that at least a proportion p of the population falls above the limit while a $(p,1-\alpha)$ one-sided upper TL is constructed so that at least a proportion p of the population falls below the limit. For earlier work, details and numerous applications of TIs, see the book by Guttman (1970), and the recent book by Krishnamoorthy and Mathew (2009).

To define a $(p,1-\alpha)$ TI formally, let $\boldsymbol{X}$ be a sample from a continuous distribution, and let X follow the same distribution independently of $\boldsymbol{X}$. A $(p,1-\alpha)$ TI $(L(\boldsymbol{X}),U(\boldsymbol{X}))$ is constructed so that

$$P_{\boldsymbol{X}}\{P_X(L(\boldsymbol{X})\le X\le U(\boldsymbol{X})|\boldsymbol{X})\ge p\}=1-\alpha. \tag{1}$$

A $(p,1-\alpha)$ upper TL $U_1(\boldsymbol{X})$ is determined so that

$$P_{\boldsymbol{X}}\{P_X(X\le U_1(\boldsymbol{X})|\boldsymbol{X})\ge p\}=P_{\boldsymbol{X}}(q_p\le U_1(\boldsymbol{X}))=1-\alpha, \tag{2}$$

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where q_p is the p quantile of the sampled population. Note that $U_1(\boldsymbol{X})$ is a $1-\alpha$ upper confidence limit for q_p. A $(p,1-\alpha)$ equal-tailed TI $(L_e(\boldsymbol{X}),U_e(\boldsymbol{X}))$ controlling percentages in both tails is determined by

$$P_{\boldsymbol{X}}(L_e(\boldsymbol{X}) \le q_{(1-p)/2} \text{ and } q_{(1+p)/2} \le U_e(\boldsymbol{X})) = 1-\alpha. \tag{3}$$

That is, the interval $[L_e(\boldsymbol{X}),U_e(\boldsymbol{X})]$ is constructed so that it includes the interval $(q_{(1-p)/2},q_{(1+p)/2})$ with confidence $1-\alpha$. Thus, one-sided TLs and equal-tailed TIs are confidence limits or simultaneous confidence limits for appropriate population quantiles; however, it should be noted that the computation of $L(\boldsymbol{X})$ and $U(\boldsymbol{X})$ that satisfy (1) does not reduce to the computation of confidence limits for certain percentiles.

All types of TIs are well documented for the normal distribution. The problem of constructing one-sided TLs is also well addressed in the literature for some continuous distributions such as the Laplace (double exponential; Kappenman, 1977; Shyu and Owen, 1986a, 1986b), Weibull (Thoman et al., 1970) and logistic (Hall, 1975). One-sided TLs for other distributions can be found in the books by Lawless (2003) and Krishnamoorthy and Mathew (2009). One-sided TLs are useful in many applications, and they are also easier to obtain than two-sided TIs. In reliability analysis one-sided lower TLs are used to assess the minimum survival time of an item and thereby to setup guarantee time; also a one-sided TL can be used to find a lower confidence limit for the survival probability. Two-sided TIs based on uncensored samples are available only for the normal and Laplace distributions. Recently, Fernandez (2010) has provided a method of constructing two-sided TIs for an exponential distribution with location parameter zero. A two-sided TI can be used to find a conservative estimate of the proportion of the population that falls between two specified values. For example, engineering components are usually required to meet certain tolerance specifications. In this case, one wants to find the actual proportion of the components that meets the specifications. If a $(p,1-\alpha)$ TI based on a sample of components falls within the specifications, then we can conclude that at least a proportion p of the components meets the specifications with confidence $1-\alpha$. Two-sided TIs are also desired to setup reference intervals for a population. In particular, equal-tailed TIs are used in clinical studies to capture the central $100p\%$ of the population, and is referred to as the reference interval (see, Harris and Boyd, 1995; Trost, 2006). For constructing two-sided TIs in random effects model and their applications, see Liao and Iyer (2004), and Liao et al. (2005).

To compute one-sided TLs, we first note that one-sided TLs are one-sided confidence limits of appropriate population quantiles. So the problem of constructing one-sided TLs simplifies to setting confidence limits for a parametric function, and pivotal quantity-based approach has been commonly used for estimating a quantile of a continuous distribution. However, the problem of computing two-sided TIs cannot be simplified to the problem of estimating some population quantiles, and so the usual approach for computing one-sided TLs is not applicable to compute two-sided TIs. A purpose of this article is to provide a general approach for constructing TIs for a location-scale family of distributions based on a censored or uncensored sample. We shall mainly address the problems of constructing two-sided TIs and equal-tailed TIs based on type II singly left censored samples except for the logistic distribution for which tolerance factors are not available even for the case of uncensored samples. Solutions to right censored samples can be easily obtained from those for the left censored samples (see Remark 2). Although some life tests are designed with type II censoring, type I censoring is much more common in planned experiments. Type I singly left censored samples also arise while assessing pollution levels in a workplace or environment where pollution levels that are below a threshold value (for example, below the detection limit of a sampling device) are not measured, and they are only known to be below the threshold value. However, sampling properties of statistical procedures based on a type I censored sample are not tractable. It has been noted in the literature (e.g., Schmee et al., 1985; Krishnamoorthy et al., 2009) that some inferential procedures based on a type II censored sample can be used as approximation for type I censored samples.

The rest of the article is organized as follows. In the following section, we outline methods of computing factors to find two-sided TIs and equal-tailed TIs for a symmetric location-scale family of distributions. In Section 3, we illustrate the methods for the normal, logistic and Laplace distributions assuming that the samples are type II singly left censored. Factors for computing both types of TIs are obtained using Monte Carlo simulation. Accuracy of the factors based on Monte Carlo simulation is evaluated in Section 4. In Section 5, we suggest an adjustment to the factors based on type II censored samples so that they can be used as approximations for constructing TIs based on type I censored samples. The accuracy of the approximations is also evaluated. The methods are illustrated using practical examples in Section 6. Some concluding remarks are given in Section 7.

2. Tolerance intervals for a location-scale distribution

A family of distributions is referred to as the location-scale family if its probability density function can be expressed in the form

$$f(x|\mu,\sigma) = \frac{1}{\sigma} f\left(\frac{x-\mu}{\sigma}\right), \quad -\infty < x < \infty, \ -\infty < \mu < \infty, \ \sigma > 0, \tag{4}$$

where μ is the location parameter and σ is the scale parameter. Let $\hat{\mu}$ and $\hat{\sigma}$ be equivariant estimators of μ and σ, respectively, based on a type II singly left censored sample in which we observe only the largest $n-r$ observations,

$x_1 < \cdots < x_{n-r}$. Then $(\hat{\mu}-\mu)/\sigma$, $\hat{\sigma}/\sigma$ and $(\hat{\mu}-\mu)/\hat{\sigma}$ are all pivotal quantities (see Lawless, 2003, Theorem E2). That is, their distributions do not depend on any parameters.

Let $h(\mu,\sigma)$ be an equivariant function. That is, $h(b\mu+a,b\sigma)=bh(\mu,\sigma)+a$ for all a and $b>0$. Then

$$\frac{\hat{\mu}-h(\mu,\sigma)}{\hat{\sigma}}=\frac{\hat{\mu}-[\sigma h(0,1)+\mu]}{\hat{\sigma}}=\frac{\hat{\mu}-\mu}{\hat{\sigma}}-h(0,1)\frac{\sigma}{\hat{\sigma}}$$

is a pivotal quantity. A hypothesis test on $h(\mu,\sigma)$ or a confidence interval (CI) for $h(\mu,\sigma)$ can be obtained using the percentiles of the above pivotal quantity. Specifically, we note that

$$\frac{\hat{\mu}-h(\mu,\sigma)}{\hat{\sigma}}\sim\frac{\hat{\mu}^*-h(0,1)}{\hat{\sigma}^*}, \tag{5}$$

where the notation "$\sim$" means "distributed as" and $(\hat{\mu}^*,\hat{\sigma}^*)$ are the equivariant estimators based on a sample from the distribution with $\mu=0$ and $\sigma=1$. Therefore, the percentiles of the above pivotal quantity can be obtained either by using a numerical method or by Monte Carlo simulation. For example, if k_1 and k_2 satisfy $P(k_1\le(\hat{\mu}^*-h(0,1))/\hat{\sigma}^*\le k_2)=1-\alpha$, then $(\hat{\mu}-k_2\hat{\sigma},\hat{\mu}-k_1\hat{\sigma})$ is a $1-\alpha$ CI for $h(\mu,\sigma)$.

As the quantiles of many commonly used location-scale distributions are equivariant, one-sided confidence limits (equivalently, one-sided TLs) based on an uncensored or a censored sample for a location-scale distribution can be easily obtained using the procedure described in the preceding paragraph.

2.1. Factors for computing two-sided tolerance intervals

To construct a $(p,1-\alpha)$ two-sided TI for a symmetric location-scale family, let us consider TIs of the form $\hat{\mu}\pm k\hat{\sigma}$, where the factor k is to be determined so that

$$P_{\hat{\mu},\hat{\sigma}}\{P_X(\hat{\mu}-k\hat{\sigma}\le X\le\hat{\mu}+k\hat{\sigma}\,|\,\hat{\mu},\hat{\sigma})\ge p\}=1-\alpha, \tag{6}$$

where X also follows the same location-scale distribution independently of $(\hat{\mu},\hat{\sigma})$. After standardizing X and other quantities, (6) can be expressed as

$$P_{\hat{\mu}^*,\hat{\sigma}^*}\{P_Z(\hat{\mu}^*-k\hat{\sigma}^*\le Z\le\hat{\mu}^*+k\hat{\sigma}^*\,|\,\hat{\mu}^*,\hat{\sigma}^*)\ge p\}=1-\alpha, \tag{7}$$

where $Z=(X-\mu)/\sigma$ and $(\hat{\mu}^*,\hat{\sigma}^*)=((\hat{\mu}-\mu)/\sigma,\hat{\sigma}/\sigma)$. The distribution of Z and the joint distribution of $(\hat{\mu}^*,\hat{\sigma}^*)$ do not depend on any parameters, and so the tolerance factor k can be estimated by Monte Carlo simulation. However, two nested "do loops," one for the inner probability and another for outer probability in (7), are required to estimate k. The Monte Carlo method, which is similar to the one for the multivariate normal case described in Algorithm 1 of Krishnamoorthy and Mondal (2006), is time consuming and not stable. Instead of using (7), we can further simplify and show that the factor k is the $1-\alpha$ quantile of a random quantity whose distribution does not depend on any unknown parameters.

Let $F_Z(\cdot)$ denote the cumulative distribution function of Z defined in (7). Then, Eq. (7) can be expressed as

$$P_{\hat{\mu}^*,\hat{\sigma}^*}\{F_Z(\hat{\mu}^*+k\hat{\sigma}^*)-F_Z(\hat{\mu}^*-k\hat{\sigma}^*)\ge p\}=1-\alpha. \tag{8}$$

Note that, for a fixed $\hat{\mu}^*$, $F_Z(\hat{\mu}^*+x)-F_Z(\hat{\mu}^*-x)$ is an increasing function of x, and so the inequality $F_Z(\hat{\mu}^*+k\hat{\sigma}^*)-F_Z(\hat{\mu}^*-k\hat{\sigma}^*)\ge p$ holds if and only if $k\hat{\sigma}^*\ge\nu$, where $\nu\equiv\nu(\hat{\mu}^*,p)$ is the solution of the equation

$$F_Z(\hat{\mu}^*+\nu)-F_Z(\hat{\mu}^*-\nu)=p. \tag{9}$$

Using the above equivalent relations, we see that the factor that satisfies (7) is the solution of the equation

$$P_{\hat{\mu}^*,\hat{\sigma}^*}(\nu(\hat{\mu}^*,p)/\hat{\sigma}^*\le k)=1-\alpha. \tag{10}$$

Thus, k is the $100(1-\alpha)$ percentile of $\nu(\hat{\mu}^*,p)/\hat{\sigma}^*$. Explicit expressions for $\nu(\hat{\mu}*,p)$, when the samples are uncensored, are available only for a few distributions, namely, the normal and Laplace distributions. For other distributions, a root finding method such as the Newton–Raphson or a root-bracketing (bisection) method can be used to find ν that satisfies (9). Finally, as the distribution of $\nu(\hat{\mu}^*,p)/\hat{\sigma}^*$ does not depend on any parameters, its percentiles can be estimated by Monte Carlo method as shown in the following algorithm.

Algorithm 1.

1. Generate a sample of size n from a symmetric location-scale distribution with $\mu=0$ and $\sigma=1$.
2. Discard the smallest r observations, and compute the MLEs (or equivariant estimators) of μ and σ based on the largest $n-r$ samples, say, $z_1,\ldots,z_{n-r}$; denote these estimators by $\hat{\mu}^*$ and $\hat{\sigma}^*$.
3. For a given p, and using $\hat{\mu}^*$ computed in step 2, find the root $\nu(\hat{\mu}^*,p)$ of Eq. (9).
4. Set $Q=\nu(\hat{\mu}^*,p)/\hat{\sigma}^*$.
5. Repeat the steps 1–4 for a large number of times, say, N.
6. The $100(1-\alpha)$ percentile of $\{Q_1,\ldots,Q_N\}$ is a Monte Carlo estimate of the tolerance factor k that satisfies (6).

2.2. Factors for constructing equal-tailed tolerance intervals

Recall that a $(p,1-\alpha)$ equal-tailed TI (L,U) is constructed so that it would contain the interval $(\mu-q_{(1+p)/2}\sigma,\mu+q_{(1+p)/2}\sigma)$ with confidence $1-\alpha$. A natural choice for (L,U) is $(\hat{\mu}-k_e\hat{\sigma},\hat{\mu}+k_e\hat{\sigma})$, where k_e is to be determined such that

$$P_{\hat{\mu},\hat{\sigma}}(\hat{\mu}-k_e\hat{\sigma}<\mu-q_{(1+p)/2}\sigma \text{ and } \mu+q_{(1+p)/2}\sigma<\hat{\mu}+k_e\hat{\sigma})=1-\alpha. \tag{11}$$

In terms of $\hat{\mu}^*=(\hat{\mu}-\mu)/\sigma$ and $\hat{\sigma}^*=\hat{\sigma}/\sigma$, the above expression can be written as

$$P_{\hat{\mu}^*,\hat{\sigma}^*}(\hat{\mu}^*<-q_{(1+p)/2}+k_e\hat{\sigma}^* \text{ and } \hat{\mu}^*>q_{(1+p)/2}-k_e\hat{\sigma}^*)=1-\alpha. \tag{12}$$

The inequalities in (12) holds only if $q_{(1+p)/2}-k_e\hat{\sigma}^*<-q_{(1+p)/2}+k_e\hat{\sigma}^*$, or equivalently, $\hat{\sigma}^*>q_{(1+p)/2}/k_e$. Thus, the tolerance factor k_e is the solution of the equation

$$P_{\hat{\mu}^*,\hat{\sigma}^*}\left(q_{(1+p)/2}-k_e\hat{\sigma}^*<\hat{\mu}^*<-q_{(1+p)/2}+k_e\hat{\sigma}^* \text{ and } \hat{\sigma}^*>\frac{q_{(1+p)/2}}{k_e}\right)=1-\alpha. \tag{13}$$

Notice that the above probability distribution does not depend on any unknown parameters, and so it can be evaluated numerically or estimated using Monte Carlo simulation. For the normal case, the factor k_e can be obtained as the solution of an integral equation (see Owen, 1964; Krishnamoorthy and Mathew, 2009, Section 2.3.2). For other distributions, the Monte Carlo method given in the following algorithm can be used to estimate the tolerance factor k_e that satisfies (13).

Algorithm 2. For a given n, p and $1-\alpha$,

1. Generate a sample of size n from the location-scale distribution with $\mu=0$ and $\sigma=1$.
2. Compute the MLEs $\hat{\mu}^*$ and $\hat{\sigma}^*$ based on the largest $n-r$ observations.
3. Repeat steps 1 and 2 a large number of times, say, N, and assign these MLEs in the arrays $\hat{\mu}^*[N]$ and $\hat{\sigma}^*[N]$.
4. For an assumed value of k_e, let $g(k_e;p,\alpha,\hat{\mu}^*[N],\hat{\sigma}^*[N])$ denote the Monte Carlo estimate of the probability in (13).
5. Solve the equation $h(k_e)=g(k_e;p,\alpha,\hat{\mu}^*[N],\hat{\sigma}^*[N])-(1-\alpha)=0$ for k_e using a “root-bracketing” method.

Note that a root-bracketing method requires two initial values, say, k_{el} and k_{eu}, so that $h(k_{el})h(k_{eu})$ is negative. It follows from the definition of equal-tailed TI that k_e must be greater than $q_{(1+p)/2}$, so we can choose k_{el} to be $q_{(1+p)/2}$, and k_{eu} to be some value larger than k_{el}.

3. Tolerance intervals

We shall illustrate the methods of constructing tolerance factors given in the preceding section for the normal, logistic and Laplace distributions assuming that the samples are type II singly left censored. As we assume symmetric models, the factors for the left censored case are also valid for right censored samples. Let $x_1<\cdots<x_{n-r}$ be the uncensored observations in a sample of size n from a location-scale distribution with location parameter μ and the scale parameter σ. Let $x_1^*=x_1$ if the samples are type II censored, and x_0 if the samples are type I censored with censoring value $x_0<x_1$. Let $z_i=(x_i-\mu)/\sigma$, $i=1,\ldots,n-r$, and let $z_1^*=(x_1^*-\mu)/\sigma$.

3.1. Normal distribution

For the normal case, Cohen (1959, 1961) has provided a method of finding the MLEs of μ and σ. A simpler alternative approach is as follows. The log-likelihood function, apart from a constant, is given by

$$\ln L(\mu,\sigma;\boldsymbol{x})=-(n-r)\ln\sigma-\frac{1}{2}\sum_{i=1}^{n-r}z_i^2+r\ln\Phi(z_1^*).$$

The MLEs are the roots of the equations

$$\frac{\partial \ln L}{\partial\mu}=0 \iff \sum_{i=1}^{n-r}z_i-r\frac{\phi(z_1^*)}{\Phi(z_1^*)}=0,$$

$$\frac{\partial \ln L}{\partial\sigma}=0 \iff -(n-r)+\sum_{i=1}^{n-r}z_i^2-rz_1^*\frac{\phi(z_1^*)}{\Phi(z_1^*)}=0, \tag{14}$$

where ϕ and Φ denote the density function and the cumulative distribution function of the standard normal random variable, respectively. From the first equation of (14), we have $r\phi(z_1^*)/\Phi(z_1^*)=\sum_{i=1}^{n-r}z_i$. Using this relation in the second equation of (14), and solving the resulting equation for σ^2, we see that

$$\hat{\sigma}^2(\mu)=s_r^2+(\overline{x}_r-\mu)^2-(x_1^*-\mu)(\overline{x}_r-\mu), \tag{15}$$

where $\bar{x}_r = (1/(n-r))\sum_{i=1}^{n-r} x_i$ and $s_r^2 = (1/(n-r))\sum_{i=1}^{n-r}(x_i-\bar{x}_r)^2$. Substituting $\hat{\sigma}(\mu)$ for σ in the first equation of (14), we get

$$\frac{(n-r)(\bar{x}_r-\mu)}{\hat{\sigma}(\mu)} - r\frac{\phi(\hat{z}_1^*)}{\Phi(\hat{z}_1^*)} = 0, \tag{16}$$

where $\hat{z}_1^* = (x_1^*-\mu)/\hat{\sigma}(\mu)$. Note that, for a given sample, the above equation is a function of μ only. The value of μ that satisfies (16) is the MLE $\hat{\mu}$ of μ, and the corresponding $\hat{\sigma}(\hat{\mu})$ is the MLE of σ. The root of (16) can be found using a root bracketing (bisection) method with the bracketing interval, for example, $(\bar{x}_r-3s_r,\bar{x}_r)$.

Factors for computing normal TIs based on uncensored samples are widely available (e.g., Odeh et al., 1977, the PC calculator *StatCalc* by Krishnamoorthy, 2006; Krishnamoorthy and Mathew, 2009). For censored samples, we computed factors to find two-sided TIs using Algorithm 1, and factors to find equal-tailed TIs using Algorithm 2, and presented them in Table 4.

3.2. Logistic distribution

The cumulative distribution function of a logistic distribution with the location parameter μ and the scale parameter σ is given by

$$F(x;\mu,\sigma) = \left[1+\exp\left\{-\left(\frac{x-\mu}{\sigma}\right)\right\}\right]^{-1}. \tag{17}$$

Let $z_i = (x_i-\mu)/\sigma$, $f(z_i) = \exp(-z_i)/[1+\exp(-z_i)]^2$ and $F(z_i) = [1+\exp(-z_i)]^{-1}$, $i=1,\dots,n-r$. The log-likelihood function, apart from a constant, is given by

$$-\sum_{i=1}^{n-r} z_i + 2\sum_{i=1}^{n-r}\ln F(z_i) - (n-r)\ln(\sigma) + r\ln F(z_1^*). \tag{18}$$

Let $h(z) = \exp(-z)/[1+\exp(-z)]$. The MLEs are the roots of the equations

$$f_1(\mu,\sigma) = (n-r) - 2\sum_{i=1}^{n-r} h(z_i) - rh(z_1^*) = 0,$$

$$f_2(\mu,\sigma) = \sum_{i=1}^{n-r} z_i - 2\sum_{i=1}^{n-r} z_i h(z_i) - (n-r) - rz_1^* h(z_1^*) = 0, \tag{19}$$

which can be obtained as a special case from Harter and Moore (1967). Newton–Raphson iterative method can be used to find the MLEs satisfying the above equations. Antle et al. (1970) have proposed the following equivariant estimators

$$\tilde{\mu} = \frac{1}{n-r}\sum_{i=1}^{n-r} x_i \quad \text{and} \quad \tilde{\sigma} = \frac{1}{\tilde{n}}(x_{n-r}-x_1), \tag{20}$$

where $\tilde{n} = 2\sum_{i=1}^{n-r} 1/i$, as initial values for the Newton–Raphson method. Partial derivatives to implement Newton–Raphson method and computational details are given in the Appendix.

3.2.1. One-sided tolerance limits

The p quantile of a logistic (μ,σ) distribution is given by $\mu+q_p\sigma$, where $q_p=\ln[p/(1-p)]$. A $(p,1-\alpha)$ one-sided tolerance factor can be computed using (5). Let k_{lg} denote the $100(1-\alpha)$ percentile of $(\hat{\mu}^*-q_{1-p})/\hat{\sigma}^*$, where $\hat{\mu}^*$ and $\hat{\sigma}^*$ are the MLEs satisfying (19) with $z_1,\dots,z_n$ being observations from a logistic(0,1) distribution. Then $\hat{\mu}-k_{lg}\hat{\sigma}$ is a $(p,1-\alpha)$ one-sided lower TL for the logistic (μ,σ) distribution. Using the symmetric property of the logistic(0,1) distribution, it can be shown that $\hat{\mu}+k_{1g}\hat{\sigma}$ is a $(p,1-\alpha)$ one-sided upper TL. For the case of complete samples, we computed the one-sided tolerance factors for values of n ranging from 5 to 100 and for all possible pairs $(p,1-\alpha)$ from the set {0.90,0.95,0.99}, and presented them in Table 5.

Remark 1. Hall (1975) has tabulated factors for constructing one-sided TLs for a logistic distribution based on a type II censored sample. Hall's approach is based on the best linear unbiased estimators (BLUEs) which also do not have closed form, and have to be obtained numerically like the MLEs. Harter and Moore's (1967) comparison study indicates that the MLEs and the BLUEs are comparable with respect to mean squared errors, and in some situations the MLEs are better than the BLUEs. Furthermore, it appears that the MLEs are commonly used in practice, and some online calculators (e.g., www.wessa.net/rwasp_fitdistrlogistic.wasp) compute the MLEs for logistic distributions. For these reasons, factors are given in Table 5 to compute one-sided TLs based on the MLEs.

3.2.2. Two-sided tolerance intervals

Factors for computing two-sided TIs can be obtained using the general results for a location-scale family given in Section 2. Specifically, let $\nu(\hat{\mu}^*,p)$ be the root of the equation in (9) with $F(z)=[1+\exp(-z)]^{-1}$. Then the $(p,1-\alpha)$ two-sided tolerance factor is the $100(1-\alpha)$ percentile of $\nu(\hat{\mu}^*,p)/\hat{\sigma}^*$. We estimated the two-sided tolerance factors using Monte Carlo simulation with 100,000 runs as described in Algorithm 1. The factors were estimated for values of n ranging from 5 to 100 and for all possible pairs $(p,1-\alpha)$ from the set {0.90,0.95,0.99}, and presented them in Table 6. Factors for equal-tailed TIs are also estimated using Algorithm 2, and presented in Table 6.

Factors for constructing two-sided TIs and equal-tailed TIs based on type II singly left censored samples are given in Table 7, for some selected values of n and r.

3.3. Laplace distribution

We shall now consider the Laplace distribution with the cumulative distribution function

$$F(x|\mu,\sigma)=\begin{cases}1-\frac{1}{2}\exp\left(-\frac{x-\mu}{\sigma}\right) & \text{for } x\geq\mu,\\ \frac{1}{2}\exp\left(\frac{x-\mu}{\sigma}\right) & \text{for } x<\mu.\end{cases} \tag{21}$$

Let $x_1,\ldots,x_{n-r}$ be the largest order statistics for a sample of size n from a Laplace distribution. The MLEs can be obtained as a special case of a general result in Childs and Balakrishnan (1997), and they are as follows:

$$\hat{\mu}=\begin{cases}x_1^*-\hat{\sigma}\ln\left(\frac{n}{2(n-r)}\right) & \text{if } r\geq\frac{n}{2},\\ x_{(n+1)/2} & \text{if } r\leq\frac{n}{2}-1 \text{ and } n \text{ is odd},\\ \frac{x_{n/2}+x_{(n+1)/2}}{2} & \text{if } r\leq\frac{n}{2}-1 \text{ and } n \text{ is even}\end{cases} \tag{22}$$

and

$$\hat{\sigma}=\begin{cases}\frac{1}{n-r}\left[\sum_{i=r+2}^{n}x_i-(n-r-1)x_1^*\right] & \text{if } r\geq\frac{n}{2},\\ \frac{1}{n-r}\left[\sum_{i=(n+1)/2+1}^{n}x_i-\sum_{i=r+2}^{(n-1)/2}x_i-(r+1)x_1^*\right] & \text{if } r\leq\frac{n}{2}-1 \text{ and } n \text{ is odd},\\ \frac{1}{n-r}\left[\sum_{i=n/2+1}^{n}x_i-\sum_{i=r+2}^{n/2}x_i-(r+1)x_1^*\right] & \text{if } r\leq\frac{n}{2}-1 \text{ and } n \text{ is even},\end{cases} \tag{23}$$

where x_1^* is x_1 if the samples are type II censored, and is the censoring value x_0 if the samples are type I censored.

3.3.1. Tolerance intervals

It follows from (8) that the $(p,1-\alpha)$ tolerance factor k for constructing a Laplace TI is determined by

$$P_{\hat{\mu}^*,\hat{\sigma}^*}\{F_Z(\hat{\mu}^*+k\hat{\sigma}^*)-F_Z(\hat{\mu}^*-k\hat{\sigma}^*)\geq p\}=1-\alpha, \tag{24}$$

where

$$F_Z(z)=\begin{cases}1-\frac{1}{2}e^{-z}, & z>0,\\ \frac{1}{2}e^{z}, & z<0.\end{cases} \tag{25}$$

Let $\nu(\hat{\mu}^*,p)$ be the root of the equation $F_Z(\hat{\mu}^*+r)-F_Z(\hat{\mu}^*-r)=p$. Then, the factor k is the $1-\alpha$ quantile of $\nu(\hat{\mu}^*,p)/\hat{\sigma}^*$. For the case of complete samples, Shyu and Owen (1986b) obtained an explicit expression for the root as

$$\nu(\mu^*,p)=-\ln\left(\frac{2(1-p)}{\exp(-\mu^*)+\exp(\mu^*)}\right)$$

Table 1
Point estimates and 95% CIs of tolerance factors for constructing $(p,1-\alpha)$ two-sided TIs based on a type II censored samples.

$(p,1-\alpha)=$		Normal (0.95, 0.95)		Laplace (0.90, 0.90)		Logistic (0.90, 0.95)	
n	k	pt. est.	CI	pt. est.	CI	pt. est.	CI
10	3	4.55	(4.54, 4.56)	4.77	(4.75, 4.79)	6.93	(6.89, 6.97)
20	3	3.00	(2.99, 3.01)	3.44	(3.43, 3.45)	4.58	(4.56, 4.60)
30	3	2.67	(2.66, 2.68)	3.11	(3.10, 3.12)	4.09	(4.08, 4.11)
20	6	3.23	(3.22, 3.24)	3.63	(3.61, 3.64)	4.96	(4.93, 4.98)
20	10	3.85	(3.84, 3.86)	4.22	(4.20, 4.23)	5.87	(5.82, 5.91)
20	15	7.44	(7.40, 7.47)	7.24	(7.18, 7.27)	11.56	(11.45, 11.66)
30	10	2.91	(2.90, 2.92)	3.30	(3.29, 3.31)	4.45	(4.42, 4.47)
30	20	4.02	(4.01, 4.04)	4.35	(4.34, 4.36)	6.21	(6.16, 6.25)

provided $p>0.5$. The above root is also valid for type II censored samples because of the equivariant property of the MLEs. If factors for $0<p<0.5$ are desired, then they can be obtained using Algorithm 1.

As the MLEs are in closed-form, and no root finding method is involved, factors for computing two-sided TIs and equal-tailed TIs can be estimated using Algorithms 1 and 2, respectively. These factors are not provided in this article, but available in Xie (2011). For the censored case, factors for constructing two-sided TIs are given in Table 8.

Remark 2. If a sample is right censored, then the procedure for a left censored sample can be easily modified using the symmetric property of the distributions. For example, to find the MLEs based on a right censored sample, multiply the

Table 2
Coverage probabilities of $(p,1-\alpha)$ two-sided TIs based on type I censored samples; $P_{x_0}=P(X\le x_0)$.

P_{x_0}	Normal, $\mu=1$, $\sigma=3$					
	n=20		n=30		n=40	
	(0.90, 0.90)	(0.95, 0.95)	(0.90, 0.90)	(0.95, 0.95)	(0.90, 0.90)	(0.95, 0.95)
0.10	0.904	0.954	0.903	0.952	0.901	0.951
0.15	0.901	0.951	0.904	0.948	0.897	0.950
0.20	0.892	0.944	0.896	0.947	0.904	0.953
0.30	0.906	0.950	0.901	0.950	0.912	0.953
0.50	0.916	0.952	0.913	0.955	0.910	0.961
0.70	0.919	0.960	0.920	0.960	0.921	0.960
$\mu=1$, $\sigma=7$						
0.10	0.902	0.951	0.903	0.952	0.903	0.950
0.15	0.903	0.947	0.901	0.951	0.899	0.951
0.20	0.895	0.946	0.896	0.947	0.901	0.953
0.30	0.902	0.950	0.901	0.950	0.910	0.951
0.50	0.917	0.952	0.911	0.955	0.915	0.959
0.70	0.922	0.960	0.920	0.960	0.918	0.960
	Logistic, $\mu=1$, $\sigma=2$					
	n=20		n=30		n=40	
	(0.90, 0.90)	(0.95, 0.95)	(0.90, 0.90)	(0.95, 0.95)	(0.90, 0.90)	(0.95, 0.95)
0.10	0.903	0.953	0.905	0.957	0.901	0.950
0.15	0.904	0.954	0.907	0.948	0.902	0.951
0.20	0.901	0.948	0.897	0.946	0.896	0.946
0.30	0.900	0.948	0.901	0.952	0.900	0.951
0.50	0.900	0.949	0.907	0.954	0.908	0.949
0.70	0.904	0.952	0.911	0.960	0.910	0.955
$\mu=1$, $\sigma=5$						
0.10	0.901	0.961	0.905	0.957	0.901	0.950
0.15	0.902	0.954	0.902	0.949	0.902	0.947
0.20	0.895	0.947	0.897	0.946	0.903	0.953
0.30	0.897	0.949	0.901	0.952	0.900	0.951
0.50	0.901	0.948	0.907	0.954	0.908	0.949
0.70	0.902	0.950	0.911	0.960	0.920	0.958
	Laplace, $\mu=1$, $\sigma=2$					
	n=20		n=30		n=40	
	(0.90, 0.90)	(0.95, 0.95)	(0.90, 0.90)	(0.95, 0.95)	(0.90, 0.90)	(0.95, 0.95)
0.10	0.906	0.960	0.905	0.957	0.901	0.950
0.15	0.903	0.946	0.902	0.945	0.901	0.949
0.20	0.901	0.948	0.897	0.946	0.896	0.946
0.30	0.897	0.947	0.901	0.952	0.900	0.951
0.50	0.900	0.948	0.907	0.954	0.908	0.949
0.70	0.904	0.952	0.911	0.960	0.910	0.955
$\mu=1$, $\sigma=5$						
0.10	0.901	0.961	0.902	0.953	0.901	0.950
0.15	0.895	0.950	0.904	0.949	0.895	0.953
0.20	0.895	0.947	0.896	0.948	0.899	0.947
0.30	0.897	0.949	0.900	0.952	0.903	0.951
0.50	0.901	0.948	0.907	0.953	0.908	0.950
0.70	0.902	0.950	0.913	0.960	0.915	0.958

samples by -1, apply the method for the left censored case to find the MLEs, and then multiply the MLE $\hat{\mu}$ by -1. Note that the MLE $\hat{\sigma}$ based on the negative transformed samples and the one based on the original samples are the same.

4. Accuracy studies

To judge the margin of error of the Monte Carlo estimates of the tolerance factors given in the preceding sections, we shall use a nonparametric approach to construct a CI for a quantile of a continuous distribution (see Dudewicz and van der Meulen, 1984). For a given sample size n and content level p, let $t_1,\ldots,t_N$ be simulated values of $\nu(\hat{\mu}^*,p)/\hat{\sigma}^*$. Let $t_{(1)} < \cdots < t_{(N)}$ be the ordered values. Let $\lfloor x \rfloor$ denote the largest integer less than or equal to x. The 100q percentile of $t_1,\ldots,t_N$, that is, $t_{(\lfloor Nq \rfloor)}$, is a Monte Carlo estimate of the q quantile of $\nu(\hat{\mu}^*,p)/\hat{\sigma}^*$. Define

$$m = \lfloor -z_{(1+\gamma)/2}\sqrt{Nq(1-q)}+Nq+0.5 \rfloor \quad \text{and} \quad s = \lfloor z_{(1+\gamma)/2}\sqrt{Nq(1-q)}+Nq+1+0.5 \rfloor,$$

where z_α is the α quantile of a standard normal distribution. The interval $(t_{(m)},t_{(s)})$ is an approximate CI for the q quantile of $\nu(\hat{\mu}^*,p)/\hat{\sigma}^*$ with confidence at least γ. The width of the CI can be used to assess the precision of a Monte Carlo estimate.

Since we used Monte Carlo simulation with 100,000 runs to estimate $(p,1-\alpha)$ tolerance factors in earlier sections, we have N=100,000 and $q=1-\alpha$. We noted the values of m and s (while we computing the factors in the preceding sections) so that the interval $(t_{(m)},t_{(s)})$ includes the $1-\alpha$ quantile of $\nu(\hat{\mu}^*,p)/\hat{\sigma}^*$ with confidence $\gamma=0.95$. These CIs along with the point estimates of the tolerance factors are reported in Table 1. Examination of the CIs in Table 1 indicates that the width of CIs increases with increasing point estimates, and in most cases the point estimates are expected to be accurate up to two decimals. The maximum relative error among the reported values is $0.21 \times 100/11.56$, which is less than 2% (see logistic, (n,k)=(20,15)).

Table 3
Failure mileages (in 1000) of locomotive controls.

22.5	37.5	46.0	48.5	51.5	53.0	54.5	57.5	66.5	68.0
69.5	76.5	77.0	78.5	80.0	81.5	82.0	83.0	84.0	91.5
93.5	102.5	107.0	108.5	112.5	113.5	116.0	117.0	118.5	119.0
120.0	122.5	123.0	127.5	131.0	132.5	134.0			

Table 4
Factors for constructing two-sided TIs and equal-tailed TIs (in parentheses) for a normal distribution based on a type II censored sample of size n with r censored observations.

n	r	p=0.90			p=0.95			p=0.99		
		$1-\alpha$			$1-\alpha$			$1-\alpha$		
		0.90	0.95	0.99	0.90	0.95	0.99	0.90	0.95	0.99
10	1	2.83(3.15)	3.21(3.58)	4.17(4.70)	3.36(3.66)	3.82(4.16)	4.99(5.47)	4.38(4.66)	4.97(5.29)	6.46(6.99)
	3	3.27(3.67)	3.85(4.33)	5.44(6.03)	3.87(4.24)	4.55(5.00)	6.40(6.99)	5.05(5.35)	5.94(6.31)	8.35(9.00)
	6	5.70(6.44)	7.71(8.74)	14.10(17.0)	6.67(7.36)	8.98(9.99)	16.9(19.0)	8.69(9.26)	11.7(12.4)	21.6(24.0)
15	1	2.43(2.68)	2.66(2.94)	3.21(3.60)	2.88(3.12)	3.16(3.42)	3.81(4.15)	3.78(3.98)	4.15(4.37)	4.97(5.27)
	3	2.56(2.84)	2.85(3.17)	3.52(4.00)	3.04(3.30)	3.38(3.67)	4.22(4.60)	4.00(4.22)	4.45(4.69)	5.49(5.89)
	6	2.92(3.26)	3.35(3.77)	4.46(5.01)	3.47(3.78)	3.97(4.36)	5.25(5.89)	4.55(4.82)	5.22(5.54)	6.92(7.47)
	9	3.87(4.37)	4.78(5.36)	7.38(8.50)	4.57(5.08)	5.63(6.26)	8.57(9.83)	5.96(6.33)	7.24(7.81)	10.9(12.0)
20	1	2.25(2.47)	2.42(2.67)	2.81(3.13)	2.67(2.87)	2.88(3.11)	3.36(3.61)	3.51(3.69)	3.78(3.98)	4.36(4.62)
	3	2.32(2.54)	2.52(2.77)	2.97(3.30)	2.76(2.97)	3.00(3.23)	3.54(3.85)	3.61(3.81)	3.92(4.14)	4.61(4.91)
	6	2.46(2.71)	2.72(3.00)	3.32(3.73)	2.93(3.16)	3.23(3.51)	3.94(4.31)	3.85(4.06)	4.25(4.47)	5.18(5.47)
	10	2.83(3.15)	3.22(3.62)	4.25(4.92)	3.37(3.67)	3.85(4.22)	5.03(5.66)	4.40(4.66)	5.03(5.33)	6.56(6.92)
	15	4.86(5.67)	6.29(7.36)	10.61(12.5)	5.72(6.47)	7.44(8.33)	12.8(14.4)	7.38(7.95)	9.47(10.1)	16.0(17.3)
25	1	2.14(2.34)	2.29(2.50)	2.61(2.85)	2.55(2.73)	2.72(2.92)	3.10(3.35)	3.35(3.52)	3.58(3.76)	4.06(4.30)
	3	2.19(2.38)	2.35(2.57)	2.69(2.98)	2.61(2.79)	2.79(3.00)	3.21(3.45)	3.43(3.58)	3.67(3.85)	4.20(4.39)
	6	2.27(2.48)	2.45(2.70)	2.87(3.21)	2.70(2.89)	2.92(3.14)	3.41(3.69)	3.55(3.73)	3.84(4.04)	4.50(4.73)
	9	2.39(2.62)	2.61(2.88)	3.13(3.50)	2.84(3.05)	3.11(3.37)	3.75(4.09)	3.73(3.91)	4.07(4.31)	4.91(5.23)
	12	2.56(2.84)	2.86(3.20)	3.57(4.06)	3.05(3.31)	3.41(3.71)	4.25(4.73)	4.00(4.24)	4.46(4.72)	5.56(6.00)
	15	2.89(3.26)	3.33(3.76)	4.46(5.04)	3.43(3.76)	3.96(4.34)	5.28(5.89)	4.48(4.78)	5.15(5.48)	6.83(7.39)
30	1	2.08(2.25)	2.20(2.40)	2.47(2.71)	2.48(2.64)	2.63(2.80)	2.94(3.17)	3.25(3.40)	3.44(3.60)	3.85(4.05)
	3	2.11(2.29)	2.24(2.44)	2.53(2.76)	2.51(2.68)	2.67(2.86)	3.01(3.26)	3.29(3.45)	3.50(3.67)	3.95(4.14)
	6	2.16(2.35)	2.31(2.52)	2.66(2.92)	2.58(2.75)	2.76(2.94)	3.15(3.40)	3.38(3.53)	3.61(3.78)	4.14(4.35)
	10	2.26(2.46)	2.45(2.67)	2.85(3.19)	2.68(2.88)	2.91(3.13)	3.40(3.68)	3.53(3.70)	3.82(4.00)	4.46(4.68)
	15	2.45(2.71)	2.71(3.03)	3.33(3.78)	2.93(3.16)	3.24(3.51)	3.96(4.41)	3.84(4.05)	4.24(4.49)	5.17(5.54)
	20	2.94(3.33)	3.40(3.89)	4.61(5.26)	3.49(3.86)	4.02(4.49)	5.42(6.03)	4.54(4.86)	5.24(5.63)	6.98(7.66)

5. Approximate tolerance intervals based on type I censored samples

As noted in the Introduction, the factors for constructing TIs for the case of type II censored samples can also be used to find approximate TIs for the case of type I censored samples. Specifically, if $x_1 < \cdots < x_{n-r}$ are uncensored observations that are below the censoring value (or the detection limit) x_0, then the factor $k_{n,r,p,1-\alpha}$ for the type II censored case can be used to find an approximate TI. Our extensive simulation studies, however, indicated that the resulted TIs are too conservative when the proportion of the censored samples $P_{x_0} = P(X \le x_0)$ is around 0.20 or more. To overcome this conservative problem, we used the factor $k_{n,r,p,1-\alpha}$ for $P_{x_0} < 0.20$ and the factor $k_{n,r-1,p,1-\alpha}$ for $P_{x_0} \ge 0.20$. The coverage probabilities of the TIs with these adjusted factors are estimated for the normal, logistic and Laplace distributions for various sample size and parameter configurations, and only part of the results are presented in Table 2. Overall, our studies indicated the following. The coverage probabilities depend on the parameters only via P_{x_0}. For smaller sample sizes around 15 or below, the coverage probabilities are smaller than the nominal levels when P_{x_0} is small, and they are slightly larger than the nominal levels when P_{x_0} is 0.30 or more. For samples of size around 20 or more, the coverage probabilities are close to the nominal confidence levels except when P_{x_0} is around 0.70 or above (see Table 2). In particular, the above approximation yields conservative TIs if P_{x_0} is around 0.70 or more, regardless of the sample size.

Overall, the approximation in the preceding paragraph yields satisfactory TIs based on type I censored samples as long as the sample size is around 20 or more, and P_{x_0} is not more than 0.70. In applications, P_{x_0} is unknown, and the choice between the factors $k_{n,r,p,1-\alpha}$ and $k_{n,r-1,p,1-\alpha}$ can be decided upon the sample proportion of censored observations.

6. Examples

Example 1 (*Normal TIs*). The data in Table 3 represent failure mileages (in units of 1000 miles) of different locomotive controls in a life test involving 96 locomotive controls. The test was terminated after 135,000 miles, and by then 37 controls had failed. This example is discussed in Schmee and Nelson (1977), and also in Lawless (2003, Section 5.3). These authors noted that a lognormal distribution gives a good fit to the data. In this type of situations, a lower TL is desired to assess the reliability of the controls, and to estimate the lifetime at certain mileages.

Table 5
Factors for constructing $(p,1-\alpha)$ one-sided TLs for a logistic distribution based on an uncensored sample of size n.

n	p =0.90			p=0.95			p =0.99		
				$1-\alpha$					
	0.90	0.95	0.99	0.90	0.95	0.99	0.90	0.95	0.99
5	5.32	6.55	10.22	6.69	8.43	12.55	10.16	12.28	21.28
6	4.83	5.78	9.53	6.21	7.33	11.59	8.94	11.04	16.98
7	4.28	5.23	7.40	5.56	6.88	9.84	8.80	10.84	15.49
8	4.11	4.79	6.67	5.35	6.19	9.02	8.18	9.71	12.98
9	4.03	4.70	6.33	4.99	5.75	7.74	7.71	8.89	11.41
10	3.84	4.31	5.49	5.01	5.60	7.02	7.35	8.15	11.30
11	3.73	4.23	5.30	4.88	5.47	7.04	7.21	7.98	9.73
12	3.63	4.13	5.11	4.60	5.21	7.17	7.18	7.90	10.64
13	3.51	3.91	4.88	4.50	5.13	6.97	6.99	7.86	9.62
14	3.43	3.98	5.23	4.45	4.96	5.87	6.74	7.28	8.92
15	3.39	3.77	4.56	4.47	5.02	5.98	6.77	7.35	8.50
17	3.28	3.65	4.51	4.25	4.57	5.74	6.57	7.17	8.66
20	3.17	3.51	4.19	4.13	4.60	5.40	6.25	6.68	8.00
25	3.05	3.30	3.89	4.02	4.39	5.17	5.95	6.40	7.43
30	2.96	3.19	3.80	3.86	4.07	4.69	5.92	6.36	7.07
35	2.92	3.12	3.52	3.76	4.01	4.53	5.82	6.18	6.67
40	2.84	3.01	3.40	3.70	3.91	4.31	5.65	6.05	7.04
50	2.72	2.85	3.15	3.62	3.80	4.12	5.48	5.76	6.42
60	2.68	2.80	3.07	3.52	3.71	4.08	2.68	2.81	3.14
70	2.65	2.77	3.09	3.47	3.64	3.89	2.63	2.77	3.04
80	2.61	2.74	2.99	3.43	3.56	3.83	2.63	2.75	2.96
90	2.59	2.70	2.92	3.44	3.58	3.91	5.26	5.42	5.84
100	2.56	2.67	2.93	3.40	3.54	3.83	5.20	5.39	5.72
150	2.48	2.55	2.71	3.29	3.38	3.59	5.07	5.22	5.52
300	2.40	2.46	2.60	3.19	3.25	3.36	4.94	5.03	5.21
600	2.32	2.36	2.43	3.12	3.17	3.24	4.85	4.92	5.02
∞	2.19	2.19	2.19	2.94	2.94	2.94	4.59	4.59	4.59

Notice that the sample is type I right censored with censoring mileage of 135,000 or x_0=135 (in 1000 mile unit). Since the lognormal distribution is applicable here, the MLEs based on the log-transformed data are computed as $\hat{\mu}=5.117$ and $\hat{\sigma}=0.705$. To compute a (0.90, 0.90) two-sided TI, we use the factor for the type II censored case with n=96 and $k-1$=58 (in view of coverage studied in Section 5) as 2.06. Thus, the TI is $5.117 \pm 2.06 \times 0.705$=(3.665, 6.569). By taking exponentiation, we get (39.056, 712.657). Thus we are 90% confident that at least 90% of locomotive controls survive 39,056 to 712,657 miles. We also note that the (0.90, 0.95) one-sided TLs reported in Krishnamoorthy and Mathew (2009, p. 334) are 54,820 miles and 646,780 miles.

Example 2 (*Logistic TIs*). Lawless (2003, p. 232) has noted that the locomotive failure data in Table 3 also fit a log-logistic model, and used the model to estimate the reliability at 80,000 miles. We shall use the same data to compute a (0.90, 0.90) TI based on a log-logistic model. The MLEs based on the log-transformed data were computed as $\hat{\mu}=5.083$ and $\hat{\sigma}=0.384$. To compute a (0.90, 0.90) two-sided TI, we computed the factor (with n=96 and $k-1$=58) as 3.74. Thus, the TI is $5.083 \pm 3.74 \times 0.384$=(3.647, 6.519). By taking exponentiation, we get (38.359, 677.900). Thus we are 90% confident that at least 90% of locomotive controls survive 38,359 to 677,900 miles. Note that this TI is shorter than the one based on the normal model given in the preceding paragraph.

Example 3 (*Laplace TIs*). The following data are breaking strengths of 100 yarns reported in Puig and Stephens (2000). These authors showed that a normal distribution does not fit the data, but a Laplace distribution fits the samples well. We shall use these samples to construct a (0.90, 0.95) TI for the breaking strength of yarns.

62	66	78	79	80	84	84	85	85	86	86	87	88	88	89
89	91	91	91	91	92	92	92	92	93	94	94	94	95	95
95	96	96	96	96	96	97	97	97	97	97	97	98	98	98
98	98	98	98	99	99	99	99	99	100	100	100	100	100	101
101	101	101	102	102	102	102	102	102	102	103	103	103	104	104
104	104	104	104	104	105	105	106	107	107	109	110	111	111	111
111	114	115	117	122	132	132	137	137	138					

The MLEs based on all 100 measurements are $\hat{\mu}=99$ and $\hat{\sigma}=8.33$. The (0.90, 0.95) tolerance factor is computed as 2.76. So the TI for breaking strength is (76.01, 121.99).

Table 6
Factors for constructing (p,$1-\alpha$) two-sided TIs and equal-tailed TIs (given parentheses) for a logistic distribution based on a uncensored sample of size n.

n	p =0.90			p =0.95			p =0.99		
				$1-\alpha$					
	0.90	0.95	0.99	0.90	0.95	0.99	0.90	0.95	0.99
5	6.96(7.88)	8.28(9.70)	11.99(15.27)	8.93(9.45)	10.65(11.63)	17.36(18.65)	12.12(13.02)	14.60(16.00)	22.79(24.99)
6	6.19(6.99)	7.29(8.33)	10.89(12.33)	7.92(8.41)	9.96(10.02)	15.88(14.89)	10.95(11.58)	13.19(13.86)	20.06(20.52)
7	5.64(6.41)	6.66(7.51)	8.67(10.50)	7.13(7.71)	8.48(9.00)	11.19(12.51)	10.02(10.67)	12.00(12.49)	17.82(17.66)
8	5.26(6.00)	5.96(6.92)	8.02(9.34)	6.52(7.26)	7.70(8.35)	10.23(11.22)	9.28(10.07)	10.75(11.55)	13.78(15.72)
9	5.21(5.75)	6.01(6.56)	7.75(8.63)	6.26(6.91)	7.19(7.89)	9.46(10.32)	8.84(9.69)	10.16(11.06)	13.55(14.67)
10	4.85(5.54)	5.47(6.26)	6.95(8.12)	6.09(6.69)	6.75(7.57)	9.12(9.81)	8.69(9.27)	9.65(10.48)	12.08(13.69)
11	4.82(5.35)	5.45(6.03)	6.65(7.67)	5.83(6.48)	6.45(7.28)	7.82(9.34)	8.41(9.03)	9.23(10.14)	12.26(13.00)
12	4.62(5.20)	5.07(5.84)	6.00(7.45)	5.63(6.31)	6.21(7.07)	7.92(9.00)	8.18(8.79)	9.36(9.87)	11.80(12.40)
13	4.43(5.08)	4.92(5.68)	6.26(7.09)	5.48(6.17)	6.22(6.89)	7.53(8.67)	7.88(8.60)	8.62(9.62)	10.92(12.20)
14	4.48(4.98)	4.87(5.54)	6.21(7.00)	5.44(6.02)	6.10(6.70)	7.47(8.49)	7.54(8.46)	8.28(9.39)	10.18(11.92)
15	4.29(4.88)	4.78(5.42)	5.78(6.78)	5.32(5.92)	5.90(6.57)	7.07(8.28)	7.58(8.30)	8.38(9.19)	10.04(11.60)
17	4.13(4.72)	4.39(5.20)	5.26(6.53)	5.18(5.74)	5.61(6.31)	6.90(7.95)	7.39(8.05)	7.94(8.87)	9.34(11.22)
20	4.07(4.54)	4.39(4.98)	5.03(6.12)	4.97(5.54)	5.33(6.06)	6.27(7.56)	7.23(7.76)	7.88(8.48)	9.24(10.52)
25	3.88(4.33)	4.13(4.70)	4.74(5.91)	4.94(5.28)	5.30(5.72)	6.02(7.02)	6.83(7.42)	7.39(8.05)	8.40(9.99)
30	3.78(4.18)	4.00(4.51)	4.53(5.52)	4.69(5.10)	4.90(5.52)	5.69(6.84)	6.83(7.21)	7.20(7.79)	8.21(9.57)
35	3.65(4.07)	3.86(4.37)	4.25(5.34)	4.57(4.96)	4.85(5.33)	5.33(6.44)	6.60(7.04)	7.06(7.55)	8.14(9.17)
40	3.67(3.98)	3.85(4.25)	4.16(5.21)	4.56(4.87)	4.82(5.21)	5.25(6.34)	6.48(6.90)	6.89(7.37)	7.50(9.00)
50	3.53(3.85)	3.67(4.09)	4.07(4.96)	4.41(4.72)	4.63(5.02)	5.06(6.17)	6.31(6.71)	6.54(7.13)	7.06(8.73)
60	3.49(3.77)	3.65(3.99)	3.89(4.80)	4.30(4.62)	4.47(4.88)	4.91(5.88)	6.19(6.56)	6.45(6.94)	6.88(8.34)
70	3.44(3.69)	3.57(3.90)	3.84(4.70)	4.22(4.53)	4.40(4.78)	4.77(5.70)	6.18(6.45)	6.41(6.79)	6.91(8.21)
80	3.39(3.64)	3.51(3.83)	3.70(4.58)	4.18(4.47)	4.33(4.69)	4.66(5.61)	6.12(6.37)	6.33(6.69)	6.67(8.18)
90	3.35(3.59)	3.47(3.77)	3.72(4.52)	4.17(4.41)	4.36(4.64)	4.64(5.44)	5.96(6.29)	6.19(6.60)	6.62(7.72)
100	3.33(3.56)	3.45(3.73)	3.68(4.47)	4.15(4.38)	4.28(4.60)	4.47(5.57)	6.01(6.23)	6.16(6.53)	6.54(7.70)
150	3.25(3.36)	3.33(3.47)	3.48(4.14)	4.03(4.15)	4.13(4.35)	4.35(5.12)	5.84(5.97)	5.98(6.33)	6.33(7.49)
300	3.15(3.29)	3.19(3.45)	3.31(4.05)	3.92(4.11)	3.98(4.30)	4.11(4.99)	5.65(5.92)	5.75(5.24)	5.98(7.31)
600	3.09(3.19)	3.12(3.27)	3.20(3.81)	3.83(3.96)	3.88(4.06)	3.97(4.74)	5.55(5.70)	5.60(5.85)	5.73(6.73)
∞	2.94	2.94	2.94	3.64	3.64	3.64	5.29	5.29	5.29

Table 7
Factors for constructing two-sided TIs and equal-tailed TIs (in parentheses) for a logistic distribution based on a type II censored sample of size n with r censored observations.

n	r	p=0.90			p=0.95			p=0.99		
		1−α			1−α			1−α		
		0.90	0.95	0.99	0.90	0.95	0.99	0.90	0.95	0.99
10	1	5.11(5.68)	5.83(6.48)	7.69(8.44)	6.31(6.80)	7.20(7.76)	9.35(9.89)	9.05(9.61)	10.35(10.95)	13.50(14.1)
	3	5.90(6.46)	6.93(7.48)	9.76(10.4)	7.24(7.79)	8.54(9.03)	12.0(12.4)	10.34(10.79)	12.22(12.43)	17.21(17.8)
	6	10.2(10.4)	13.7(13.7)	25.1(25.1)	12.54(12.5)	16.78(16.1)	31.6(31.7)	17.61(16.84)	23.57(21.63)	43.82(38.7)
15	1	4.41(4.89)	4.86(5.40)	5.92(6.57)	5.47(5.93)	6.03(6.53)	7.28(7.85)	7.85(8.35)	8.66(9.08)	10.54(11.0)
	3	4.64(5.27)	5.18(5.78)	6.48(7.14)	5.75(6.40)	6.43(7.07)	8.04(8.56)	8.24(8.95)	9.22(9.82)	11.45(12.0)
	6	5.29(6.31)	6.08(7.02)	8.14(8.79)	6.54(7.56)	7.53(8.32)	10.2(10.5)	9.36(10.45)	10.76(11.57)	14.32(14.2)
	9	7.01(8.20)	8.65(9.46)	13.09(13.7)	8.60(9.69)	10.62(11.1)	16.2(16.4)	12.28(13.09)	15.01(15.10)	22.94(21.9)
20	1	4.10(4.53)	4.45(4.92)	5.25(5.83)	5.08(5.52)	5.50(5.95)	6.45(6.96)	7.29(7.75)	7.90(8.43)	9.29(9.93)
	3	4.20(4.81)	4.58(5.20)	5.43(6.10)	5.22(5.84)	5.70(6.29)	6.77(7.40)	7.50(8.16)	8.17(8.85)	9.68(10.40)
	6	4.47(5.43)	4.96(5.91)	6.06(6.91)	5.55(6.60)	6.15(7.18)	7.52(8.43)	7.96(9.20)	8.79(10.00)	10.75(11.53)
	10	5.13(6.69)	5.87(7.39)	7.71(8.92)	6.38(7.56)	7.29(8.32)	9.59(9.81)	9.14(10.90)	10.5(11.94)	13.69(14.56)
	15	8.91(10.2)	11.56(12.1)	19.50(19.5)	10.95(9.03)	14.1(14.1)	23.5(23.6)	15.25(15.87)	19.5(19.5)	32.92(30.00)
25	1	3.92(4.29)	4.20(4.62)	4.80(5.41)	4.86(5.26)	5.21(5.64)	5.98(6.53)	6.98(7.41)	7.49(7.95)	8.60(9.12)
	3	3.98(4.53)	4.29(4.82)	4.97(5.52)	4.94(5.50)	5.32(5.89)	6.17(6.71)	7.10(7.76)	7.64(8.33)	8.83(9.49)
	6	4.12(4.99)	4.48(5.38)	5.29(6.25)	5.12(6.08)	5.57(6.52)	6.55(7.46)	7.37(8.57)	8.01(9.20)	9.41(10.32)
	9	4.33(5.62)	4.75(6.08)	5.78(7.09)	5.37(6.77)	5.90(7.37)	7.13(8.40)	7.72(9.32)	8.51(10.1)	10.22(11.71)
	12	4.65(6.36)	5.21(6.95)	6.56(8.12)	5.79(7.54)	6.48(8.27)	8.11(9.67)	8.32(10.31)	9.31(11.1)	11.7(13.0)
	15	5.26(7.35)	6.04(8.17)	8.12(9.69)	6.53(8.71)	7.49(9.55)	9.94(11.4)	9.33(11.67)	10.68(12.81)	14.1(15.6)
30	1	3.79(4.14)	4.04(4.43)	4.57(5.04)	4.70(5.09)	5.00(5.45)	5.67(6.19)	6.78(7.16)	7.22(7.65)	8.16(8.74)
	3	3.83(4.36)	4.09(4.65)	4.65(5.27)	4.77(5.31)	5.09(5.65)	5.80(6.33)	6.85(7.46)	7.31(7.94)	8.34(8.82)
	6	3.92(4.75)	4.22(5.09)	4.85(5.78)	4.87(5.73)	5.23(6.16)	6.01(6.98)	7.01(8.04)	7.54(8.63)	8.67(9.65)
	10	4.10(5.32)	4.45(5.78)	5.26(6.65)	5.09(6.45)	5.53(7.01)	6.51(8.03)	7.33(9.02)	7.96(9.77)	9.38(11.1)
	15	4.48(6.34)	4.97(6.92)	6.10(8.09)	5.56(7.56)	6.18(8.24)	7.57(9.50)	7.97(10.4)	8.83(11.32)	10.8(13.2)
	20	5.36(7.84)	6.21(8.59)	8.34(10.4)	6.63(9.22)	7.68(10.2)	10.3(12.1)	9.45(12.4)	10.9(13.53)	14.7(15.9)

Table 8
Factors for constructing two-sided TIs and equal-tailed TIs (in parentheses) for a Laplace distribution based on a type II censored sample of size n with r censored observations.

n	r	p=0.90			p=0.95			p=0.99		
		1−α			1−α			1−α		
		0.90	0.95	0.99	0.90	0.95	0.99	0.90	0.95	0.99
10	1	4.25(4.62)	4.94(5.36)	6.67(7.32)	5.49(5.85)	6.41(6.82)	8.63(9.28)	8.39(8.73)	9.75(10.19)	13.28(13.89)
	3	4.77(5.19)	5.72(6.20)	8.21(9.00)	6.16(6.61)	7.36(7.91)	10.7(11.4)	9.46(9.87)	11.28(11.88)	16.19(17.10)
	6	8.80(9.77)	11.80(13.2)	22.1(26.0)	11.3(12.2)	15.2(16.5)	28.4(32.0)	17.02(17.99)	22.84(24.18)	42.80(46.97)
15	1	3.64(3.93)	4.08(4.40)	5.11(5.58)	4.72(5.01)	5.29(5.61)	6.57(7.00)	7.23(7.51)	8.11(8.42)	10.21(10.50)
	3	3.81(4.11)	4.31(4.66)	5.50(6.00)	4.94(5.24)	5.60(5.91)	7.24(7.59)	7.55(7.83)	8.56(8.88)	10.92(11.38)
	6	4.28(4.63)	5.00(5.37)	6.79(7.37)	5.54(5.89)	6.47(6.87)	8.84(9.38)	8.53(8.82)	9.92(10.25)	13.48(14.00)
	9	5.86(6.47)	7.28(8.00)	11.3(12.4)	7.57(8.17)	9.39(10.1)	14.6(16.0)	11.50(12.06)	14.25(14.93)	22.00(23.00)
20	1	3.35(3.58)	3.69(3.94)	4.42(4.77)	4.35(4.60)	4.78(5.05)	5.77(6.08)	6.67(6.87)	7.34(7.60)	8.82(9.07)
	3	3.44(3.69)	3.80(4.07)	4.62(4.97)	4.46(4.69)	4.93(5.19)	6.00(6.34)	6.85(7.07)	7.58(7.81)	9.29(9.52)
	6	3.63(3.88)	4.05(4.34)	5.11(5.47)	4.69(4.94)	5.27(5.54)	6.63(7.00)	7.22(7.42)	8.09(8.33)	10.19(10.42)
	10	4.22(4.56)	4.88(5.29)	6.49(7.20)	5.47(5.78)	6.30(6.70)	8.44(9.00)	8.37(8.68)	9.67(10.07)	12.91(13.63)
	15	7.24(8.40)	9.35(0.96)	15.70(19.0)	9.22(10.4)	11.9(13.6)	20.1(23.0)	13.79(15.00)	17.73(19.33)	30.13(32.98)
25	1	3.19(3.40)	3.47(3.70)	4.09(4.35)	4.14(4.35)	4.50(4.73)	5.30(5.54)	6.34(6.55)	6.90(7.12)	8.11(8.33)
	3	3.24(3.46)	3.54(3.78)	4.20(4.48)	4.21(4.42)	4.60(4.82)	5.43(5.69)	6.47(6.67)	7.06(7.29)	8.34(8.65)
	6	3.34(3.56)	3.67(3.90)	4.41(4.73)	4.35(4.56)	4.78(5.01)	5.74(6.05)	6.66(6.86)	7.34(7.57)	8.85(9.14)
	9	3.49(3.72)	3.89(4.13)	4.78(5.06)	4.54(4.75)	5.05(5.27)	6.20(6.51)	6.94(7.16)	7.74(7.97)	9.57(9.81)
	12	3.78(4.04)	4.28(4.57)	5.49(5.88)	4.90(5.17)	5.54(5.85)	7.01(7.52)	7.52(7.76)	8.51(8.77)	10.78(11.20)
	15	4.32(4.69)	5.01(5.46)	6.68(7.39)	5.58(5.96)	6.46(6.93)	8.74(9.38)	8.55(8.97)	9.90(10.33)	13.28(14.00)
30	1	3.08(3.27)	3.32(3.52)	3.83(4.04)	4.00(4.17)	4.31(4.50)	4.97(5.26)	6.13(6.30)	6.60(6.79)	7.65(7.87)
	3	3.11(3.30)	3.36(3.57)	3.92(4.16)	4.05(4.23)	4.38(4.57)	5.08(5.32)	6.20(6.39)	6.69(6.90)	7.80(8.03)
	6	3.18(3.37)	3.44(3.66)	4.06(4.30)	4.12(4.31)	4.48(4.69)	5.25(5.50)	6.34(6.52)	6.88(7.08)	8.10(8.31)
	10	3.30(3.49)	3.62(3.83)	4.33(4.65)	4.29(4.49)	4.70(4.93)	5.66(5.89)	6.58(6.77)	7.21(7.46)	8.66(9.00)
	15	3.64(3.86)	4.06(4.32)	5.08(5.44)	4.71(4.96)	5.26(5.53)	6.56(6.98)	7.23(7.45)	8.09(8.36)	10.07(10.45)
	20	4.35(4.82)	5.07(5.63)	6.78(7.83)	5.62(6.07)	6.51(7.10)	8.76(9.82)	8.59(9.03)	9.97(10.54)	13.40(14.25)

For the sake of illustration, suppose that a measuring device cannot measure the strength of a yarn if it is below 90. In this case, only $n-r$=84 yarn strengths are recorded in a sample of size n=100 and the MLEs are $\hat{\mu}=99$ and $\hat{\sigma}=8.45$. The (0.90, 0.95) tolerance factor when n=100 and k=16 is 2.81, and the TI is (75.26, 122.74). Note that the TI based on the censored sample is wider, but still close to the one based on all 100 measurements. The factor for computing (0.90, 0.95) one-sided lower TL, when n=100 and k=16, is 2.01, and the limit is 99–2.01 × 8.45 = 82.02.

7. Concluding remarks

In this article, we have provided methods for constructing two-sided TIs and equal-tailed TIs for symmetric location-scale families of distributions. The methods are exact except for the simulation errors, and these errors also negligible for practical purpose as noted in Table 1. Our simulation studies also indicate that satisfactory TIs can be constructed when the samples are type I censored, as long as the proportion of censored observation is no more than 0.70. We note that our methods are applicable for any setup as long as the pivotal quantities mentioned in Section 2 are valid. For example, if the samples are type II doubly censored, then the pivotal quantities based on the MLEs are valid.

The method for constructing a two-sided TI in Section 2 is also applicable to construct a two-sided TI for an asymmetric location-scale distribution such as the extreme-value distribution and the two-parameter exponential distribution. However, for an asymmetric distribution the TIs of the form $\hat{\mu}\pm k\hat{\sigma}$ are not appropriate, and they could be unnecessarily wide. In fact, for an asymmetric distribution, TIs of the form $(\hat{\mu}+k_1\hat{\sigma},\hat{\mu}+k_2\hat{\sigma})$ are appropriate. At present it is not clear as to the method of constructing TIs for an asymmetric distribution. Finally, we note that in many practical situations, such as environmental assessment and workplace pollution assessment, samples are often multiply censored or include measurements with multiple detection limits. In these situations, it may not be difficult to find one-sided TLs, but finding two-sided TIs could be difficult. We are currently investigating the problem of finding TLs based on samples with multiple detection limits.

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Appendix

Recall that $z_i=(x_i-\mu)/\sigma$, i=1,…,$n-r$, and $h(z_i)=\exp(-z_i)/[1+\exp(-z_i)]$ and the pdf $f(z)=\exp(-z)/[1+\exp(-z)]^2$. The likelihood equations are

$$f_1(\mu,\sigma)=(n-r)-2\sum_{i=1}^{n-r}h(z_i)-rh(z_1^*)=0,$$

$$f_2(\mu,\sigma)=\sum_{i=1}^{n-r}z_i-2\sum_{i=1}^{n-r}z_ih(z_i)-(n-r)-rz_1^*h(z_1^*)=0, \tag{26}$$

where z_1^* is as defined in (18). Note that $dh(z)/dz=-f(z)$, $\partial z/\partial\mu=-1/\sigma$ and $\partial z/\partial\sigma=-z/\sigma$. Let $f_{1\mu}(\mu,\sigma)=\partial f_1/\partial\mu$ and define $f_{1\sigma}(\mu,\sigma)$, $f_{2\mu}(\mu,\sigma)$ and $f_{2\sigma}(\mu,\sigma)$ similarly. In terms of these notations, the partial derivatives can be expressed as follows:

$$f_{1\mu}(\mu,\sigma)=-\frac{1}{\sigma}\left[2\sum_{i=1}^{n-r}f(z_i)+rf(z_1^*)\right],$$

$$f_{1\sigma}(\mu,\sigma)=-\frac{1}{\sigma}\left[2\sum_{i=1}^{n-r}z_if(z_i)+rz_1^*f(z_1^*)\right],$$

$$f_{2\mu}(\mu,\sigma)=f_{1\sigma}(\mu,\sigma),$$

$$f_{2\sigma}(\mu,\sigma)=-\frac{1}{\sigma}\left[\sum_{i=1}^{n-r}z_i-2\sum_{i=1}^{n-r}z_ih(z_i)+2\sum_{i=1}^{n-r}z_i^2f(z_i)-rz_1^*h(z_1^*)+rz_1^{*2}f(z_1^*)\right].$$

Then, the Newton–Raphson iterative relation is given by

$$\begin{pmatrix}\mu\\ \sigma\end{pmatrix}\leftarrow\begin{pmatrix}\mu_0\\ \sigma_0\end{pmatrix}-\begin{pmatrix}f_{1\mu}(\mu_0,\sigma_0) & f_{1\sigma}(\mu_0,\sigma_0)\\ f_{2\mu}(\mu_0,\sigma_0) & f_{2\sigma}(\mu_0,\sigma_0)\end{pmatrix}^{-1}\begin{pmatrix}f_1(\mu_0,\sigma_0)\\ f_2(\mu_0,\sigma_0)\end{pmatrix}, \tag{27}$$

where μ_0 and σ_0 are the initial guess values for the roots. The equivariant estimators in (20) can be used as initial values for the above iterative scheme. We like to note that the above partial derivatives are in different forms from those given in

Harter and Moore (1967). These alternative forms are warranted because we noted some overflow errors with the expressions in Harter and Moore (1967) while no such errors were encountered with the above expressions.

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