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On combining correlated estimators of the common mean of a multivariate normal distribution

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The inferential procedures based on an optimal combination of correlated estimators of the common mean of a multivariate normal distribution are considered. Exact properties of the conditional and unconditional confidence intervals due to Halperin [Halperin, 1961, Almost linearly-optimum combination of unbiased estimates. *Journal of the American Statistical Association*, **56**, 36–43] are numerically evaluated. Our numerical studies show that the conditional confidence interval is slightly shorter than the unconditional confidence interval. A condition under which the conditional approach is advantageous over the best of the t procedures based on individual components is discussed. The methods are illustrated using an example.

Keywords: Concomitant variable; Expected length; Maximum likelihood estimator; Noncentral t distribution; Multiple correlation coefficient; Power

1. Introduction

The problem of combining independent estimators for the common mean of several normal populations is well-known and has been well addressed in the literature. An important result in the common mean problem is due to Graybill and Deal [1] who first showed, for the two-sample case, that the weighted average of the sample means with weights inversely proportional to their variances has smaller variance than either sample mean provided the sample sizes are greater than 10. Since, then many authors improved and extended this result to the case of more than two populations, and developed several methods for hypothesis testing and interval estimation for the common mean. For a good exposition of the work in this area, we refer to Cohen and Sackrowitz [2], Zhou and Mathew [3], Yu *et al.* [4] and Krishnamoorthy and Lu [5], and the references therein. However, the results on combining the correlated estimators in the normal case are very limited. Halperin [6] seems to be the first paper to address this problem. Halperin pointed out that the problem of estimating the common mean of a multivariate normal population arises when several alike neutron transportation problems are considered. Halperin

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derived the maximum likelihood estimator (MLE) and developed two interval estimates for the common mean of a multivariate normal population.

We shall now describe the setup of the problem as given in Halperin [6]. Let $\mathbf{U} \sim N_p(\mathbf{e}\mu, \Sigma)$, where $\mathbf{e}$ denotes the vector of ones. Let $\bar{\mathbf{U}}$ and S_u denote respectively the mean and covariance matrix based a sample of n observations from $N_p(\mathbf{e}\mu, \Sigma)$. The MLE of μ due to Halperin [6] is given by

$$\hat{\mu} = \frac{\mathbf{e}' S_u^{-1} \bar{\mathbf{U}}}{\mathbf{e}' S_u^{-1} \mathbf{e}}. \tag{1}$$

If Σ is known, then the best linear unbiased estimator (BLUE) of μ is given by $\mathbf{e}'\Sigma^{-1}\bar{\mathbf{U}}/(\mathbf{e}'\Sigma^{-1}\mathbf{e})$ and it has variance $(n\mathbf{e}'\Sigma^{-1}\mathbf{e})^{-1}$. If Σ is unknown, replacing Σ by its estimate S_u, we can get the MLE. The variance of the MLE is given by

$$\operatorname{Var}(\hat{\mu}) = \left(1 + \frac{p-1}{n-p-1}\right) \frac{1}{n\mathbf{e}'\Sigma^{-1}\mathbf{e}}, \tag{2}$$

which approaches the variance of the BLUE as $n \to \infty$.

The form of the MLE in equation (1) is not conducive to develop a confidence interval for μ. To derive the distribution of the MLE, Halperin suggested using the following transformation. Let $A = (a_{ij})$ be a $p \times p$ matrix such that $a_{i1} = 1$ for $i = 1, \ldots, p$, $a_{ii} = -1$ for $i = 2, \ldots, p$, and $a_{ij} = 0$ elsewhere. Then $A\mathbf{U} = (y, x_1, \ldots, x_{p-1})' = (y, X')'$ follows a p-variate normal distribution with mean vector $(\mu, 0, \ldots, 0)'$ and covariance matrix

$$A\Sigma A' = \begin{pmatrix} \sigma_{yy} & \sigma'_{Xy} \\ \sigma_{Xy} & \Sigma_{XX} \end{pmatrix}_{p\times p}. \tag{3}$$

Thus, estimation of the common mean μ is equivalent to estimation of the mean of y given that the mean of $X = 0_{p-1}$. Let $(y_1, X_1), \ldots, (y_n, X_n)$ be independent observations on (y, X). Define

$$(\bar{y}, \bar{X}') = \frac{1}{n} \sum_{i=1}^{n} (y_i, X_i') \tag{4}$$

and

$$\mathbf{W}_{p\times p} = \begin{pmatrix} w_{yy} & W'_{Xy} \\ W_{Xy} & W_{XX} \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^n (y_i - \bar{y})^2 & \sum_{i=1}^n (y_i - \bar{y})(X_i - \bar{X})' \\ \sum_{i=1}^n (y_i - \bar{y})(X_i - \bar{X}) & \sum_{i=1}^n (X_i - \bar{X})(X_i - \bar{X})' \end{pmatrix}, \tag{5}$$

so that W_{XX} is a $(p-1) \times (p-1)$ matrix. Let $\mathbf{a} = (a_1, \ldots, a_{p-1})'$ be a vector of real numbers, and $\beta = \Sigma_{XX}^{-1}\sigma_{Xy}$. Consider the class of estimators of the form $\bar{y}(\mathbf{a}) = \bar{y} - \sum_{i=1}^{p-1} a_i \bar{x}_i$. It can be easily shown that $\operatorname{Var}(\bar{y}(\mathbf{a}))$ is minimized when $\mathbf{a} = \beta$. Thus, if β is known, then $\bar{y}(\beta)$ is the BLUE of μ. If β is unknown, then replacing it by $\mathbf{b} = W_{XX}^{-1} W_{Xy}$ we get

$$\hat{\mu} = \bar{y} - \mathbf{b}'\bar{X}. \tag{6}$$

This is an alternative form of the MLE in equation (1). The expression for the variance of the MLE can be written as

$$\operatorname{Var}(\hat{\mu}) = \left(1 + \frac{p-1}{n-p-1}\right) \frac{\sigma_{yy\cdot X}}{n}, \quad \text{for } n > p+1, \tag{7}$$

where $\sigma_{yy\cdot X} = \sigma_{yy}(1 - \rho^2_{y\cdot X})$, and $\rho^2_{y\cdot X} = (\sigma'_{Xy}\Sigma_{XX}^{-1}\sigma_{Xy})/\sigma_{yy}$ is the squared multiple correlation coefficient between y and X. It follows from equation (7) that the variance of the MLE is smaller than that of $\bar{y} = \bar{u}_1$ if and only if

$$\rho^2_{y\cdot X} > \frac{p-1}{n-2}.$$

Recall that the transformation we used is $(u_1, u_1 - u_2, \dots, u_1 - u_p) = (y, x_1, \dots, x_{p-1})$. If we let $(u_2, u_2 - u_1, \dots, u_2 - u_p) = (y, x_1, \dots, x_{p-1})$, then the MLE has smaller variance than $\bar{u}_2$ if and only if $\rho^2_{u_1\cdot(u_1-u_2),\dots,(u_1-u_p)} > (p-1)/(n-2)$. Proceeding this way, we see that the MLE has smaller variance than the $\min\{\mathrm{Var}(\bar{u}_1), \dots, \mathrm{Var}(\bar{u}_p)\}$ if and only if

$$\min\{\rho^2_{u_1\cdot(u_1-u_2),\dots,(u_1-u_p)}, \dots, \rho^2_{u_p\cdot(u_p-u_1),\dots,(u_p-u_{p-1})}\} > \frac{p-1}{n-2}. \tag{8}$$

Krishnamoorthy and Rohatgi [7] showed that $\hat{\mu}$ can be improved using the fact that the mean of X is known to be zero. In particular, they suggested using $W_{X0} = \sum_{i=1}^n X_i X_i'$ to estimate β. This leads to the estimator $\hat{\mu}_1 = \bar{y} - \mathbf{b}_0'\bar{X}$, where $\mathbf{b}_0 = W_{X0} W_{Xy}$ and W_{Xy} is defined in equation (5). Krishnamoorthy and Rohatgi [7] showed that $\hat{\mu}_1$ has smaller variance than $\hat{\mu}$ over a wide range of parameter space.

The problem of estimating the mean of y given that the mean of X is $0_{p-1\times 1}$ has been considered by Berry [8], Tan and Gleser [9] and Jin and Berry [10]. These authors refer to the vector X as concomitant control vector for estimating the mean μ of y. This problem is equivalent to the common mean problem with the transformed variables. However, the main interest in the common mean problem is to develop a better inferential procedure, based on a combination of the correlated estimators, than the best of the t procedures based on individual estimators whereas there is no such interest in the problem of estimating μ with a concomitant control vector.

In this article, we are mainly interested in comparing three confidence intervals, including the t-interval based on the marginal distribution of y, that are given in Halperin [6]. In the following section, we describe the conditional interval and the unconditional interval due to Halperin, and present expressions for their expected lengths. The expected lengths are compared numerically. Our comparison studies in section 3 show that the conditional intervals are either slightly shorter than or almost close to the unconditional intervals for all the cases considered. We also discuss a condition under which the expected length of the conditional confidence interval is shorter than the best of the t-intervals based on individual means. For the sake of completeness, we also present the test based on the conditional approach, and its power function. The methods are illustrated using a simulated data set.

2. Interval estimation and expected lengths

In the following lemma, we present some basic distributional results related to the statistics defined in the previous section. These results can be found, for example, in Muirhead [11, chapter 3].

LEMMA 2.1

(i) *The conditional distribution of* $\mathbf{b} = W_{Xy}' W_{XX}^{-1}$ *given* $(X_1, \dots, X_n)$ *is* $N_{p-1}(\beta, \sigma_{yy\cdot X} W_{XX}^{-1})$.
(ii) $n\bar{X}\Sigma_{XX}^{-1}\bar{X} = (\sqrt{n}\bar{X}'\Sigma^{-1/2})(\Sigma^{-1/2}\bar{X}\sqrt{n}) = Z'Z \sim \chi^2_{p-1}$.
(iii) $V = (\bar{X}\Sigma_{XX}^{-1}\bar{X}/(\bar{X}W_{XX}^{-1}\bar{X})) \sim \chi^2_{n-p+1}$ *independently of* $\bar{X}$ *(or* Z*)*

(iv) $Q = n\bar{X}'W_{XX}^{-1}\bar{X} = Z'Z/V \sim ((p-1)/(n-p+1))F_{p-1,n-p+1}$, *where* $F_{a,b}$ *denotes the* F *random variable with the numerator df* $= a$ *and the denominator df* $= b$.

(v) *The sample conditional variance of* y *given* X *is defined as* $\hat{\sigma}_{yy\cdot X} = ((w_{yy} - W'_{Xy}W_{XX}^{-1}W_{Xy})/(n-p))$ *and is distributed as* $(\sigma_{yy\cdot X}/(n-p))\chi^2_{n-p}$ *independently of* Q.

We shall now present the confidence intervals that will be considered for comparison.

2.1 *The t-interval*

The usual t-interval based on the marginal distribution of y is given by

$$\bar{y} \pm t_{n-1,1-\alpha/2}\sqrt{\frac{s_{yy}}{n}}, \tag{9}$$

where s_{yy} is the sample variance of y and $t_{m,\alpha}$ denotes the αth quantile of the Student's t distribution. The expected length of the t-interval is given by

$$\mathrm{EL}_1 = 2t_{n-1,1-\alpha/2}E\left(\sqrt{\frac{s_{yy}}{n}}\right) = 2t_{n-1,1-\alpha/2}\sqrt{\frac{2}{n}}\frac{\Gamma(n/2)}{\Gamma(n-1)/2}\sqrt{\frac{\sigma_{yy}}{n-1}}, \tag{10}$$

where $\Gamma(\cdot)$ denotes the gamma function.

2.2 *The conditional interval*

We shall now describe the conditional confidence interval due to Halperin [6]. Using the results of Lemma 2.1, it can be readily verified that the conditional distribution of $\hat{\mu}$ given $X_1, \ldots, X_n$ is normal with mean μ and variance $\sigma_{yy\cdot X}(1+Q)$, where $Q = n\bar{X}'W_{XX}^{-1}\bar{X}$ defined in Lemma 2.1(v). We write

$$\hat{\mu}|(X_1, \ldots, X_n) \sim N\left(\mu, \frac{\sigma_{yy\cdot X}(1+Q)}{n}\right). \tag{11}$$

Notice that $(n-p)\hat{\sigma}_{yy\cdot X}/\sigma_{yy\cdot X} \sim \chi^2_{n-p}$. Using this result, we see that, conditionally given Q, the pivotal quantity

$$\frac{\sqrt{n}(\hat{\mu}-\mu)}{\sqrt{\hat{\sigma}_{yy\cdot X}\,(1+Q)}} \sim t_{n-p}. \tag{12}$$

This leads to the conditional confidence interval

$$\hat{\mu} \pm t_{n-p,1-\alpha/2}\,(1+Q)^{1/2}\,\sqrt{\frac{\hat{\sigma}_{yy\cdot X}}{n}}. \tag{13}$$

It follows from Lemma 2.1(iv) that $(1+Q)$ is distributed as U^{-1}, where U is a beta random variable with parameters $(n-p+1)/2$ and $(p-1)/2$. Using this result and the fact that $\bar{X}'W_{XX}^{-1}\bar{X}$ and $\hat{\sigma}_{yy\cdot X}$ are independent, it is easy to see that the expected length of the conditional confidence interval in equation (13) is

$$\mathrm{EL}_2 = 2t_{n-p,1-\alpha/2}\sqrt{\frac{2}{n}}\frac{\Gamma\,(n/2)}{\Gamma(n-1)/2}\left(\frac{\sigma_{yy}(1-\rho^2_{y\cdot X})}{n-p}\right)^{1/2}. \tag{14}$$

It should be noted that the formula for EL_2 given in Halperin [6] is incorrect.

2.3 *The unconditional confidence interval*

It follows from equation (12) and Lemma 2.1(iv) that

$$T = \frac{\sqrt{n}(\bar{y}(\mathbf{b}) - \mu_0)}{\sqrt{\hat{\sigma}_{yy\cdot X}}} \sim t_{n-p}\left(1 + \frac{p-1}{n-p+1}F_{p-1,n-p+1}\right)^{1/2}. \tag{15}$$

The percentiles of T can be used to form a $(1-\alpha)$ confidence interval for μ. Using some standard methods, it can be shown that the $(1-\alpha)$th quantile k of T is the solution of the equation

$$\frac{\Gamma(n/2)}{\Gamma((p-1)/2)\Gamma((n-p+1)/2)} \times \int_0^1 G\left(k\sqrt{1-x}; n-p\right) x^{(p-1)/2-1}(1-x)^{(n-p+1)/2-1}\,\mathrm{d}x = 1-\alpha, \tag{16}$$

where $G(\cdot; m)$ denotes the Student's t cdf with df $= m$. To get equation (16), we used the fact that $F_{a,b}$ is distributed as $bU/(a(1-U))$, where U is a beta$(a/2, b/2)$ random variable. Noting that the Student's t distribution is symmetric about zero, it follows from equation (16) that the distribution of T is also symmetric about zero. Let T_α denote the αth quantile of T.

Table 1. Critical points $T_{1-\alpha/2}$ for constructing unconditional confidence intervals.

	$p=2$		$p=3$		$p=4$		$p=5$	
n	$\alpha=0.05$	$\alpha=0.10$	$\alpha=0.05$	$\alpha=0.10$	$\alpha=0.05$	$\alpha=0.10$	$\alpha=0.05$	$\alpha=0.10$
6	3.1900	2.4198	4.3705	3.1631	7.6337	5.0175	–	–
7	2.8661	2.2340	3.5565	2.6903	4.9204	3.5130	8.5522	5.5248
8	2.6894	2.1210	3.1478	2.4336	3.9086	2.9316	5.3637	3.8168
9	2.5590	2.0442	2.8964	2.3029	3.3856	2.6167	4.2050	3.1612
10	2.4664	1.9856	2.7330	2.1722	3.1018	2.4282	3.6301	2.7853
11	2.3969	1.9402	2.6243	2.1005	2.9069	2.3003	3.2978	2.5723
12	2.3539	1.9061	2.5375	2.0489	2.7699	2.2075	3.0582	2.4240
13	2.3085	1.8863	2.4677	2.0041	2.6603	2.1428	2.8958	2.3143
14	2.2762	1.8584	2.4163	1.9641	2.5822	2.0904	2.7835	2.2307
15	2.2470	1.8430	2.3681	1.9327	2.5180	2.0388	2.6854	2.1731
16	2.2291	1.8280	2.3356	1.9116	2.4651	2.0074	2.6121	2.1184
17	2.2051	1.8124	2.3033	1.8897	2.4211	1.9749	2.5499	2.0772
18	2.1922	1.8007	2.2836	1.8714	2.3850	1.9503	2.5011	2.0383
19	2.1759	1.7964	2.2625	1.8602	2.3505	1.9265	2.4584	2.0115
20	2.1647	1.7856	2.2418	1.8405	2.3222	1.9103	2.4234	1.9884
21	2.1549	1.7749	2.2229	1.8342	2.3058	1.8951	2.3883	1.9641
22	2.1434	1.7652	2.2074	1.8238	2.2758	1.8804	2.3664	1.9422
23	2.1360	1.7644	2.1966	1.8102	2.2666	1.8657	2.3412	1.9243
24	2.1240	1.7586	2.1866	1.8070	2.2502	1.8605	2.3173	1.9143
25	2.1166	1.7471	2.1735	1.7963	2.2377	1.8388	2.2951	1.8973
30	2.0857	1.7346	2.1301	1.7711	2.1749	1.8003	2.2252	1.8452
35	2.0617	1.7151	2.1048	1.7482	2.1370	1.7791	2.1750	1.8127
40	2.0525	1.7047	2.0809	1.7348	2.1148	1.7549	2.1429	1.7895
45	2.0364	1.6993	2.0622	1.7217	2.0964	1.7467	2.1228	1.7674
50	2.0347	1.6920	2.0597	1.7151	2.0775	1.7337	2.1034	1.7563
60	2.0219	1.6868	2.0360	1.7037	2.0614	1.7176	2.0731	1.7341
70	2.0076	1.6788	2.0226	1.6956	2.0390	1.7088	2.0590	1.7221
80	2.0081	1.6765	2.0195	1.6889	2.0319	1.6966	2.0440	1.7096
90	1.9979	1.6743	2.0095	1.6842	2.0210	1.6899	2.0352	1.7008
100	1.9959	1.6676	2.0085	1.6825	2.0182	1.6872	2.0255	1.6991
1000	1.9627	1.6512	1.9628	1.6478	1.9654	1.6494	1.9635	1.6490

Then, the unconditional $(1-\alpha)$ confidence interval for μ is given by

$$\hat{\mu} \pm T_{1-\alpha/2}\sqrt{\frac{\hat{\sigma}_{yy\cdot X}}{n}}. \tag{17}$$

The expected length of the unconditional confidence interval is given by

$$\mathrm{EL}_3 = 2T_{1-\alpha/2}E\left(\sqrt{\frac{\hat{\sigma}_{yy\cdot X}}{n}}\right) = 2T_{1-\alpha/2}\sqrt{\frac{2}{n}}\frac{\Gamma((n-p+1)/2)}{\Gamma((n-p)/2)}\left(\frac{\sigma_{yy}(1-\rho^2_{y\cdot X})}{n-p}\right)^{1/2}. \tag{18}$$

Remark 1 Using equation (16), we computed the values of $T_{1-\alpha/2}$ when $\alpha = 0.05$ and 0.1, $p = 2, 3, 4$ and 5, and values of n ranging from 6 to 1000. These critical values are presented in table 1. We also found that the distribution of T in equation (15) can be approximated by ct_{n-p}, where $c = \sqrt{(n-2)/(n-p-1)}$. The constant c was obtained by solving the equation $E(c^2 t^2_{n-p}) = E(T^2)$. Using this approximation, we have $T_{1-\alpha/2} \doteq t_{n-p,1-\alpha/2}\sqrt{(n-2)/(n-p-1)}$. This approximation is satisfactory as long as $(n-p) \geq 4$.

3. Comparison of expected lengths

It is clear from the expressions of EL_1, EL_2 and EL_3, that the ratios $\mathrm{EL}_2/\mathrm{EL}_1$ and $\mathrm{EL}_3/\mathrm{EL}_1$ depend on the parameter space only through $\rho^2_{y.X}$. Using this fact, direct comparison between EL_2 and EL_1 shows that the expected length of the conditional confidence interval is shorter than the expected length of the usual t interval based on y observations alone if and only if

$$\rho^2_{y\cdot X} > 1 - \left(\frac{t_{n-1,1-\alpha/2}}{t_{n-p,1-\alpha/2}}\right)^2\left(\frac{n-p}{n-1}\right). \tag{19}$$

The above inequality is different from the one given in Halperin [6, p. 41], because, as we already pointed out, Halperin's formula for the expected length of the conditional interval is incorrect. For fixed p, the right-hand side of equation (19) approaches zero as $n \to \infty$. This implies that, for large n, EL_2 is smaller than EL_1 for all practically meaningful values of $\rho^2_{y\cdot X}$. However, this does not mean that EL_2 is smaller than the expected length of the shortest of the individual t-intervals. The above condition merely implies that EL_2 is smaller than the expected length of the t-interval based on $\bar{u}_1 = \bar{y}$. For EL_2 to be shorter than the t-interval based on $\bar{u}_2$, we should have

$$\rho^2_{u_2\cdot(u_2-u_1),\ldots,(u_2-u_p)} > 1 - \left(\frac{t_{n-1,1-\alpha/2}}{t_{n-p,1-\alpha/2}}\right)^2\left(\frac{n-p}{n-1}\right),$$

where $\rho^2_{u_2\cdot(u_2-u_1),\ldots,(u_2-u_p)}$ is the squared multiple correlation coefficient between U_2 and $((U_2-U_1),\ldots,(U_2-U_p))$. Proceeding this way, we see that EL_2 is shorter than the shortest of the t-intervals if and only if

$$\min\{\rho^2_{u_1\cdot(u_1-u_2),\ldots,(u_1-u_p)},\ldots,\rho^2_{u_p\cdot(u_p-u_1),\ldots,(u_p-u_{p-1})}\} > 1 - \left(\frac{t_{n-1,1-\alpha/2}}{t_{n-p,1-\alpha/2}}\right)^2\left(\frac{n-p}{n-1}\right). \tag{20}$$

Comparison between EL_2 and EL_3 shows that the ratio $EL_2/EL_3 > 1$ if and only if

$$t_{n-p,1-\alpha/2}\frac{\Gamma(n/2)}{\Gamma((n-1)/2)} < T_{1-\alpha/2}\frac{\Gamma((n-p+1)/2)}{\Gamma((n-p)/2)}. \quad (21)$$

We numerically evaluated EL_2/EL_1 and EL_3/EL_1 and presented them in table 2. It is clear from the table values that EL_2 is, in general, either very close to EL_3 or smaller than EL_3, and the difference between them decreases as n increases. Thus, we see that the conditional confidence interval is not only simple to construct but also narrower than the unconditional confidence interval. Furthermore, if $n-p$ is small and $\rho^2_{y\cdot X}$ is small, then the usual t interval is shorter than both conditional and unconditional intervals (see the values in table 2 when $n=6$, $p=3$ and $n=6$, $p=5$). Thus, the conditional combined method is preferable to the best of the t procedures only when condition (20) holds and/or $n-p$ is moderately large.

Table 2. Ratios of the expected lengths of 95% confidence intervals.

	$n=6$		$n=10$		$n=15$		$n=20$		$n=30$	
$\rho^2_{y\cdot X}$	EL_2/EL_1	EL_3/EL_1	EL_2/EL_1	EL_3/EL_1	EL_2/EL_1	EL_3/EL_1	EL_2/EL_1	EL_3/EL_1	EL_2/EL_1	EL_3/EL_1
					$(p=2)$					
0.05	1.1770	1.2121	1.0538	1.0518	1.0188	1.0211	1.0052	1.0143	0.9935	0.9900
0.10	1.1456	1.1510	1.0257	1.0157	0.9916	0.9867	0.9784	0.9657	0.9670	0.9655
0.20	1.0801	1.1151	0.9671	0.9641	0.9349	0.9255	0.9224	0.9206	0.9117	0.9140
0.30	1.0103	1.0275	0.9046	0.9113	0.8746	0.8719	0.8628	0.8652	0.8528	0.8542
0.40	0.9354	0.9507	0.8375	0.8488	0.8097	0.8075	0.7988	0.8032	0.7895	0.7882
0.50	0.8539	0.8773	0.7645	0.7630	0.7391	0.7367	0.7292	0.7299	0.7207	0.7176
0.60	0.7637	0.7843	0.6838	0.6855	0.6611	0.6551	0.6522	0.6501	0.6447	0.6442
0.70	0.6614	0.6741	0.5922	0.5931	0.5725	0.5733	0.5649	0.5633	0.5583	0.5610
0.80	0.5400	0.5509	0.4835	0.4833	0.4675	0.4684	0.4612	0.4609	0.4558	0.4581
0.90	0.3819	0.3863	0.3419	0.3410	0.3305	0.3310	0.3261	0.3279	0.3223	0.3217
0.95	0.2700	0.2746	0.2418	0.2441	0.2337	0.2345	0.2306	0.2301	0.2279	0.2281
ρ^2_L	0.3142		0.1446		0.0848		0.0597		0.0375	
	$n=6$		$n=10$		$n=20$		$n=30$		$n=40$	
$\rho^2_{y\cdot X}$	EL_2/EL_1	EL_3/EL_1	EL_2/EL_1	EL_3/EL_1	EL_2/EL_1	EL_3/EL_1	EL_2/EL_1	EL_3/EL_1	EL_2/EL_1	EL_3/EL_1
					$(p=3)$					
0.05	1.5578	1.6084	1.1552	1.1651	1.0695	1.0737	1.0387	1.0408	1.0134	1.0147
0.10	1.5163	1.5652	1.1244	1.1423	1.0410	1.0461	1.0110	1.0132	0.9864	0.9884
0.20	1.4296	1.4853	1.0601	1.0722	0.9814	0.9850	0.9532	0.9565	0.9300	0.9298
0.30	1.3372	1.3845	0.9917	1.0051	0.9180	0.9205	0.8916	0.8957	0.8699	0.8715
0.40	1.2380	1.2763	0.9181	0.9293	0.8499	0.8564	0.8255	0.8270	0.8054	0.8049
0.50	1.1302	1.1726	0.8381	0.8480	0.7759	0.7793	0.7535	0.7550	0.7352	0.7356
0.60	1.0108	1.0451	0.7496	0.7589	0.6940	0.6985	0.6740	0.6760	0.6576	0.6586
0.70	0.8754	0.9035	0.6492	0.6562	0.6010	0.6045	0.5837	0.5859	0.5695	0.5687
0.80	0.7148	0.7394	0.5301	0.5363	0.4907	0.4942	0.4766	0.4770	0.4650	0.4640
0.90	0.5054	0.5256	0.3748	0.3809	0.3470	0.3490	0.3370	0.3378	0.3288	0.3291
0.95	0.3574	0.3702	0.2650	0.2688	0.2454	0.2467	0.2383	0.2391	0.2325	0.2325
ρ^2_L	0.6100		0.2882		0.1694		0.1194		0.0749	
					$(p=5)$					
0.05	10.7729	10.3750	1.4860	1.5312	1.1171	1.1263	1.0571	1.0594	1.0326	1.0348
0.10	10.4855	10.0999	1.4463	1.4949	1.0873	1.0912	1.0289	1.0328	1.0051	1.0078
0.20	9.8859	9.5459	1.3636	1.4038	1.0251	1.0321	0.9701	0.9730	0.9476	0.9489
0.30	9.2474	8.9105	1.2755	1.3155	0.9589	0.9638	0.9074	0.9100	0.8864	0.8886
0.40	8.5614	8.2577	1.1809	1.2170	0.8878	0.8960	0.8401	0.8425	0.8207	0.8211
0.50	7.8155	7.5403	1.0780	1.1085	0.8104	0.8146	0.7669	0.7671	0.7492	0.7505
0.60	6.9904	6.6793	0.9642	0.9894	0.7249	0.7289	0.6859	0.6863	0.6701	0.6729
0.70	6.0538	5.8581	0.8350	0.8609	0.6278	0.6333	0.5940	0.5940	0.5803	0.5815
0.80	4.9429	4.7997	0.6818	0.7039	0.5126	0.5147	0.4850	0.4858	0.4738	0.4736
0.90	3.4952	3.3807	0.4821	0.4977	0.3624	0.3651	0.3430	0.3437	0.3350	0.3342
0.95	2.4715	2.3903	0.3409	0.3505	0.2563	0.2578	0.2425	0.2433	0.2369	0.2376
ρ^2_L	0.9918		0.5698		0.2387		0.1499		0.1091	

Note: ρ^2_L is the lower bound given in the right-hand side of equation (20); $EL_2 < EL_1$ when equation (20) holds.

4. Power function

We observed in the preceding section that the conditional method performs better than the unconditional method, and hence we consider only the power function of the conditional test based on equation (12). Consider the hypotheses

$$H_0\colon \mu \le \mu_0 \quad vs. \quad H_a\colon \mu > \mu_0.$$

The conditional non-null distribution (given T^2) is the noncentral t distribution with df $= n - p - 1$ and the noncentrality parameter

$$\delta(Q) = \frac{\sqrt{n}(\mu - \mu_0)}{\sqrt{\sigma_{yy\cdot X}}\sqrt{1+Q}}, \tag{22}$$

where μ is the true value and μ_0 is the specified value of the mean. The unconditional power of a right-tail test can be expressed as

$$E_Q\left[P(t_{n-p-1}(\delta(Q)) > t_{n-p-1,1-\alpha})\right]. \tag{23}$$

Again, using the fact that $1 + Q$ is distributed as the reciprocal of a beta $((n-p)/2, p/2)$ random variable, the power can be computed using the numerical integration

$$1 - \frac{\Gamma(n/2)}{\Gamma(p/2)\Gamma((n-p)/2)} \int_0^1 G\left(c_1; n-p-1, \delta(u^{-1}-1)\right) u^{p/2-1}(1-u)^{(n-p)/2-1}\,\mathrm{d}u, \tag{24}$$

where $c_1 = t_{n-p-1,1-\alpha}$ and $G(x; m, d)$ denotes the cdf of a noncentral t random variable with df = m and the noncentrality parameter d.

Although it is not difficult to compute equation (20), a simple approximate power expression can be obtained from equation (19), and is given by

$$P\left(t_{n-p-1}(\delta(E(Q)) > t_{n-p-1,1-\alpha}\right). \tag{25}$$

Noting that $E(Q) = (n-2)/(n-p-2)$, for a given level of significance α, p and $\eta = \sqrt{}(\mu - \mu_0)/\sigma_{yy\cdot X}$, an approximate sample size n that is required to attain a power of $1-\beta$ satisfies

$$P\left(t_{n-p-1}(\delta_1) > t_{n-p-1,1-\alpha}\right) = 1 - \beta, \tag{26}$$

where

$$\delta_1 = \frac{\sqrt{n}(\mu - \mu_0)}{\sqrt{\sigma_{yy\cdot X}}}\left(\frac{n-p-2}{n-2}\right)^{1/2}. \tag{27}$$

In order to understand the validity of the approximation, we computed the exact power using equation (24) and the approximate power based on equation (26) for various values of n, $\eta = \sqrt{}(\mu - \mu_0)/\sigma_{yy\cdot X}$ and $p = 2, 3$ and 4. These powers are presented in table 3. We see from the table values that the approximate powers are close to the exact powers provided n is moderately large in comparison to p. Our extensive numerical studies for various values of p (not reported here) showed that the approximation is very satisfactory for values of $n \ge 5p$. An advantage of this approximation is that it only involves the computation of the noncentral t cdf with fixed noncentrality parameter (when η, n and p are given), and so the power computation can be carried out using freely available PC calculators such as StatCalc (http://www.ucs.louisians.edu/~kxk4695) or online calculator available at http://calculators.stat.ucla.edu/.

Table 3. Exact powers (24) and approximate powers (25) of the conditional test when $\alpha = 0.05$.

	$n = 6$		$n = 8$		$n = 12$		$n = 16$		$n = 20$	
$\eta = \mu - \mu_0/\sqrt{\sigma_{yy\cdot X}}$	Exact	Approximate	Exact	Approximate	Exact	Approximate	Exact	Approximate	Exact	Approximate
($p = 2$)										
0.00	0.050	0.050	0.050	0.050	0.050	0.050	0.050	0.050	0.050	0.050
0.10	0.079	0.080	0.085	0.086	0.091	0.091	0.096	0.096	0.101	0.101
0.20	0.120	0.121	0.137	0.137	0.153	0.153	0.168	0.168	0.182	0.182
0.30	0.174	0.175	0.206	0.207	0.237	0.237	0.266	0.267	0.295	0.295
0.40	0.239	0.242	0.291	0.293	0.341	0.342	0.387	0.388	0.431	0.432
0.50	0.314	0.321	0.389	0.393	0.458	0.461	0.520	0.522	0.576	0.578
0.60	0.396	0.407	0.494	0.501	0.577	0.583	0.649	0.653	0.711	0.714
0.70	0.480	0.498	0.597	0.607	0.689	0.697	0.763	0.768	0.820	0.824
0.80	0.565	0.588	0.693	0.706	0.785	0.794	0.852	0.858	0.899	0.903
0.90	0.645	0.673	0.776	0.792	0.861	0.870	0.915	0.920	0.949	0.952
1.00	0.718	0.750	0.844	0.860	0.916	0.924	0.955	0.959	0.977	0.979

	$n = 8$		$n = 12$		$n = 16$		$n = 20$		$n = 24$	
$\eta = \mu - \mu_0/\sqrt{\sigma_{yy\cdot X}}$	Exact	Approximate	Exact	Approximate	Exact	Approximate	Exact	Approximate	Exact	Approximate
($p = 3$)										
0.00	0.050	0.050	0.050	0.050	0.050	0.050	0.050	0.050	0.050	0.050
0.10	0.084	0.086	0.089	0.090	0.099	0.099	0.108	0.108	0.117	0.117
0.20	0.133	0.137	0.148	0.149	0.176	0.177	0.204	0.205	0.231	0.232
0.30	0.196	0.205	0.226	0.229	0.283	0.284	0.338	0.339	0.390	0.390
0.40	0.271	0.288	0.322	0.328	0.413	0.416	0.495	0.497	0.568	0.570
0.50	0.353	0.383	0.428	0.441	0.551	0.557	0.651	0.656	0.733	0.736
0.60	0.439	0.484	0.536	0.558	0.681	0.691	0.785	0.791	0.857	0.861
0.70	0.524	0.585	0.640	0.669	0.791	0.804	0.882	0.888	0.935	0.938
0.80	0.605	0.679	0.732	0.767	0.874	0.887	0.942	0.948	0.974	0.977
0.90	0.679	0.762	0.809	0.846	0.930	0.941	0.975	0.979	0.992	0.993
1.00	0.745	0.831	0.870	0.905	0.964	0.972	0.991	0.993	0.998	0.998

(*continued*)

Table 3. Continued.

$\eta = \mu - \mu_0/\sqrt{\sigma_{yy\cdot x}}$	$n = 8$ Exact	$n = 8$ Approximate	$n = 12$ Exact	$n = 12$ Approximate	$n = 16$ Exact	$n = 16$ Approximate	$n = 20$ Exact	$n = 20$ Approximate	$n = 24$ Exact	$n = 24$ Approximate
					$(p = 4)$					
0.00	0.050	0.050	0.050	0.050	0.050	0.050	0.050	0.050	0.050	0.050
0.10	0.094	0.096	0.103	0.103	0.111	0.112	0.120	0.120	0.128	0.128
0.20	0.162	0.165	0.187	0.188	0.214	0.214	0.240	0.240	0.265	0.266
0.30	0.252	0.260	0.304	0.306	0.355	0.357	0.405	0.406	0.452	0.453
0.40	0.358	0.377	0.442	0.448	0.519	0.523	0.588	0.591	0.650	0.652
0.50	0.470	0.504	0.585	0.597	0.677	0.685	0.752	0.758	0.812	0.816
0.60	0.580	0.630	0.714	0.733	0.807	0.817	0.872	0.878	0.917	0.920
0.70	0.680	0.743	0.819	0.841	0.898	0.908	0.944	0.949	0.970	0.972
0.80	0.766	0.833	0.894	0.915	0.952	0.960	0.979	0.982	0.991	0.992
0.90	0.835	0.900	0.943	0.959	0.980	0.985	0.993	0.995	0.998	0.998
1.00	0.888	0.944	0.972	0.983	0.993	0.995	0.998	0.999	1.000	1.000

5. An example

To illustrate the methods of this article, we generated a sample of 20 observations from $N(\mu, \Sigma)$, where

$$\mu = \begin{pmatrix} 4 \\ 4 \\ 4 \end{pmatrix} \quad \text{and} \quad \Sigma = \begin{pmatrix} 3 & 0.9798 & 2.0785 \\ 0.9789 & 2 & 1.9799 \\ 2.0785 & 1.9799 & 4 \end{pmatrix}.$$

The data points are given in table 4. The summary statistics are

$$(\bar{y}, \bar{x}_1, \bar{x}_2) = (3.9064, 4.1602, 4.2912),$$

$$\begin{pmatrix} w_{yy} & W_{yy.X} \\ W'_{yy.X} & W_{XX} \end{pmatrix} = \begin{pmatrix} 57.9621 & 24.7029 & 22.7379 \\ - & 28.4999 & 11.6107 \\ - & - & 55.7325 \end{pmatrix},$$

$$W_{XX}^{-1} = \begin{pmatrix} 3.83421\text{E}-2 & -7.98701\text{E}{-3} \\ - & 1.960696\text{E}{-2} \end{pmatrix},$$

$$\hat{\mu} = 4.0953, \ \hat{\sigma}_{yy\cdot X} = 1.9648 \quad \text{and} \quad \rho^2_{y.X} = 0.4238.$$

The standard deviations are $s_{u_1} = 1.747$, $s_{u_2} = 1.397$ and $s_{u_3} = 1.895$. The critical points $t_{19,0.975} = 2.0930$ and $T_{u,0.975} = 2.2418$. Using these statistics, we computed the following confidence intervals for μ.

The 95% t-intervals

(a) $\bar{u}_1 \pm t_{19,0.975}\, s_{u_1}/\sqrt{n} = 4.2912 \pm 0.8174;\ \hat{\rho}^2_{u_1\cdot(u_1-u_2,u_1-u_3)} = 0.4238,$
(b) $\bar{u}_2 \pm t_{19,0.975}\, s_{u_2}/\sqrt{n} = 4.1602 \pm 0.6536;\ \hat{\rho}^2_{u_2\cdot(u_2-u_1,u_2-u_3)} = 0.09864,$
(c) $\bar{u}_3 \pm t_{19,0.975}\, s_{u_3}/\sqrt{n} = 3.9064 \pm 0.8868;\ \hat{\rho}^2_{u_3\cdot(u_3-u_2,u_3-u_1)} = 0.5104.$

Table 4. Simulated data ($n = 20$, $p = 3$).

	u_1	u_2	u_3	$y = u_1$	$x_1 = u_1 - u_2$	$x_2 = u_1 - u3$
1	0.9913	2.3451	2.4441	0.9913	−1.3538	−1.4528
2	1.8680	3.5647	1.1850	1.8680	−1.6967	0.6830
3	4.5551	4.7550	5.2841	4.5551	−0.1999	−0.7290
4	4.0409	2.9949	6.6683	4.0409	1.0460	−2.6274
5	5.0872	5.3200	5.1576	5.0872	−0.2328	−0.0704
6	3.4412	5.6166	3.1288	3.4412	−2.1754	0.3124
7	6.1288	5.5200	7.6320	6.1288	0.6088	−1.5032
8	5.7505	4.4716	2.6449	5.7505	1.2789	3.1056
9	4.9089	3.7319	6.7311	4.9089	1.1770	−1.8222
10	3.4542	2.9163	2.1313	3.4542	0.5379	1.3229
11	5.4352	3.6849	3.7720	5.4352	1.7503	1.6632
12	3.7283	5.6323	4.5298	3.7283	−1.9040	−0.8015
13	4.8681	4.7071	2.5703	4.8681	0.1610	2.2978
14	5.0751	4.4576	4.2097	5.0751	0.6175	0.8654
15	2.4324	2.1372	1.0812	2.4324	0.2952	1.3512
16	4.8654	4.1743	4.5202	4.8654	0.6911	0.3452
17	9.1629	7.4908	5.5794	9.1629	1.6721	3.5835
18	3.7241	4.3609	1.7385	3.7241	−0.6368	1.9856
19	3.4018	1.6481	2.5608	3.4018	1.7537	0.8410
20	2.9055	3.6752	4.5584	2.9055	−0.7697	−1.6529

The conditional interval in equation (13) $= 4.0953 \pm 0.6793$. The unconditional interval in equation (17) $= 4.0953 \pm 0.7026$.

The interval (b) is the shortest among all the intervals. We also note that, for the conditional interval to be the shortest, we must have

$$\min\{\rho^2_{u_1\cdot(u_1-u_2),(u_1-u_2)}, \rho^2_{u_1\cdot(u_1-u_2),(u_1-u_2)}, \rho^2_{u_1\cdot(u_1-u_2),(u_1-u_2)}\} > 0.1194 \qquad (28)$$

(see table 2, $n = 20$, $p = 3$). Since the minimum of the sample squared multiple correlation coefficients is 0.09864, we do not have any evidence in favor of equation (28). Therefore, as already observed, the conditional approach did not produce the shortest interval. We also see that, among all the point estimates, the MLE is very close to the true mean 4.

6. Concluding remarks

We observed from the preceding sections that the unconditional approach is not only simple to use but also better than the unconditional method for constructing confidence interval for the common mean μ. Furthermore, if the sample size is sufficiently large, then the conditional approach may yield better results than the ones based on the individual t procedures. For a fixed p and $\alpha = 0.05$, we computed the least value of n for which $\rho_L^2 = 1 - \left(t_{n-1,1-\alpha/2}/t_{n-p,1-\alpha/2}\right)^2 ((n-p)/(n-1)) > 0.05$. Based on a linear fit of these pairs of (n, p), we found that $\rho_L^2 > 0.05$ for any $n > 20p - 15$. This implies that n must be at least $20p - 15$ for the conditional approach offers improvement over the best of the t procedures for any $\rho^2_{y\cdot X} > 0.05$. For moderate sample sizes, to check if the conditional combined approach is superior to the best of individual t methods, one should test whether the minimum of the squared sample multiple correlation coefficients is greater than $\rho_L^2 = 1 - \left(t_{n-1,1-\alpha/2}/t_{n-p,1-\alpha/2}\right)^2 ((n-p)/(n-1))$. It is difficult to obtain an exact test to verify this condition. Therefore, in practice one may want to compute all the t-intervals based on the individual components and the conditional confidence interval, and then choose the shortest of the intervals for applications.

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