Normal-Based Methods for a Gamma Distribution: Prediction and Tolerance Intervals and Stress-Strength Reliability

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In this article we propose inferential procedures for a gamma distribution using the Wilson–Hilferty (WH) normal approximation. Specifically, using the result that the cube root of a gamma random variable is approximately normally distributed, we propose normal-based approaches for a gamma distribution for (a) constructing prediction limits, one-sided tolerance limits, and tolerance intervals; (b) for obtaining upper prediction limits for at least l of m observations from a gamma distribution at each of r locations; and (c) assessing the reliability of a stress-strength model involving two independent gamma random variables. For each problem, a normal-based approximate procedure is outlined, and its applicability and validity for a gamma distribution are studied using Monte Carlo simulation. Our investigation shows that the approximate procedures are very satisfactory for all of these problems. For each problem considered, the results are illustrated using practical examples. Our overall conclusion is that the WH normal approximation provides a simple, easy-to-use unified approach for addressing various problems for the gamma distribution.

KEY WORDS: Confidence limits; Coverage probability; Quantile; Survival probability; Tolerance limits; Wilson–Hilferty approximation.

1. INTRODUCTION

The gamma distribution is one of the waiting time distributions that may offer a good fit to time to failure data. Even though this distribution is not widely used as a lifetime distribution model, it is used in many other important practical problems. For example, gamma-related distributions are widely used to model the amounts of daily rainfall in a region (Das 1955; Stephenson et al. 1999) and to fit hydrological data sets (Ashkar and Bobée 1988; Ashkar and Ouarda 1998; Aksoy 2000). In particular, Ashkar and Ouarda (1998) used a two-parameter gamma distribution to fit annual maximum flood series to construct confidence intervals for a quantile. Two-parameter gamma tolerance limits and prediction limits are used in monitoring and control problems. For example, in environmental monitoring, upper tolerance limits are often constructed based on background data (regional surface water, groundwater, or air monitoring data) and used to determine whether a potential source of contamination (e.g., landfill by a waste management facility, hazardous material storage facility, or factory) has adversely impacted the environment (Bhaumik and Gibbons 2006). The gamma distribution has also found a number of applications in occupational and industrial hygiene. In a recent article, Maxim et al. (2006) observed that the gamma distribution is a possible distribution for concentrations of carbon/coke fibers in plants that produce green or calcined petroleum coke. In a study of tuberculosis risk and incidence, Ko, Burge, Nardell, and Thompson (2001) noted that the gamma distribution is appropriate for modeling the time spent in the waiting room at primary care sites. Earlier, Nieuwenhuijsen et al. (1995) used a gamma distribution to model determinants of exposure to rat urinary aeroallergen.

In this article we consider the problems of constructing tolerance limits and prediction limits, as well as some related problems, for a two-parameter gamma distribution. Specifically, we address the following problems: (a) constructing prediction limits, one-sided tolerance limits, and tolerance intervals; (b) finding upper prediction limits (UPLs) for at least l of m observations from a gamma distribution at each of r locations; and (c) assessing the reliability of a stress-strength model involving two independent gamma random variables. We first briefly review the relevant literature and practical situations for each of these problems.

Bain, Engelhardt, and Shiue (1984) proposed approximate tolerance limits for a gamma distribution for the purpose of finding lower tolerance limits for the endurance of deep-groove ball bearings. They obtained these tolerance limits by assuming first that the scale parameter b is known and the shape parameter a is unknown, and then replacing the scale parameter by

TECHNOMETRICS, FEBRUARY 2008, VOL. 50, NO. 1
DOI 10.1198/004017007000000353

its sample estimate. Ashkar and Ouarda (1998) developed an approximate method of setting confidence limits for the gamma quantile by transforming the tolerance limits for the normal distribution. Toward this end, they used the result that X is distributed as $F_X^{-1}(F_Y(Y))$, where X is a gamma random variable with the distribution function F_X and Y is a normal random variable with the distribution function F_Y. This transformation generally is not independent of the parameters, and eventually the unknown parameters must be replaced by estimates to obtain approximate tolerance limits. We note that the problem of setting confidence limits for a gamma quantile is also of interest for exposure data analysis in industrial hygiene applications, because a parameter of interest in such applications is the proportion of exposure measurements that exceed an occupational exposure limit. Clearly, inference concerning this parameter can be reduced to inference concerning a quantile. Aryal, Bhaumik, Mathew, and Gibbons (2006) argued that the distribution of X can be approximated by a normal distribution for large values of a. Their suggestion is to use normal-based tolerance limits if the maximum likelihood estimator (MLE) is $\hat{a} > 7$. For $0 < \hat{a} \leq 7$, they provided tabular values to construct tolerance factors.

The second problem arises in monitoring and control problems where the future samples, to be collected periodically during the operation of a process, are compared with some past background data to determine whether a change in the process has occurred on each sampling occasion. This type of process monitoring is also practiced in groundwater quality detection monitoring in the vicinity of hazardous waste management facilities (HWMFs). For example, to monitor groundwater quality, a series of samples from each of several monitoring wells in the vicinity of a HWMF are often compared with statistical prediction limits based on a sample of measurements obtained from one or more upgradient sampling locations of the facility. Davis and McNichols (1987) addressed this problem assuming normality. Bhaumik and Gibbons (2006) argued that the normal model seldom offers a good fit for such environmental data, and that the gamma distribution generally characterizes the data well. Assuming a gamma distribution, these authors proposed an approximate method for constructing prediction limits for the aforementioned purpose.

The third problem that we address is assessing the reliability in a stress-strength model. This model involves two independent random variables, X_1 and X_2, where X_1 represents the strength variable of a component and X_2 represents the stress variable to which the component is subjected. If $X_1 \leq X_2$, then either the component fails or the system that uses the component may malfunction. The reliability parameter R of the unit can be expressed as $R = P(X_1 > X_2)$. Assuming normality of X_1 and X_2, Hall (1984), Reiser and Guttman (1986), and Guo and Krishnamoorthy (2004) proposed approximate methods for computing confidence limits for R. Several authors considered the problem of estimating R when X_1 and X_2 are independent gamma random variables. Basu (1981) and Constantine, Karson, and Tse (1986, 1989, 1990) considered point and interval estimation of R. As pointed out by Constantine et al. (1990), many investigators assume that the shape parameters are known and are integer-valued. If the shape parameters are known, then it is not difficult to obtain exact confidence limits for R (see Sec. 5). Several approximate procedures have been proposed for situations when the shape parameters are unknown. Reiser and Rocke (1993) compared several procedures for computing lower limits for R and recommended two procedures, the delta method on logits and the bootstrap percentile test inversion. It should be noted that the parameter R arises in application areas other than reliability. Wolfe and Hogg (1971) introduced R as a general measure of difference; Hauck, Hyslop, and Anderson (2000) considered its usefulness in clinical trial applications; and Reiser (2000) proposed applications to the analysis of receiver operating characteristic curves.

In this article we propose simple approximate solutions for the problems mentioned in the preceding paragraphs using a normal approximation due to Wilson and Hilferty (1931). Those authors developed the normal approximation for a chi-squared random variable, from which the normal approximation of the gamma distribution can be easily derived for the chi-squared distribution. Specifically, the Wilson–Hilferty (WH) approximation states that if X follows a two-parameter gamma distribution, then the distribution of $X^{1/3}$ can be approximated by a normal distribution. We investigated the accuracy of this approximation and made a comparison with a more recent approximation due to Hawkins and Wixley (1986), which states that $X^{1/4}$ can be approximated by a normal distribution. Specifically, we compared the quantiles of a gamma distribution with those obtained using the WH and Hawkins–Wixley (HW) approximations. Such a comparison shows that the normal approximations are quite accurate, and the that WH approximation is preferred overall. Because prediction intervals, tolerance intervals, stress-strength reliability problems, and other aspects have been well investigated for the normal distribution, we can immediately adopt the corresponding results for the gamma distribution by applying the WH normal approximation. This is precisely what we have done in the present article. Furthermore, in each case we have evaluated the performance of the resulting approximate procedures using appropriate simulations. The overall conclusion is that the normal-based approximate procedures are quite accurate for the gamma distribution.

We want to point out that the normal approximations appear to be not useful for carrying out inferences concerning the gamma parameters. Rather, the approximations become relevant and useful for the computation of prediction intervals and tolerance intervals and for inference concerning stress-strength reliability parameters. Indeed, for such problems, the approximations provide a simple, accurate, and unified methodology for the two-parameter gamma distribution.

2. NORMAL APPROXIMATIONS AND THE PROPOSED METHOD

Let X_a denote a gamma variable with shape parameter a and scale parameter 1. Wilson and Hilferty (1931) provided a normal approximation to the cube root of a chi-squared variable using the moment-matching method. As X_a is distributed as $\chi^2_{2a}/2$, we explain the moment-matching approach for approximating the distribution of a gamma variate raised to the λ power. Toward this end, we note that the mean and variance of X_a^λ are given by

$$\mu_\lambda = \frac{\Gamma(a+\lambda)}{\Gamma(a)} \quad \text{and} \quad \sigma_\lambda^2 = \frac{\Gamma(a+2\lambda)}{\Gamma(a)} - \mu_\lambda^2. \tag{1}$$

Wilson and Hilferty's (1931) choice for λ is $\frac{1}{3}$, and in this case $X_a^{1/3} \sim N(\mu_{1/3}, \sigma_{1/3}^2)$ approximately (see also Hernandez and Johnson 1980, sec. 3.2, for a justification of the choice $\lambda = \frac{1}{3}$). Hawkins and Wixley (1986) have argued that the approximation can be improved for smaller values of a by using $\lambda = \frac{1}{4}$.

In what follows, we compare the foregoing two methods for approximating gamma quantiles. Note that both aforementioned articles provided approximations to the mean and variance in (1) to avoid computation of the gamma functions. However, to evaluate the merits of these approximations, we use the exact expressions in (1). Note that the WH method approximates the pth quantile of a gamma$(a, 1)$ distribution as $\max\{0, (\mu_{1/3} + z_p\sigma_{1/3})^3\}$, whereas the HW method approximates it as $[\max\{0, \mu_{1/4} + z_p\sigma_{1/4}\}]^4$.

Table 1 presents the quantiles of a gamma$(a, 1)$ distribution for values $a = 1, 2, 5$, and 20 based on an exact method (IMSL routine GAMIN) and the WH and HW approximations. We observe from the reported quantiles that for smaller values of a, the HW approximation performs better than the WH approximation at the lower quantiles, whereas the converse is true at the upper quantiles. In general, the WH approximation performs better than the HW approximation even for smaller values of a (e.g., $a = 1$ and 2, $p = .05$ and .10).

In view of the foregoing discussion and comparison, we use the WH cube root approximation to develop inferential procedures for a gamma distribution with the shape parameter a and the scale parameter b, say, gamma(a, b). We first note that if $X_{a,b}$ denotes such a gamma random variable, then $X_{a,b}$ is distributed as bX_a. The WH approximation now states that $X_{a,b}^{1/3}$ is approximately normal with mean and variance

$$\mu = \frac{b^{1/3}\Gamma(a + 1/3)}{\Gamma(a)}$$

and

$$\sigma^2 = \frac{b^{2/3}\Gamma(a + 2/3)}{\Gamma(a)} - \mu^2.$$

In the procedures developed in this article, we ignore the functional forms of μ and σ^2 as functions of a and b. Thus if $X_1, \ldots, X_n$ is a sample from a gamma(a, b) distribution, then we simply consider the transformed sample $Y_1 = X_1^{1/3}, \ldots, Y_n = X_n^{1/3}$ as a sample from a normal distribution with an arbitrary mean μ and arbitrary variance σ^2, and then develop tolerance intervals, prediction intervals, and so on, as though we have a sample from a normal distribution. In particular, this will result in procedures that are not functions of the complete sufficient statistics for the gamma distribution, namely the arithmetic mean and the geometric mean among $X_1, \ldots, X_n$. Here we ignore this aspect in view of the accuracy of the WH approximation and the simplicity of the normal-based procedures.

3. PREDICTION AND TOLERANCE LIMITS

Let $X_1, \ldots, X_n$ be a sample from a gamma(a, b) distribution. To apply the WH approximation, write $Y_i = X_i^{1/3}$, $i = 1, \ldots, n$. Let

$$\bar{Y} = \frac{1}{n}\sum_{i=1}^{n} Y_i \qquad \text{and} \qquad S_y^2 = \frac{1}{n-1}\sum_{i=1}^{n}(Y_i - \bar{Y})^2.$$

3.1 Prediction Limits

A $1 - \alpha$ UPL for a future measurement is given by

$$PL_u = \bar{Y} + t_{n-1,1-\alpha}S_y\sqrt{1 + \frac{1}{n}}, \tag{2}$$

where $t_{m,c}$ denotes the cth quantile of the Student t distribution with degrees of freedom df $= m$. Then $(PL_u)^3$ is an approximate $1 - \alpha$ UPL for a future observation from the gamma distribution. The lower prediction limit (LPL) can be obtained by replacing the plus sign in (2) by the minus sign. Note that if the LPL so computed turns out to be negative, then the LPL will be taken to be equal to 0; this convention also applies to the tolerance limits discussed later.

Table 1. WH and HW approximations to gamma$(a, 1)$ quantiles

	$a = 1$			$a = 2$			$a = 5$			$a = 20$		
p	Exact	WH	HW	Exact	WH	HW	Exact	WH	HW	Exact	WH	HW
.001	.0010	0	.0002	.0454	.0218	.0500	.7394	.6936	.8016	8.9582	8.9300	9.1401
.010	.0101	.0026	.0098	.1486	.1285	.1636	1.2791	1.2587	1.3354	11.0821	11.0760	11.1965
.050	.0513	.0463	.0568	.3554	.3513	.3732	1.9701	1.9683	2.0044	13.2547	13.2591	13.3078
.100	.1054	.1086	.1136	.5318	.5368	.5450	2.4326	2.4371	2.4501	14.5253	14.5320	14.5484
.200	.2231	.2381	.2298	.8244	.8371	.8255	3.0895	3.0977	3.0847	16.1725	16.1793	16.1644
.300	.3567	.3776	.3571	1.0973	1.1119	1.0865	3.6336	3.6417	3.6136	17.4360	17.4414	17.4107
.500	.6931	.7121	.6750	1.6783	1.6879	1.6479	4.6709	4.6746	4.6334	19.6677	19.6688	19.6271
.700	1.2040	1.2017	1.1687	2.4392	2.4346	2.4029	5.8904	5.8862	5.8552	22.0824	22.0782	22.0488
.800	1.6094	1.5858	1.5759	2.9943	2.9786	2.9704	6.7210	6.7121	6.7028	23.6343	23.6271	23.6185
.900	2.3026	2.2425	2.3059	3.8897	3.8591	3.9168	7.9936	7.9802	8.0267	25.9025	25.8926	25.9322
.950	2.9957	2.9047	3.0791	4.7439	4.7051	4.8554	9.1535	9.1405	9.2602	27.8792	27.8691	27.9677
.990	4.6052	4.4758	5.0350	6.6384	6.6090	7.0594	11.6046	11.6104	11.9514	31.8454	31.8433	32.1066
.999	6.9078	6.8149	8.2000	9.2334	9.2860	10.3403	14.7942	14.8666	15.6226	36.7010	36.7277	37.2742

3.2 Tolerance Limits

Let p denote the *content* and γ denote the *confidence level* of a tolerance interval. Such an interval should contain a proportion of at least p of the population with $100\gamma\%$ confidence, and we refer to the interval simply as a (p, γ) tolerance interval. (For details, especially in the context of the normal distribution, see Guttman 1970.)

If U is a (p, γ) upper tolerance limit based on $\bar{Y}$ and S_y^2, then U^3 is an approximate (p, γ) upper tolerance limit for the gamma(a, b) distribution. Note that for the normal distribution, a (p, γ) upper tolerance limit U is given by (see Guttman 1970)

$$U = \bar{Y} + c_1 S_y, \quad \text{with } c_1 = \frac{1}{\sqrt{n}} t_{n-1,\gamma}(z_p\sqrt{n}), \tag{3}$$

where z_p is the pth quantile of the standard normal distribution and $t_{m,\alpha}(\delta)$ denotes the αth quantile of a noncentral t distribution with df = m and noncentrality parameter δ. The quantity c_1 is referred to as a tolerance factor. A lower tolerance limit can be obtained by replacing the plus sign in (3) by the minus sign. Note that a (p, γ) upper tolerance limit also provides a $100\gamma\%$ upper confidence limit for the pth quantile. Similarly, a (p, γ) lower tolerance limit also provides a $100\gamma\%$ lower confidence limit for the $(1 - p)$th quantile. Thus, in particular, the upper and lower tolerance limits derived earlier also provide approximate confidence limits for the appropriate percentile of the gamma distribution. Such confidence limits were required in the application discussed by Ashkar and Ouarda (1998).

An exact two-sided tolerance interval for the normal distribution is given by $\bar{Y} \pm cS_y$, where c is the tolerance factor. Odeh (1978) computed the exact tolerance factor c for $n = 2(1)98, 100$; $p = .75, .90, .95, .975, .99, .995, .999$; and $\gamma = .5, .75, .90, .95, .975, .99, .995$. Eberhardt, Mee, and Reeve (1989) provided a Fortran program to compute the values of c. An online calculator from the UCLA Department of Statistics (available at *http://calculators.stat.ucla.edu/cdf*) can be used to compute the percentiles of a noncentral t distribution. The PC calculator that accompanies the book by Krishnamoorthy (2006) computes the one-sided limits and exact tolerance intervals for a normal distribution. This calculator is free and can be downloaded from *http://www.ucs.louisiana.edu/~kxk4695*.

An accurate approximation for the factor c, which is due to Wald and Wolfowitz (1946), is given by

$$c = \left(\frac{(n-1)\chi^2_{1,p}(1/n)}{\chi^2_{n-1,1-\gamma}} \right)^{1/2},$$

where $\chi^2_{1,p}(1/n)$ denotes the pth quantile of a noncentral chi-squared distribution with df $= 1$ and noncentrality parameter $1/n$ and $\chi^2_{n-1,1-\gamma}$ denotes the $(1 - \gamma)$th quantile of a central chi-squared distribution with df $= n - 1$, the df associated with the sample variance. This approximation is extremely satisfactory even for small sample sizes (as small as 3) if p and γ are $\geq .9$.

3.3 Assessing Survival Probability

Suppose that we want to estimate the survival probability (reliability) at time t based on a sample of lifetime data $X_1, \ldots, X_n$ from a gamma distribution. Because the survival probability is $S_t = P(X > t) = P(X^{1/3} > t^{1/3})$, the normal approximation method can be used to make inferences about S_t. Indeed, the one-sided tolerance limits discussed earlier can be used to find a lower confidence limit for S_t. For example, if a (p, γ) lower tolerance limit for the gamma(a, b) distribution is $> t^{1/3}$, then we can conclude that S_t is at least p with confidence γ. Consequently, an approximate one-sided $100\gamma\%$ lower confidence limit for S_t is given by

$$\max\left\{ p : \bar{Y} - \frac{1}{\sqrt{n}} t_{n-1,\gamma}(z_p\sqrt{n}) S_y > t^{1/3} \right\}. \tag{4}$$

Using the argument that the γth quantile of a noncentral t distribution is an increasing function of the noncentrality parameter, we can state that the lower tolerance limit is a decreasing function of p. Therefore, a lower confidence limit for S_t can be obtained as the solution (with respect to p) to the equation

$$t_{n-1,\gamma}(z_p\sqrt{n}) = \frac{\bar{Y} - t^{1/3}}{S_y/\sqrt{n}}. \tag{5}$$

Once n, γ, and the quantity on the right side of (5) are given, the foregoing equation can be solved using the PC calculators mentioned in the preceding section (see Sec. 3.5 for an example).

3.4 Validity of the Normal Approximation for Constructing Gamma Tolerance Limits

Using Monte Carlo simulation, we evaluated the accuracy of the foregoing approximate procedures for computing tolerance limits for the gamma distribution. We carried out the simulation study as follows. For a given shape parameter a, n random numbers were generated from the gamma$(a, 1)$ distribution. After taking the cube root, normal-based (p, γ) upper tolerance limits were constructed. The simulation was done with 100,000 runs, and the proportion of the 100,000 upper limits that were greater than the pth quantile of gamma$(a, 1)$ was computed. For a good procedure, this proportion (i.e., coverage probability) should be close to the nominal confidence level γ. The coverage probabilities of two-sided tolerance intervals were estimated similarly. Table 2 gives the estimated coverage probabilities of upper tolerance limits based on the WH approximation for $n = 3, 7$, and 12; (p, γ) in $\{.9, .95, .99\}$; and a few values of the shape parameter ranging from .5 to 9. Table 3 gives estimated coverage probabilities of two-sided tolerance intervals. It is evident from the tabulated values in Tables 2 and 3 that the WH approximation provides satisfactory coverage probabilities except when a is small.

It should be noted that a (p, γ) upper tolerance limit based on the cube root transformation is indeed a γ-level upper confidence limit for $A_p = (\mu_{1/3} + z_p\sigma_{1/3})^3$, where μ_λ and σ_λ are as defined in (1), which is an approximation to the true upper quantile Q_p of a gamma$(a, 1)$ distribution. Because A_p is smaller than Q_p for smaller values of a (see, e.g., the quantiles in Table 1 for $a = 1$ and $p \geq .80$), in this case the coverage probabilities of an upper tolerance limit are expected to be smaller than the nominal level γ. On the other hand, the coverage probabilities of a lower tolerance limit are expected to be larger than the nominal level γ for smaller values of a, because in this case lower approximate quantiles are smaller than the corresponding

Table 2. Estimated coverage probabilities of the upper tolerance limits based on the WH approximation with confidence levels $\gamma_1 = .90$, $\gamma_2 = .95$, and $\gamma_3 = .99$

	$p = .90$			$p = .95$			$p = .99$		
a	$\widehat{\gamma}_1$	$\widehat{\gamma}_2$	$\widehat{\gamma}_3$	$\widehat{\gamma}_1$	$\widehat{\gamma}_2$	$\widehat{\gamma}_3$	$\widehat{\gamma}_1$	$\widehat{\gamma}_2$	$\widehat{\gamma}_3$
$n = 3$									
.5	.875	.935	.987	.878	.936	.987	.881	.939	.987
1.5	.898	.949	.990	.898	.949	.990	.898	.949	.989
3.0	.899	.949	.990	.900	.950	.991	.900	.951	.990
9.0	.901	.950	.990	.900	.951	.990	.900	.950	.990
$n = 7$									
.5	.859	.923	.982	.868	.923	.982	.862	.926	.984
1.5	.893	.946	.989	.895	.948	.990	.898	.949	.990
3.0	.898	.949	.990	.901	.950	.990	.901	.950	.990
9.0	.899	.949	.990	.901	.950	.990	.902	.951	.991
$n = 12$									
.5	.857	.925	.988	.865	.932	.978	.878	.937	.980
1.5	.891	.943	.988	.892	.945	.989	.899	.949	.990
3.0	.897	.949	.990	.899	.949	.990	.902	.952	.990
9.0	.901	.950	.990	.900	.949	.989	.903	.952	.990

true gamma quantiles (see the quantiles in Table 1 for $a = 1$ and 2 and $p \leq .05$).

The foregoing findings indicate that two-sided tolerance intervals based on the WH approximation should be satisfactory regardless of the values of a. This is also evident from the numerical results in Table 3. More detailed tables (similar to Tables 2 and 3), available in the work of Mukherjee (2007), further confirm our findings.

3.5 An Example

In this example we use the data reported by Gibbons (1994, p. 261), which were also used for illustrative purposes by Aryal et al. (2006). The measurements represent alkalinity concentrations in groundwater obtained from a "greenfield" site (i.e., the site of a waste disposal landfill before disposal of waste) and are reproduced here in Table 4.

To apply the WH approximation, define $Y_i = X_i^{1/3}$, $i = 1, \ldots, 27$. The mean is $\bar{Y} = 3.8274$, and the standard deviation is $S_y = .4298$.

Tolerance Limits. Table 5 presents 95% one-sided tolerance limits and two-sided tolerance intervals along with the corresponding tolerance factors.

Prediction Limits. Using formula (2), we computed the 90% prediction limit as 85.353 mg/L and the 95% prediction limit as 95.690 mg/L.

Probability of Exceeding a Threshold Value. Suppose that we want to find a 95% lower limit for the probability that a sample alkalinity concentration exceeds 41 mg/L, that is,

Table 3. Estimated coverage probabilities of two-sided tolerance intervals based on the WH approximation with confidence levels $\gamma_1 = .90$, $\gamma_2 = .95$, and $\gamma_3 = .99$

	$p = .90$			$p = .95$			$p = .99$		
a	$\widehat{\gamma}_1$	$\widehat{\gamma}_2$	$\widehat{\gamma}_3$	$\widehat{\gamma}_1$	$\widehat{\gamma}_2$	$\widehat{\gamma}_3$	$\widehat{\gamma}_1$	$\widehat{\gamma}_2$	$\widehat{\gamma}_3$
$n = 3$									
.5	.904	.951	.990	.907	.953	.990	.911	.955	.991
1.5	.904	.952	.991	.904	.952	.990	.910	.955	.991
3.0	.901	.952	.991	.905	.952	.991	.907	.953	.990
9.0	.903	.952	.991	.902	.952	.990	.904	.953	.990
$n = 7$									
.5	.910	.953	.990	.913	.955	.990	.917	.957	.991
1.5	.903	.952	.990	.907	.953	.991	.917	.959	.991
3.0	.900	.950	.990	.903	.951	.990	.910	.955	.991
9.0	.899	.949	.990	.901	.950	.989	.903	.951	.990
$n = 12$									
.5	.920	.958	.991	.922	.959	.991	.917	.956	.991
1.5	.903	.952	.990	.908	.955	.991	.924	.962	.993
3.0	.901	.950	.990	.903	.951	.990	.912	.956	.992
9.0	.898	.948	.989	.901	.950	.990	.902	.950	.990

Table 4. Alkalinity concentrations in groundwater

X:	58	82	42	28	118	96	49	54	42	51
	66	89	40	51	54	55	59	42	39	40
	60	63	59	70	32	52	79			

$P(Y > 41^{1/3})$. Using (5), we get

$$t_{26,.95}(z_p\sqrt{27}) = \frac{3.8274 - 41^{1/3}}{.4298/\sqrt{27}} = 4.584.$$

Solving for the noncentrality parameter, we get $z_p\sqrt{27} = 2.601$. This implies that $z_p = .5006$ or $p = .692$. Thus the probability that alkalinity concentration exceeds 41 mg/L in a sample is at least .692 with confidence 95%.

4. ONE–SIDED PREDICTION LIMITS FOR AT LEAST l OF m OBSERVATIONS FROM A GAMMA DISTRIBUTION AT EACH OF r LOCATIONS

As already mentioned in Section 1, this problem arises, for example, in groundwater quality detection in the vicinity of HWMFs. For example, in groundwater quality monitoring near waste disposal facilities, a series of m sample observations from each of r monitoring wells located hydraulically downgradient of the HWMF are often compared with statistical prediction limits based on n measurements obtained from one or more upgradient sampling locations. Given a random sample of size n, the statistical problem is to construct an UPL so that l of m sample values are below the limit at each of r downgradient monitoring wells. Discussions of this problem and strategies for monitoring groundwater quality have been provided by Davis and McNichols (1987), Gibbons (1994), Gibbons and Coleman (2001), and Bhaumik and Gibbons (2006). Davis and McNichols (1987) developed an exact method for constructing the UPL assuming normality, and Bhaumik and Gibbons (2006) proposed an approximate method assuming a gamma distribution.

4.1 Normal-Based Upper Prediction Limit

We outline Davis and McNichols' (1987) approach for normally distributed samples. Let $Y_1, \ldots, Y_n$ be a sample from a normal population, and let $\bar{Y}$ and S_y denote the mean and standard deviation of $Y_1, \ldots, Y_n$. Then the UPL is of the form $\bar{Y} + k_u S_y$, where k_u is chosen so that at least l of m future observations are below $\bar{Y} + k_u S_y$ on each of r locations, with probability γ. Davis and McNichols showed that the factor k_u for constructing a γ-level UPL can be obtained as the solution of

$$\int_0^1 F\big(\sqrt{n}k_u; n-1, \sqrt{\pi}\Phi^{-1}(x)\big) r(I(x; l, m+1-l))^{r-1} \times \frac{x^{l-1}(1-x)^{m-l}}{B(l, m+1-l)}\, dx = \gamma, \quad (6)$$

where $F(x; \nu, \delta)$ denotes the cumulative distribution function (cdf) of the noncentral t random variable with df $= \nu$ and the noncentrality parameter δ, $B(a, b)$ denotes the usual beta function, and $I(x; a, b)$ denotes the cdf of a beta random variable with parameters a and b. Davis and McNichols tabulated values of k_u for some selected values of (n, r, l) and for $\gamma = .95$.

Assuming that $X_1, \ldots, X_n$ is a sample from a gamma distribution, we apply the WH approximation. Setting $Y_i = X_i^{1/3}$, $i = 1, \ldots, n$, we first construct the normal-based UPL based on Y_i's, and we can then use the cubic power of the UPL as a γ-level UPL limit for the gamma distribution. Apart from simplicity, another advantage of the normal approach is that the factor k_u does not depend on any sample statistic, whereas the factor k based on the approach of Bhaumik and Gibbons depends on the MLE of a, which makes the tabulation of tolerance factors difficult. Another advantage is that we can compute the UPL using the already tabulated values of k_u given by Davis and McNichols (1987).

4.2 Accuracy of the Normal Approximation

To appraise the accuracy of the normal-based UPL given previously using the WH approximation, we estimated the coverage probabilities using Monte Carlo simulation. We first generated n random numbers from a gamma$(a, 1)$ distribution. After transforming the random numbers by taking cube root, we constructed normal-based UPL U. Then we generated r sets of m random numbers, say, X_{ij}, $j = 1, \ldots, m$, $i = 1, \ldots, r$, from the gamma$(a, 1)$ distribution. We computed $X^* = \max\{X_{1(l)}, \ldots, X_{r(l)}\}$, where $X_{i(l)}$ is the lth smallest of X_{ij} for each i. We repeated the procedure 100,000 times, and used the proportion of times X^* less than U^3 as an estimate of the coverage probability. The accuracy of the normal approximation can be judged by the closeness of the coverage probabilities to the nominal level γ.

Table 6 gives the estimated coverage probabilities of 95% UPLs for values of the shape parameter a from the set $\{.4, .8, 1.3, 2, 5\}$. The values of (r, l, m, k_u) are selected from Table 2 ($n = 6$) and table 4 ($n = 15$) of Davis and McNichols (1987). We observe from the values in Table 5 that the coverage probabilities are very close to (or coincide with) the nominal confidence level .95, showing the accuracy of the WH normal approximation for this problem.

4.3 An Example

To illustrate our method described earlier, we consider the data given in table 1 of Bhaumik and Gibbons (2006). The data, reproduced here in Table 7, represent vinyl chloride concentrations collected from clean upgradient monitoring wells. A quantile–quantile plot of Bhaumik and Gibbons showed an excellent fit of these data to a gamma distribution.

Table 5. Tolerance limits based on the WH approximation

(p, γ)	Factor for one-sided	Lower limit	Upper limit	Factor for two-sided	Two-sided tolerance interval
(.9, .95)	1.8114	28.341	97.7129	2.1841	(24.104, 108.27)
(.95, .95)	2.2601	23.296	110.507	2.6011	(19.890, 120.95)
(.99, .95)	3.1165	15.400	137.94	3.4146	(13.141, 148.46)

Table 6. Coverage probabilities of UPLs that contain at least l of m observations at each of r locations

$n=6$									$n=12$								
				a									a				
r	l	m	k_u	.4	.8	1.3	2	5	r	l	m	k_u	.4	.8	1.3	2	5
1	1	2	1.07	.95	.95	.95	.95	.95	1	1	2	.87	.95	.95	.95	.95	.95
1	1	3	.58	.95	.95	.95	.95	.95	1	1	3	.43	.96	.95	.95	.95	.95
1	2	3	1.58	.94	.95	.95	.95	.95	1	2	4	.81	.95	.95	.95	.95	.95
1	1	4	.29	.96	.95	.95	.95	.95	1	2	6	.32	.96	.95	.95	.95	.95
1	2	5	.72	.95	.95	.95	.95	.95	2	1	2	1.15	.94	.95	.95	.95	.95
1	2	6	.50	.96	.95	.95	.95	.95	2	1	3	.66	.95	.95	.95	.95	.95
1	3	6	1.01	.95	.95	.95	.95	.95	2	2	5	.72	.95	.95	.95	.95	.95
2	1	2	1.43	.94	.95	.95	.95	.95	2	2	6	.50	.96	.95	.95	.95	.95
2	1	3	.87	.95	.95	.95	.95	.95	4	1	2	1.41	.93	.94	.95	.95	.95
2	2	3	1.96	.93	.94	.95	.95	.95	4	1	3	.88	.95	.95	.95	.95	.95
2	1	4	.54	.95	.95	.95	.95	.95	4	1	4	.56	.95	.95	.95	.95	.95
2	2	4	1.35	.94	.95	.95	.95	.95	4	2	5	.90	.95	.95	.95	.95	.95
2	1	5	.32	.96	.95	.95	.95	.95	4	2	6	.67	.95	.95	.95	.95	.95
2	2	5	.98	.95	.95	.95	.95	.95	8	1	2	1.66	.93	.94	.95	.95	.95
2	2	6	.73	.95	.95	.95	.95	.95	8	1	3	1.09	.94	.95	.95	.95	.95
2	3	6	1.28	.94	.95	.95	.95	.95	8	1	4	.74	.95	.95	.95	.95	.95
4	1	2	1.80	.93	.94	.95	.95	.95	8	1	5	.50	.96	.95	.95	.95	.95
8	1	3	1.45	.94	.95	.95	.95	.95	8	2	6	.82	.95	.95	.95	.95	.95
8	1	6	.54	.96	.95	.95	.95	.95	16	1	2	1.90	.92	.94	.95	.95	.95
8	2	6	1.17	.94	.95	.95	.95	.95	16	1	3	1.28	.93	.94	.95	.95	.95
16	1	3	1.73	.93	.94	.95	.95	.95	16	1	4	.91	.95	.95	.95	.95	.95
16	2	5	1.73	.93	.94	.95	.95	.95	16	2	4	1.63	.92	.94	.95	.95	.95
16	1	4	1.27	.94	.95	.95	.95	.95	16	2	5	1.24	.93	.94	.95	.95	.95
16	1	6	.73	.95	.95	.95	.95	.95	16	1	6	.47	.96	.95	.95	.95	.95

We computed the mean and standard deviation of the cube root data as $\bar{Y} = 1.1022$ and $S_y = .3999$. The critical values k_u for computing UPLs when $n = 34$ were not given by Davis and McNichols (1987), so we computed them using the integral equation in (6). To compare the normal-based UPLs with those of Bhaumik and Gibbons, we chose the same combinations of (r, l, m) as given in their article. The normal-based approximate prediction limits, along with those of Bhaumik and Gibbons (2006), are given in Table 8.

For all of the cases considered in Table 8, the normal-based approximate limits using the WH approximation and the approximate limits due to Bhaumik and Gibbons (2006) are in close agreement.

5. STRESS–STRENGTH RELIABILITY INVOLVING TWO INDEPENDENT GAMMA RANDOM VARIABLES

Let $X_1 \sim \text{gamma}(a_1, b_1)$ independent of $X_2 \sim \text{gamma}(a_2, b_2)$. If X_1 is a strength variable and X_2 is a stress variable, then the reliability parameter is given by

$$R = P(X_1 > X_2) = P\left(F_{2a_1,2a_2} > \frac{a_2 b_2}{a_1 b_1}\right), \tag{7}$$

where $F_{m,n}$ denotes the F random variable with df $= (m, n)$. If a_1 and a_2 are known, then inferential procedures can be obtained readily (see Kotz, Lumelskii, and Pensky 2003, p. 114). No exact procedure is available if a_1 and a_2 are unknown.

Suppose that we are interested in testing

$$H_0 : R \le R_0 \qquad \text{versus} \qquad H_a : R > R_0, \tag{8}$$

where R_0 is a specified probability. To use the WH approximation, we note that $R = P(X_1^{1/3} - X_2^{1/3} > 0)$. Thus a level-$\alpha$ test rejects the null hypothesis when a $(R_0, 1-\alpha)$ lower tolerance limit for the distribution of $X_1^{1/3} - X_2^{1/3}$ is positive. Because $X_1^{1/3}$ and $X_2^{1/3}$ are approximately normally distributed, normal-based tolerance limits for $X_1^{1/3} - X_2^{1/3}$ can be used to test the above hypotheses.

Table 7. Vinyl chloride data from clean upgradient groundwater monitoring wells in μg/L

5.1	2.4	.4	.5	2.5	.1	6.8	1.2	.5	.6
5.3	2.3	1.8	1.2	1.3	1.1	.9	3.2	1.0	.9
.4	.6	8.0	.4	2.7	.2	2.0	.2	.5	.8
2.0	2.9	.1	4.0						

Table 8. 95% UPLs for the vinyl chloride data

r	l	m	k_u	$UPL = \bar{Y} + k_u S_y$	$(UPL)^3$	Bhaumik–Gibbons
1	1	2	.807	1.425	2.893	2.931
10	1	2	1.577	1.733	5.203	5.224
10	1	3	1.033	1.515	3.479	3.521
10	2	3	1.879	1.854	6.369	6.330

5.1 One-Sided Tolerance Limits for the Distribution of the Difference Between Two Independent Normal Variables

Approximate methods of constructing one-sided tolerance limits (or estimating the reliability) for stress-strength reliability involving two independent normal random variables have been given Hall (1984) and Reiser and Guttman (1986). Recently, Guo and Krishnamoorthy (2004) suggested a modification of Hall's approach that provides satisfactory tolerance limits even for small samples. We outline their method here.

We now explain the application of the WH normal approximation. Let $\bar{Y}_i$ and S_i^2 denote the mean and variance of $Y_{i1}, \ldots, Y_{in_i}$, where $Y_{ij} = X_{ij}^{1/3}$, $j = 1, \ldots, n_i$, $i = 1, 2$. Furthermore, let

$$\hat{q}_1 = \frac{(n_2-3)S_1^2}{(n_2-1)S_2^2},$$
$$m_1 = \frac{n_1(1+\hat{q}_1)}{\hat{q}_1 + n_1/n_2}, \qquad \text{and} \tag{9}$$
$$f_1 = \frac{(n_1-1)(\hat{q}_1+1)^2}{\hat{q}_1^2 + (n_1-1)/(n_2-1)}.$$

Then a $(R_0, 1-\alpha)$ lower tolerance limit due to Hall (1984) is given by

$$L_{1R_0} = \bar{Y}_d - t_{f_1,1-\alpha}\left(z_{R_0}\sqrt{m_1}\right)\sqrt{\frac{S_1^2+S_2^2}{m_1}}, \tag{10}$$

where $\bar{Y}_d = \bar{Y}_1 - \bar{Y}_2$. Using the fact that $t_m(\delta)$ is stochastically increasing with respect to δ, it can be easily verified that a $1-\alpha$ lower limit for R is the value of R_{1L} that satisfies

$$t_{f_1,1-\alpha}\left(z_{R_{1L}}\sqrt{m_1}\right) = \frac{\sqrt{m_1}\bar{Y}_d}{\sqrt{S_1^2+S_2^2}}. \tag{11}$$

Note that the H_0 in (8) will be rejected if $L_{1R_0} > 0$ or, equivalently,

$$t_{f_1,1-\alpha}\left(z_{R_0}\sqrt{m_1}\right) < \frac{\sqrt{m_1}\bar{Y}_d}{\sqrt{S_1^2+S_2^2}},$$

or, equivalently, if the p value is

$$P_1 = P\left(t_{f_1}\left(z_{R_0}\sqrt{m_1}\right) > \frac{\sqrt{m_1}\bar{Y}_d}{\sqrt{S_1^2+S_2^2}}\right) < \alpha. \tag{12}$$

Because the foregoing quantities depend on the labeling of the variables, we can get other tolerance limits, R_{2L}, by defining

$$\hat{q}_2 = \frac{(n_1-3)S_2^2}{(n_1-1)S_1^2},$$
$$m_2 = \frac{n_2(1+\hat{q}_2)}{\hat{q}_2 + n_2/n_1}, \qquad \text{and} \tag{13}$$
$$f_2 = \frac{(n_2-1)(\hat{q}_2+1)^2}{\hat{q}_2^2 + (n_2-1)/(n_1-1)}.$$

Then R_{2L} that satisfies

$$t_{f_2,\gamma}\left(z_{R_{2L}}\sqrt{m_2}\right) = \frac{\sqrt{m_2}\bar{Y}_d}{\sqrt{S_1^2+S_2^2}}, \tag{14}$$

is also an approximate $(1-\alpha)$-level lower limit for R. Based on extensive simulation studies, Guo and Krishnamoorthy (2004) found that

$$R_L = \min\{R_{1L}, R_{2L}\} \tag{15}$$

is a satisfactory $1-\alpha$ lower confidence limit for R in terms of providing better coverage probabilities compared with both R_{1L} and R_{2L}.

For hypothesis testing, we can compute another p value, say, P_2, by replacing (m_1, f_1) in (12) by (m_2, f_2). It turns out that the test that rejects H_0 when $\max\{P_1, P_2\} < \alpha$ has better size properties than the test based on either P_1 or P_2.

5.2 Monte Carlo Estimates of the Sizes

To assess the validity of the WH approximation for the stress-strength model involving gamma random variables, we estimated the sizes of the test based on $\max\{P_1, P_2\}$. We generated n_1 random numbers from a gamma$(a_1, 1)$ distribution and n_2 random numbers from a gamma$(a_2, 1)$ distribution. After taking cube root of the generated samples, we computed the p value as $\max\{P_1, P_2\}$, following the procedure outlined earlier. We repeated this 10,000 times, and used the proportion of the p values which were less than α as an estimate of the size of the test. The estimated sizes [as function of R in (7)] are given in Table 9 when $a_2 = 1$. We observe from the estimated sizes that for the values of R and sample sizes considered here, the WH normal approximation procedure is very satisfactory.

We also estimated the sizes for a few other values of a_2 for the sample sizes considered in Table 9. The estimated sizes were similar to those given in Table 9, and so we do not report them here.

5.3 An Illustrative Example

We use the simulated data given by Basu (1981) to illustrate the computation of a lower confidence limit for the stress-strength reliability parameter R. The data are reproduced here in Table 10. The same data were also used later by other authors (e.g., Reiser and Rocke 1993) to find a lower limit for R.

After taking the cube root transformation, we get $\bar{Y}_1 = 1.02135$, $S_1^2 = .110025$, $\bar{Y}_2 = .35363$, and $S_2^2 = .006823$. Other quantities are

$$n_1 = 15, \qquad n_2 = 15,$$
$$\hat{q}_1 = 13.82174, \qquad \hat{q}_2 = .053155,$$
$$\hat{m}_1 = 15, \qquad \hat{m}_2 = 15,$$

and

$$\hat{f}_1 = 16.01525, \qquad \hat{f}_2 = 15.4841.$$

To find a 95% lower confidence limit for R, we used (11) to get $t_{16.0153,.95}(z_{R_{1L}}\sqrt{15}) = 7.565$. Solving this equation for the noncentrality parameter, we get $z_{R_{1L}}\sqrt{15} = 4.7611$. This implies that $Z_{R_{1L}} = 1.2293$ or $R_{1L} = .891$. Similarly, using (14),

Table 9. Sizes of the test for $H_0: R \le R_0$ versus $H_a: R > R_0$; $\alpha = .05$

$R = R_0$	$n_1 = n_2 = 10$	$n_1 = 10, n_2 = 15$	$n_1 = n_2 = 15$	$n_1 = 20, n_2 = 10$	$n_1 = n_2 = 20$
.129	.058	.063	.056	.054	.052
.293	.056	.049	.054	.046	.054
.500	.050	.045	.048	.045	.049
.646	.049	.050	.051	.050	.047
.750	.052	.052	.048	.049	.050
.823	.050	.051	.050	.052	.051
.875	.053	.050	.054	.049	.053
.912	.050	.048	.052	.051	.055
.938	.052	.048	.052	.048	.053
.956	.049	.048	.054	.052	.054
.969	.047	.050	.051	.051	.051
.984	.044	.047	.050	.046	.052
.992	.048	.045	.055	.054	.056
.996	.050	.046	.053	.049	.054

we get $R_{2L} = .889$. Therefore, $\min\{R_{1L}, R_{2l}\} = .889$ is our 95% lower confidence limit for R.

Reiser and Rocke (1993) computed the lower limits using two recommended procedures, .898 (delta method on logits) and .904 (bootstrap percentile). Note the closeness of our lower limits with these two values.

6. CONCLUDING REMARKS

The WH approximation is a simple normal approximation for a gamma distribution. In this article we have exploited this approximation for several problems for the gamma distribution, including computation of prediction intervals and tolerance intervals and inference on the stress-strength reliability parameter. Because solutions are already available for the corresponding problems in the normal case, the approximation has allowed us to adopt these solutions for the gamma distribution in a straightforward manner. In each case, we also have numerically investigated the accuracy of our approximate solutions. Our approach has resulted in a unified methodology for addressing various problems when data can be modeled using the gamma distribution, most notably in the context of environmental applications, applications in industrial hygiene, and applications to lifetime data analysis.

ACKNOWLEDGMENTS

The authors are grateful to a referee and an associate editor for several helpful comments. In particular, the associate editors' comments resulted in the clarification of several ideas, and the associate editor also drew the author's attention to the work of Hernandez and Johnson (1980) and Hawkins and Wixley (1986). The research of Krishnamoorthy and Mathew was supported by National Institute of Occupational Safety and Health grant R01-OH03628.

[Received August 2006. Revised March 2007.]

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Table 10. Basu's (1981) simulated data

X_1	1.7700	.9457	1.8985	2.6121	1.0929	.0362	1.0615	2.3895	.0982	.7971
	.8316	3.2304	.4373	2.5648	.6377					
X_2	.0352	.0397	.0677	.0233	.0873	.1156	.0286	.0200	.0793	.0072
	.0245	.0251	.0469	.0838	.0796					

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