

Inference for functions of parameters in discrete distributions based on fiducial approach: binomial and Poisson cases

K. Krishnamoorthy* and Meesook Lee

*Department of Mathematics, University of Louisiana at Lafayette
Lafayette, LA 70504, U.S.A*

ABSTRACT

In this article, we propose a simple method of constructing confidence intervals for a function of binomial success probabilities and for a function of Poisson means. The method involves finding an approximate *fiducial quantity* (FQ) for the parameters of interest. A FQ for a function of several parameters can be obtained by substitution. For the binomial case, the fiducial approach is illustrated for constructing confidence intervals for the relative risk and the ratio of odds. Fiducial inferential procedures are also provided for estimating functions of several Poisson parameters. In particular, fiducial inferential approach is illustrated for interval estimating the ratio of two Poisson means and for a weighted sum of several Poisson means. Simple approximations to the distributions of the FQs are also given for some problems. The merits of the procedures are evaluated by comparing them with those of existing asymptotic methods with respect to coverage probabilities, and in some cases, expected widths. Comparison studies indicate that the fiducial confidence intervals are very satisfactory, and they are comparable or better than some available asymptotic methods. The fiducial method is easy to use and is applicable to find confidence intervals for many commonly used summary indices. Some examples are used to illustrate and compare the results of fiducial approach with those of other available asymptotic methods.

Keywords: Clopper-Pearson interval; Conditional approach; Fiducial test; Likelihood method; Logit confidence interval; Negative binomial; Score interval

Corresponding author. Tel.: +1 337 482 5283; fax: +1 337 482 5346. E-mail address: krishna@louisiana.edu (K. Krishnamoorthy).

1. Introduction

Many summary indices in comparative clinical trials can be expressed as a function of independent binomial success probabilities or of Poisson means. For instance, if p_e denotes the probability of an event (such as death or adverse symptoms) in the exposed group of individuals, and p_c denotes the same in the control group, then the ratio p_e/p_c is a measure of relative risk for the exposed group. The ratio of odds is defined by $p_e(1-p_c)/(p_c(1-p_e))$ which represents the relative odds of event in the exposed group compared to that in the control group. Another measure of importance is the relative difference defined by $(p_2 - p_1)/(1 - p_1)$, where p_1 denotes the proportion of patients who “improve” when treatment 1 is administered and p_2 is the corresponding proportion of patients treated with treatment 2 (Fleiss, 1973). There are other applications where it is of interest to estimate the product of two or more independent binomial probabilities (Buehler, 1957 and Harris, 1971). Another well-known problem of interest is comparison of two proportions via the difference $p_1 - p_2$. In their recent article, Brown and Li (2005) compared several interval estimation procedures for $p_1 - p_2$ with respect to coverage probabilities and recommended some for practical applications.

The problem of estimating a function of several Poisson parameters arises in many applications. For instance, the ratio of two independent Poisson means is used to compare the incident rates of a disease in a treatment group and a control group, where the incident rate is defined as the number of events observed divided by the time at risk during the observed period. A weighted sum of independent Poisson parameters is commonly used to assess the standardized mortality rates (Dobson et al. 1991). Confidence intervals (CIs) for a product of Poisson parameters are also used to estimate the reliability of a parallel system (Harris, 1971).

Wald method and the asymptotic method based on the likelihood approach are commonly used for many of the aforementioned problems. The Wald confidence intervals (CIs) that use the standard errors obtained with delta method often work poorly for small to moderate sample size (often conservative). Closed form formulas for score or likelihood CIs are not available, and some iterative methods are required to evaluate them. Even though the likelihood ratio or score CIs are computationally more complex than Wald intervals, these intervals often perform better than Wald CIs. For estimating the odds ratio, one can use the exact conditional method which is based on the power function of Fisher’s exact test for testing equality of two binomial proportions (Thomas and Gart, 1977). This type of conditional confidence limits are usually too conservative yielding CIs that are unnecessarily wide.

In this article, we propose simple inferential procedures for some of the aforementioned problems based on Fisher’s fiducial argument. The idea of fiducial probability and fiducial inference was introduced by Fisher (1930, 1935). As mentioned in Zabell (1992) fiducial

inference has been a subject of severe criticisms concerning the interpretation of fiducial distribution. Efron (1998) remarked in Section 8 of his paper that fiducial distribution is generally considered to be Fisher's biggest blunder; however, he concluded the section by stating "Maybe Fisher's biggest blunder will become a big hit in the 21st century!" We observed from these two review articles just cited that many criticisms about the fiducial approach are philosophical than practical. In particular, fiducial approach is a useful tool to find solutions to many complex problems with satisfactory frequentist properties. In fact, fiducial inference in many situations are now well accepted. For example, Clopper-Pearson's (1934) fiducial limits for a binomial proportion and Garwood's (1936) fiducial limits for a Poisson mean are now commonly referred to as the exact (in the frequentist sense) confidence intervals. Furthermore, the exact conditional CI for the ratio of two Poisson means by Chapman (1952), and the exact CI for the correlation coefficient of a bivariate normal distribution (see Anderson, 1984, Section 4.2) are also fiducial intervals. For other situations where fiducial inference led to exact CIs, see Dawid and Stone (1982).

Fiducial inference appears to have made a resurgence recently under the label of *generalized inference* by Tsui and Weerahandi (1989) and Weerahandi (1993). Hannig, Iyer and Patterson (2006) have noted that the generalized variable procedures are a special case of fiducial inference procedures, and are asymptotically exact in many situations. For more details and applications of the generalized inference, see the books by Weerahandi (1995, 2004). For the continuous case, the fiducial approach has been used successfully to estimate or to test a function of parameters where ordinary pivotal quantities are available for individual parameters (e.g., lognormal mean, normal quantiles and quantiles in one-way random model). We note that no ordinary pivotal quantity is available to make inference on a binomial success probability or for a Poisson mean, and so the methods for the continuous case can not be extended to these discrete distributions. Thus, we shall explore an alternative approximate approach on the basis of our observation of the method by Cox (1953) for constructing confidence limits for the ratio of two independent Poisson means.

The rest of the article is organized as follows. In the next section we describe fiducial quantities for a binomial success probability and for a Poisson mean. In Section 3, we provide approximate fiducial CIs for the relative risk and the ratio of odds. For each of the problems, we evaluate the coverage probabilities of the fiducial CI and compare them with those of popular CIs based on asymptotic methods, and provide illustrative examples. In Section 4, we provide fiducial approaches for estimating the ratio of two Poisson means and for estimating a weighted sum of Poisson means. The validity of the fiducial approach is evaluated using Monte Carlo simulation, and the results are illustrated using some examples. A test procedure based on a fiducial quantity is outlined in Section 5. Applications of the fiducial approach to other problems are given in Section 6. Some concluding remarks and limitations of the fiducial approach are given in Section 7.

2. Fiducial quantities and their validity

2.1 A Fiducial Quantity for a Binomial p

Let $X \sim \text{binomial}(n, p)$, and let $B_{a,b}$ denote the beta random variable with shape parameters a and b . It is well-known that, for an observed value k of X , $P(X \geq k|n, p) = P(B_{k,n-k+1} \leq p)$ and $P(X \leq k|n, p) = P(B_{k+1,n-k} \geq p)$. On the basis of this relation, Stevens (1950) pointed out that there is a pair of fiducial distributions for p , namely, $B_{k,n-k+1}$ for setting lower limit for p and $B_{k+1,n-k}$ for setting upper limit for p . The Clopper-Pearson (1934) CI based on this pair of fiducial variables is given by $(B_{X,n-X+1;\alpha/2}, B_{X+1,n-X;1-\alpha/2})$. Instead of having two fiducial variables, a random quantity that is “stochastically between” $B_{k,n-k+1}$ and $B_{k+1,n-k}$ can be used as a single approximate fiducial variable for p . On the basis of Cai’s (2005) result, a simple choice is $B_{k+1/2,n-k+1/2}$. The term “stochastically between” means that $P(B_{k,n-k+1} > t) \leq P(B_{k+1/2,n-k+1/2} > t) \leq P(B_{k+1,n-k} > t)$, $k = 1, \dots, n-1$, $0 < t < 1$, with strict inequality for some t . It has been noted in the literature (e.g., Brown, Cai and Das Gupta, 2001 and Cai, 2005) that $B_{k+1/2,n-k+1/2}$ is the posterior distribution with the Jeffreys prior $B_{1/2,1/2}$. The CI based on this approximate fiducial variable is given by

$$(B_{d_1/2,d_2/2;\alpha/2}, B_{d_1/2,d_2/2;1-\alpha/2}) = \left(\left(1 + \frac{d_2}{d_1} F_{d_2,d_1;1-\alpha/2}\right)^{-1}, \left(1 + \frac{d_2}{d_1} F_{d_2,d_1;\alpha/2}\right)^{-1} \right), \quad (1)$$

where $d_1 = 2X + 1$, $d_2 = 2n - 2X + 1$, $B_{a,b;\alpha}$ denotes the α quantile of a beta(a, b) distribution and $F_{f_1,f_2;\alpha}$ denotes the α quantile of an F_{f_1,f_2} distribution. Note that the above CI is also the $1 - \alpha$ Bayesian credible interval with the Jeffreys prior $B_{1/2,1/2}$.

The coverage properties of the CI (1) have been well studied by Brown et al. (2001) and Cai (2005) has studied its properties for one-sided interval estimation. Both articles showed that the above CI is very satisfactory in maintaining coverage probabilities and is better than or comparable to other well-known procedures, in particular, the popular score CI

$$\left(\frac{\hat{p} + \frac{z_{1-\alpha/2}^2}{2n}}{1 + \frac{z_{1-\alpha/2}^2}{n}} \right) \pm \frac{\frac{z_{1-\alpha/2}}{\sqrt{n}} \sqrt{\hat{p}(1-\hat{p}) + z_{1-\alpha/2}^2/(4n)}}{1 + \frac{z_{1-\alpha/2}^2}{n}}, \quad (2)$$

where $\hat{p} = X/n$ and z_α is the α quantile of the standard normal distribution.

2.2 A Fiducial Quantity for a Poisson Mean

An approximate fiducial quantity for a Poisson mean can be obtained by using the argument similar to the one given for the binomial case. Let Y_1, \dots, Y_n be a sample from a Poisson(λ) distribution, and let $Y = \sum_{i=1}^n Y_i$ so that $Y \sim \text{Poisson}(n\lambda)$. For an observed

value m of Y , we have $P(Y \geq m|\lambda) = P\left(\frac{\chi_{2m}^2}{2n} < \lambda\right)$ and $P(Y \leq m) = P\left(\frac{\chi_{2m+2}^2}{2n} > \lambda\right)$, where χ_a^2 is the chi-square random variable with degrees of freedom (df) a . Thus, we see that there is a pair of fiducial distributions for λ , namely, $\frac{\chi_{2m}^2}{2n}$ and $\frac{\chi_{2m+2}^2}{2n}$. As χ_a^2 is stochastically increasing in a , the random variable $\frac{\chi_{2m+1}^2}{2n}$, which stochastically lies between the two fiducial variables, is an approximate fiducial quantity for λ (see also Dempster, 2008). Cox (1953) used this fiducial argument to develop an approximate CI for the ratio of two Poisson means. This approximate fiducial distribution is also the posterior distribution of λ with the improper prior $1/\sqrt{\lambda}$, $0 < \lambda < \infty$ (see Cai, 2005). Thus, the approximate fiducial CI

$$\left(\frac{1}{2n}\chi_{2m+1;\alpha/2}^2, \frac{1}{2n}\chi_{2m+1;1-\alpha/2}^2\right)$$

is the $1-\alpha$ credible set based on the above improper prior. This fiducial interval is contained in the exact confidence interval $\left(\frac{\chi_{2m;\alpha/2}^2}{2n}, \frac{\chi_{2m+2;1-\alpha/2}^2}{2n}\right)$ due to Garwood (1936) for each possible value of m . Thus, the fiducial interval is narrower than the Garwood interval, and as a consequence, it could be liberal. However, our numerical comparison studies (not reported here) with respect to expected widths and coverage probabilities indicated that the fiducial interval is as good as or better than the score interval $\hat{\lambda} + \frac{z_{1-\alpha/2}^2}{2n} \pm \sqrt{\left(\hat{\lambda} + \frac{z_{1-\alpha/2}^2}{2n}\right)^2 - \hat{\lambda}^2}$, where $\hat{\lambda} = Y/n$, which seems to be a very satisfactory CI for λ (Agresti and Coull, 1988).

2.3 A FQ for a Function of Parameters

In order to find the joint fiducial distribution for a set of parameters, we shall recall Fisher's statement as quoted in Section 5.2 of Zabell (1992): "In general, it appears that if statistics T_1, T_2, T_3, \dots contain jointly the whole of the information available respecting parameters $\theta_1, \theta_2, \theta_3, \dots$, and if functions t_1, t_2, t_3, \dots of the T 's and θ 's can be found, the simultaneous distribution of which is independent of $\theta_1, \theta_2, \theta_3, \dots$, then the fiducial distribution of $\theta_1, \theta_2, \theta_3, \dots$ simultaneously may be found by substitution."

The above assertion was used in Cox's (1953) paper for finding CIs for the ratio of two Poisson means. In general, a FQ for a function $h(\lambda_1, \dots, \lambda_g)$, where $\lambda_1, \dots, \lambda_g$ are the means of g independent Poisson distributions, can be obtained by substitution as $h\left(\frac{\chi_{2m_1+1}^2}{2n_1}, \dots, \frac{\chi_{2m_g+1}^2}{2n_g}\right)$. Here, (m_1, \dots, m_g) is an observed value of (Y_1, \dots, Y_g) , where Y_1, \dots, Y_g are independent random variables with $Y_i \sim \text{Poisson}(n_i\lambda_i)$, $i = 1, \dots, g$. Similarly, a FQ for $h(p_1, \dots, p_g)$, where p_i 's are success probabilities of g binomial distributions, can be obtained by replacing the parameters by their fiducial quantities. In the following sections, we shall consider the inferential procedures based on such fiducial quantities for binomial and Poisson distributions.

3. Inference for binomial distributions

In the following we shall provide fiducial CIs for the relative risk and the ratio of odds. Throughout the section, it is assumed that $X_1 \sim \text{binomial}(n_1, p_1)$ independently of $X_2 \sim \text{binomial}(n_2, p_2)$, and (k_1, k_2) is an observed value of (X_1, X_2) .

3.1a Confidence Intervals for the Relative Risk

Score Method: To express the score CI (based on the likelihood approach), we shall use Miettinen and Nurminen's (1985) derivation which appears to be simpler than the likelihood derivation (see Gart and Nam, 1988). Let $\zeta = p_1/p_2$. The MLE \tilde{p}_2 of p_2 under the constraint $p_1 = \zeta p_2$, is the solution to the equation $a\tilde{p}_2^2 + b\tilde{p}_2 + c = 0$, where $a = (n_1 + n_2)\zeta$, $b = -[(X_2 + n_1)\zeta + X_1 + n_2]$, and $c = X_1 + X_2$. The approximate $1 - \alpha$ confidence limits are the roots to the equation (with respect to ζ)

$$\left| \frac{\hat{p}_1 - \zeta \hat{p}_2}{\sqrt{\frac{\hat{p}_1 \hat{q}_1}{n_1} + \frac{\hat{p}_2 \hat{q}_2}{n_2}}} \right| = z_{1-\alpha/2}. \quad (3)$$

Notice that the above CI is based on asymptotic normality of the term within the absolute signs. Furthermore, this method is not applicable when $X_1 = X_2 = 0$.

Fiducial Confidence Intervals : A FQ for the relative risk $\zeta = p_1/p_2$ can be obtained by substitution, and is given by

$$Q_\zeta = \frac{Q_{p_1}}{Q_{p_2}} = \frac{B_{k_1+1/2, n_1-k_1+1/2}}{B_{k_2+1/2, n_2-k_2+1/2}}, \quad (4)$$

where the beta random variables in (4) are independent. For a given (k_1, k_2) , appropriate percentiles of Q_ζ form a CI for ζ . The α quantile of Q_ζ can be estimated using Monte Carlo simulation or as the solution (with respect to q) of the integral equation

$$P(Q_\zeta \leq q) = \int_0^1 F(qx; k_1, n_1) f(x; k_2, n_2) dx = \alpha, \quad (5)$$

where $f(x; k_i, n_i)$ and $F(x; k_i, n_i)$ are the probability density function and the cumulative distribution function of the beta($k_i + 1/2, n_i - k_i + 1/2$) random variable, respectively.

3.1b Coverage Properties of Confidence Intervals for the Relative Risk

For a given set of parameters and sample sizes, exact coverage probabilities of a confidence interval $(L(k_1, k_2, n_1, n_2; \alpha), U(k_1, k_2, n_1, n_2; \alpha))$ for a function $h(p_1, p_2)$ can be computed us-

ing the expression

$$\sum_{k_1=0}^{n_1} \sum_{k_2=0}^{n_2} g(k_1; n_1, p_1) g(k_2; n_2, p_2) I_{(L(k_1, k_2, n_1, n_2; \alpha), U(k_1, k_2, n_1, n_2; \alpha))}(h(p_1, p_2)), \quad (6)$$

where $g(x; n, p)$ is the binomial probability mass function and $I_A(x)$ is the indicator function. For a satisfactory confidence interval method, the above coverage probabilities should be close to the nominal level $1 - \alpha$ for all (n_1, p_1, n_2, p_2) .

Using (6), we evaluated the coverage probabilities of the 95% fiducial confidence intervals for the sample size and parameter configurations given in Table 2 of Gart and Nam (1988). These authors provided exact coverage probabilities of the CIs based on the ln-method and the ones based on the score method, and noted that the ln-method is in general inferior to the score method with respect to coverage probabilities. So we compare the fiducial approach only with the score method with respect to coverage probabilities; expected widths are not included for comparison because the score CI fails when $X_1 = X_2 = 0$. These coverage probabilities of the score intervals along with those for the fiducial intervals are given in Table 1. We observe from Table 1 that the fiducial CI and the score CI are comparable in most cases. For all the cases considered, the coverage probabilities of the fiducial intervals are either very close to or slightly larger than the nominal level 0.95. It should be noted that the fiducial CI is easier to compute than the score CI due to Gart and Nam (1988).

Table 1: Actual coverage probabilities of 95% confidence intervals for $\zeta = p_1/p_2$

methods		$n_1 = n_2 = 15$					$n_1 = 20, n_2 = 10$				
		$\zeta = \frac{1}{2}$	1	2	4	7	$\frac{1}{2}$	1	2	4	7
$p_1 = .125$	score	.963	.966	.963	.961	.961	.931	.965	.958	.951	.961
	fiducial	.960	.947	.960	.961	.951	.945	.943	.960	.950	.960
$p_1 = .250$	score	.947	.963	.946	.947	.958	.956	.953	.957	.946	.952
	fiducial	.947	.947	.951	.966	.960	.947	.947	.949	.960	.962
$p_1 = .500$	score	.959	.961	.947	.954	.950	.953	.963	.952	.948	.939
	fiducial	.960	.951	.945	.948	.960	.951	.946	.945	.952	.960

3.1c An Example for the Risk Ratio

To assess the effectiveness of a diagnostic test for detecting a certain disease, Koopman (1984) reports that 36 out of 40 diseased persons were correctly diagnosed by the test and 16 out of 80 nondiseased persons were incorrectly diagnosed. Let p_t and p_f denote respectively the true positive diagnoses and false positive diagnoses. We shall find 95% CIs for p_t/p_f . Noticing that $\hat{p}_t = 0.9$ and $\hat{p}_f = 0.2$, we get the point estimate for the ratio as 4.5.

The 95% confidence interval based on our fiducial approach is computed by simulating Q_ζ in (4) 100,000 times. Bedrick (1987) calculated several confidence intervals from a family of CIs, and the shortest one is reported in the following table. Note that all four CIs shifted to the right of the fiducial CI, which is the shortest among these five CIs.

95% confidence intervals for p_t/p_f	
method	95% confidence intervals for p_t/p_f
ln-method	(2.87, 7.06)
likelihood-score	(2.94, 7.15)
Bailey (1987)	(2.95, 7.30)
Bedrick (1987)	(2.93, 7.10)
fiducial approach	
using simulation	(2.86, 6.96)
using integral eqn. (5)	(2.87, 6.99)

3.2a Confidence Intervals for the Odds Ratio

Logit Confidence Interval: The CI is based on the asymptotic normality of $\ln\left(\frac{\hat{p}_1}{1-\hat{p}_1}\right) - \ln\left(\frac{\hat{p}_2}{1-\hat{p}_2}\right)$. In order to handle zero counts (see Agresti, 1999), the formula for the CI is adjusted by adding one half to each cell in a 2×2 contingency table setup. Let $\eta = p_1(1 - p_2)/[p_2(1 - p_1)]$ and $\hat{\eta}_{\frac{1}{2}} = \hat{p}_{1,\frac{1}{2}}(1 - \hat{p}_{2,\frac{1}{2}})/[\hat{p}_{2,\frac{1}{2}}(1 - \hat{p}_{1,\frac{1}{2}})]$, where $\hat{p}_{i,\frac{1}{2}} = (X_i + 1/2)/(n_i + 1/2)$, $i = 1, 2$. An estimate of the standard deviation of $\ln(\hat{\eta}_{\frac{1}{2}})$ is given by

$$\hat{\sigma}\left(\hat{\eta}_{\frac{1}{2}}\right) = \left(\frac{1}{X_1 + \frac{1}{2}} + \frac{1}{n_1 - X_1 + \frac{1}{2}} + \frac{1}{X_2 + \frac{1}{2}} + \frac{1}{n_2 - X_2 + \frac{1}{2}}\right)^{\frac{1}{2}}.$$

Using the above quantities, the $1 - \alpha$ CI for the odds ratio based on the ln-method is given by

$$\hat{\eta}_{\frac{1}{2}} \left(\exp \left[-z_{\alpha/2} \hat{\sigma} \left(\hat{\eta}_{\frac{1}{2}} \right) \right], \exp \left[z_{\alpha/2} \hat{\sigma} \left(\hat{\eta}_{\frac{1}{2}} \right) \right] \right). \quad (7)$$

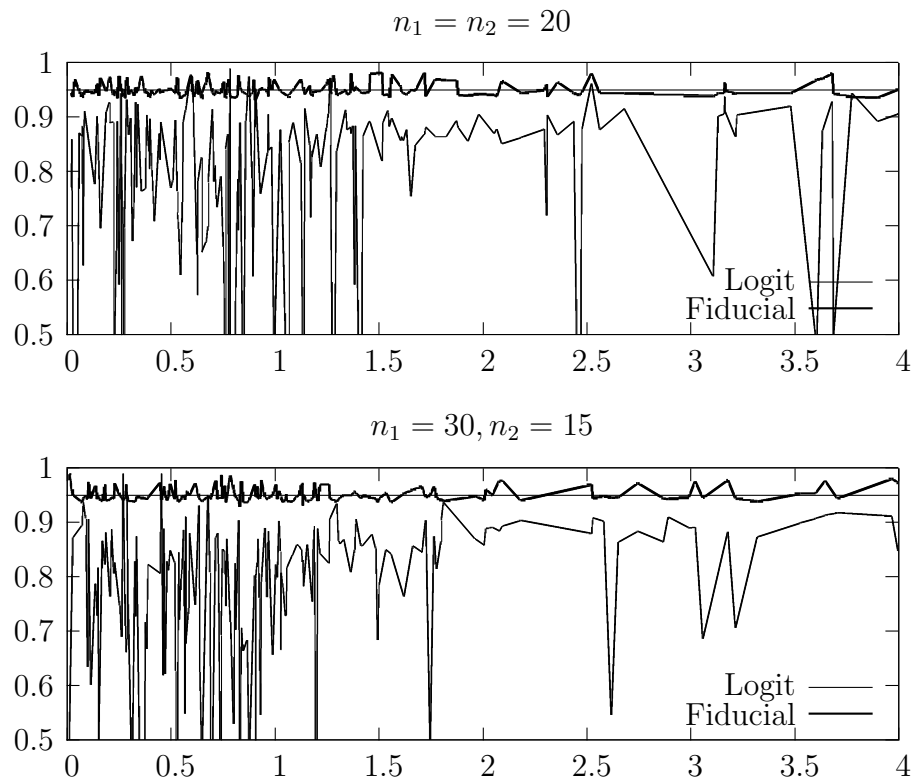
Fiducial Intervals: A FQ for the odds ratio $\eta = p_1(1 - p_2)/[p_2(1 - p_1)]$ can be obtained by substitution, and is given by

$$Q_\eta = \frac{B_{k_1+1/2, n_1-k_1+1/2}(1 - B_{k_2+1/2, n_2-k_2+1/2})}{B_{k_2+1/2, n_2-k_2+1/2}(1 - B_{k_1+1/2, n_1-k_1+1/2})}, \quad (8)$$

where all the beta random variables are mutually independent. Percentiles of Q_η , which can be estimated using Monte Carlo simulation, form a $1 - \alpha$ CI for the odds ratio.

3.2b Coverage Properties of CIs for the Odds Ratio

The coverage probabilities of the fiducial CI and the logit CI are evaluated using the expression (6) for 500 randomly generated p_1 and p_2 from a uniform(0, 1) distribution. The coverage probabilities are plotted in Figure 1 for sample sizes $(n_1, n_2) = (20, 20)$, $(30, 15)$ and $(40, 40)$. We see from these plots that the fiducial CIs perform satisfactorily even for sample size of 20 whereas the logit CIs perform poorly even for samples as large as 40. The coverage probabilities 95% logit CIs could fall well below 0.5. It appears that the logit CI works satisfactorily only when both samples are large and the values of odds ratio are large (see the plot in Figure 1 for the case $n_1 = n_2 = 40$). Overall, the fiducial approach is satisfactory and it is certainly preferable to the logit CI in terms of coverage probabilities.



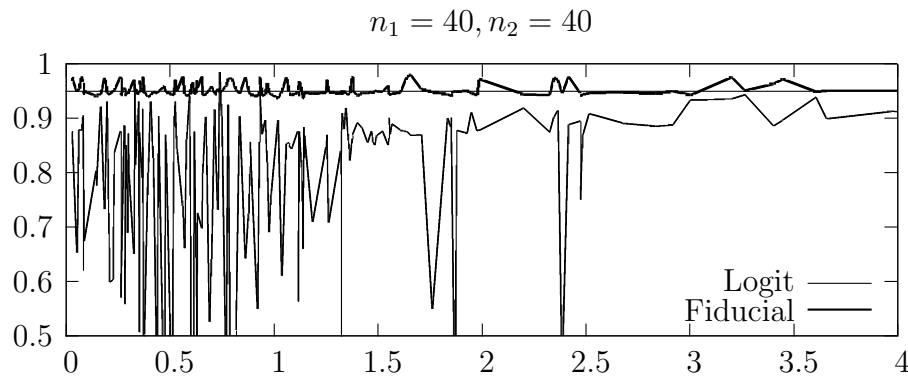


Figure 1. Coverage probabilities of logit and fiducial CIs as a function of odds ratio

3.2c An Example for the Odds Ratio

To illustrate the results for the odds ratio, let us consider the stillbirth and miscarriages data reported in Bailey (1987). In a group of 220 women who were exposed to diethylstilbestrol (DES) eight suffered a stillbirth (57 miscarriages), and in a group of 224 women who were not exposed to DES, three suffered a stillbirth (36 miscarriages).

Note that $k_1 = 8, n_1 = 220, k_2 = 3$ and $n_2 = 224$. We shall now compute 95% confidence intervals for the ratio of odds of a stillbirth in the exposed group to that in the non-exposed group. The point estimate of the odds ratio is 2.78. The 95% confidence interval using ln-method is (0.73, 10.62), and the one based on (8) with 100,000 simulation runs is (0.79, 11.60). The exact one using the conditional approach (see Thomas and Gart, 1977) is (0.65, 16.45), which is, as noted in the introduction, the widest among these three intervals.

We shall now compute 95% confidence intervals for the ratio of odds of a miscarriage in the exposed group to that in the non-exposed group. Now, $k_1 = 57, n_1 = 220, k_2 = 36$ and $n_2 = 224$. The point estimate of the odds ratio is 1.83. The 95% confidence interval using ln-method is (1.15, 2.87), and the one based on (8) with 100,000 simulation runs is (1.15, 2.93). The exact one using the conditional approach is (1.12, 3.01), which is again the widest among these three intervals.

4. Inference for Poisson distributions

4.1a CIs for the Ratio of Two Poisson Means

Let Y_i be the number of random occurrences of an event over a period of time t_i (or from a sample of n_i units) with mean rate λ_i , so that $Y_i \sim \text{Poisson}(t_i \lambda_i)$, $i = 1, 2$. The problem of interest here is to find confidence intervals for the ratio $PR = \lambda_1 / \lambda_2$. As confidence interval for the ratio can be obtained from the CI for $\frac{\lambda_1 t_1}{\lambda_2 t_2}$, we can ignore t_1 and t_2 while developing CI for the ratio of means.

Binomial-Score CI: This CI is based on the result that the conditional distribution of Y_1 given $Y_1 + Y_2 = M > 0$ is binomial(M, p_λ), where $p_\lambda = \frac{\lambda_1/\lambda_2}{1+\lambda_1/\lambda_2}$. Let $\hat{p} = Y_1/M$. Thus, we can regard Y_1 as the number of successes out of M independent trials each with success probability p_λ . A CI for λ_1/λ_2 can be obtained from the score confidence interval in (2) for p_λ . The resulting CI (PR_l, PR_u) for $PR = \lambda_1/\lambda_2$ from this score interval for p_λ can be written as follows. Let $a_1 = 2Y_1 + z_{1-\alpha/2}^2$, $a_2 = 2Y_2 + z_{1-\alpha/2}^2$ and $a_{12} = 4z_{1-\alpha/2}^2 Y_1 Y_2 / M + z_{1-\alpha/2}^4$. Then, the binomial-score CI for λ_1/λ_2 is given by

$$(PR_l, PR_u) = \left(\frac{a_1 - \sqrt{a_{12}}}{a_2 + \sqrt{a_{12}}}, \frac{a_1 + \sqrt{a_{12}}}{a_2 - \sqrt{a_{12}}} \right). \quad (9)$$

Notice that (PR_{ls}, PR_{us}) is defined for all Y_1 and Y_2 except for the case $Y_1 = Y_2 = 0$.

Sato (1990) and Graham et al. (2003) developed likelihood-score CI for λ_1/λ_2 . Their likelihood-score CI is the same as the one in (9) except that the likelihood-score CI is not defined when $Y_2 = 0$ and $Y_1 \geq 0$.

Fiducial CI for the Ratio of Poisson Means: A fiducial quantity for λ_1/λ_2 is given by $\chi_{2Y_1+1}^2/\chi_{2Y_2+1}^2$. The CI based on this fiducial quantity is

$$\left(\frac{2Y_1 + 1}{2Y_2 + 1} F_{2Y_1+1, 2Y_2+1; \frac{\alpha}{2}}, \frac{2Y_1 + 1}{2Y_2 + 1} F_{2Y_1+1, 2Y_2+1; 1-\frac{\alpha}{2}} \right), \quad (10)$$

where $F_{m,n;q}$ denotes the 100 q th percentile of an $F_{m,n}$ distribution. As noted in the introduction, Cox (1953) has proposed the above CI. It is interesting to note that the above CI coincides with the one that can be deduced from the fiducial CI of the conditional parameter $p_\lambda = \frac{\lambda_1/\lambda_2}{1+\lambda_1/\lambda_2}$ mentioned in the preceding paragraph. Specifically, the fiducial CI for p_λ is the CI in (1) with $d_1 = 2Y_1 + 1$ and $d_2 = 2Y_2 + 1$. The CI for λ_1/λ_2 that can be obtained from this CI of p_λ is $(\frac{d_1}{d_2}(F_{d_2, d_1; 1-\alpha/2})^{-1}, \frac{d_1}{d_2}(F_{d_2, d_1; \alpha/2})^{-1})$, which is the same as the one in (10).

4.1b Coverage Studies of the CIs for the Ratio of Poisson means

Graham et al. (2003) have carried out extensive simulation studies comparing their likelihood-score CI (which is essentially the same as the binomial-score CI (9)) with other asymptotic CIs, and concluded that the likelihood-score CI is the best. So we shall compare the fiducial CI only with the binomial-score CI. We computed the coverage probabilities 95% CIs for randomly generated values of λ_1 and λ_2 from uniform distributions, and plotted them in Figure 2. We see from these coverage plots that the binomial-score CI is conservative for small values of λ_1/λ_2 . In general, the fiducial CI maintains the coverage probability around the nominal level except in a few cases it could be liberal. Note that the coverage probabilities of both CIs seldom fall below 0.94.

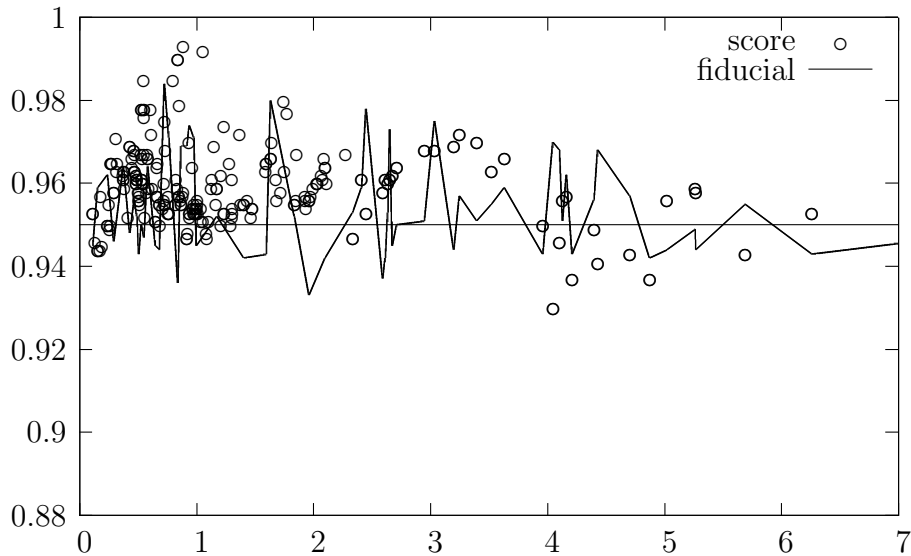


Figure 2. Coverage probabilities of 95% CIs for the ratio of Poisson means as a function of λ_1/λ_2 ; λ_1 and λ_2 are generated independently from a uniform(.5, 5) distribution

4.1c An Example for the Ratio of Poisson Means

This example is taken from Boice and Monson (1977), where two groups of women were compared to find whether those who had been examined using x-ray fluoroscopy during treatment for tuberculosis had a higher rate of breast cancer than those who had not been examined using the x-ray fluoroscopy. In the treatment group with women receiving x-ray fluoroscopy 41 cases of breast cancer in 28,010 person-years at risk were reported while in the control group of women not receiving x-ray fluoroscopy 15 cases of breast cancer in 19,017 person-years at risk were reported. Here $Y_1 = 41$, $t_1 = 28,010$, $Y_2 = 15$ and $t_2 = 19,017$, and the problem of interest is to obtain a CI for the ratio λ_1/λ_2 , where λ_1 is the mean rate of breast cancers for the treatment group and λ_2 is that for the control group. A CI for λ_1/λ_2 can be obtained from the one for $t_1\lambda_1/(t_2\lambda_2)$. The 95% fiducial confidence interval for λ_1/λ_2 is (1.047, 3.421), and the binomial-score CI is (1.036, 3.325). Note that both intervals indicate that λ_1 is significantly larger than λ_2 . We also observe that the fiducial interval is shifted to the right of the binomial-score interval, and is slightly wider of the two intervals.

4.2 Confidence Intervals for a Weighted Sum of Poisson Parameters

We shall now illustrate fiducial approach for constructing CIs for a weighted sum of Poisson parameters. Let Y_1, \dots, Y_g be independent random variables with $Y_i \sim \text{Poisson}(n_i\lambda_i)$, $i = 1, \dots, g$. We are interested in finding confidence intervals for $\sum_{i=1}^g c_i\lambda_i$, where c_i 's are known positive constants. Without loss of generality, we can assume $c_i \in (0, 1)$, $i = 1, \dots, g$ so that $\sum_{i=1}^g c_i = 1$ and $n_1 = \dots = n_g = 1$. The case of unequal sample sizes can be handled by letting $w_i = c_i/n_i$ and $\xi_i = n_i\lambda_i$, $i = 1, \dots, g$, so that $\sum_{i=1}^g w_i\xi_i = \sum_{i=1}^g c_i\lambda_i$.

4.2a Normal Based Approximate Method

Dobson et al. (1991) proposed a method that uses the normal approximation to the Poisson distribution and Garwood's (1936) exact approach based on the total count. To describe their method, let $S = \sum_{i=1}^g c_i Y_i$ so that $S \sim N(\sum_{i=1}^g c_i \lambda_i, \sum_{i=1}^g c_i^2 \lambda_i)$ approximately. Dobson et al. noted that a CI based on this approximate distribution is unsatisfactory when Y_i 's are small; they suggested an alternative approach that uses the quantity $T = a + bY$, where $Y = \sum_{i=1}^g Y_i$ is the total counts. The values of a and b are to be determined by matching the mean and variance of T with those of S . The resulting expressions for a and b are functions of parameters which have to be replaced by estimators. Using these arguments, they developed an approximate $1 - \alpha$ CI as

$$(a + bY_L, a + bY_U), \quad (11)$$

where $a = \sum_{i=1}^g c_i Y_i - [\sum_{i=1}^g Y_i \sum_{i=1}^g c_i^2 Y_i]^{\frac{1}{2}}$, $b = [\sum_{i=1}^g c_i^2 Y_i / \sum_{i=1}^g Y_i]^{\frac{1}{2}}$, $Y_L = \frac{1}{2} \chi_{2Y; \alpha/2}^2$ and $Y_U = \frac{1}{2} \chi_{2Y+2; 1-\alpha/2}^2$. Note that (Y_L, Y_U) is the Garwood exact CI for $\sum_{i=1}^g \lambda_i$ based on the total count Y .

4.2b Fiducial Interval for a Weighted Sum of Poisson Means

An approximate FQ for the weighted mean $\mu = \sum_{i=1}^g c_i \lambda_i$ is obtained by replacing the λ_i 's by their fiducial quantities. Letting $c_i^* = c_i / (2n_i)$, $i = 1, \dots, g$, we write the FQ for μ as $Q_\mu = \sum_{i=1}^g c_i^* \chi_{2m_i+1}^2$, where m_i is an observed value of Y_i , $i = 1, \dots, g$. For given sample sizes and observed counts, one can use Monte Carlo simulation to estimate the percentiles of Q_μ . As Q_μ is a linear combination of independent χ^2 variables, we can also approximate the distribution of Q_μ by $e\chi_f^2$, where e and f are to be determined by matching moments. By using this moment matching method, we find $e = \frac{\sum_{i=1}^g c_i^{*2} (2m_i+1)}{\sum_{i=1}^g c_i^* (2m_i+1)}$ and $f = \frac{(\sum_{i=1}^g c_i^* (2m_i+1))^2}{\sum_{i=1}^g c_i^{*2} (2m_i+1)}$. Thus, an approximate $1 - \alpha$ CI for μ is given by

$$(e\chi_{f; \alpha/2}^2, e\chi_{f; 1-\alpha/2}^2). \quad (12)$$

Our numerical investigation (not reported here) indicated that the CI based on Monte Carlo method and the one in (12) are practically the same.

4.2c Coverage Properties of CIs for the Weighted Sum of Poisson Means

The coverage probabilities of the normal based CI (11) and those based on the chi-square approximation (12) are estimated using Monte Carlo simulation for the case of $g = 3$, $w_1 = w_2 = w_3 = 1/3$ and $(w_1, w_2, w_3) = (.2, .2, .6)$, and for 100 randomly generated $(\lambda_1, \lambda_2, \lambda_3)$ from uniform(0.5, 5) distribution. The computed values are plotted in Figure 3. For the

case of unequal weights, both methods are expected to be liberal in some cases, and the chi-square approximate CI has better coverage property than those based on the normal approximation for small values of $\mu = \sum_{i=1}^g w_i \lambda_i$. For the case equal weight, the normal approximate method appears to be conservative, and so the CIs based on this approach are expected to be slightly wider than those of the fiducial approach (see Table 2). The fiducial CI in (12) could be slightly liberal in some cases; in general, its coverage probabilities are close to the nominal level in most of the cases.

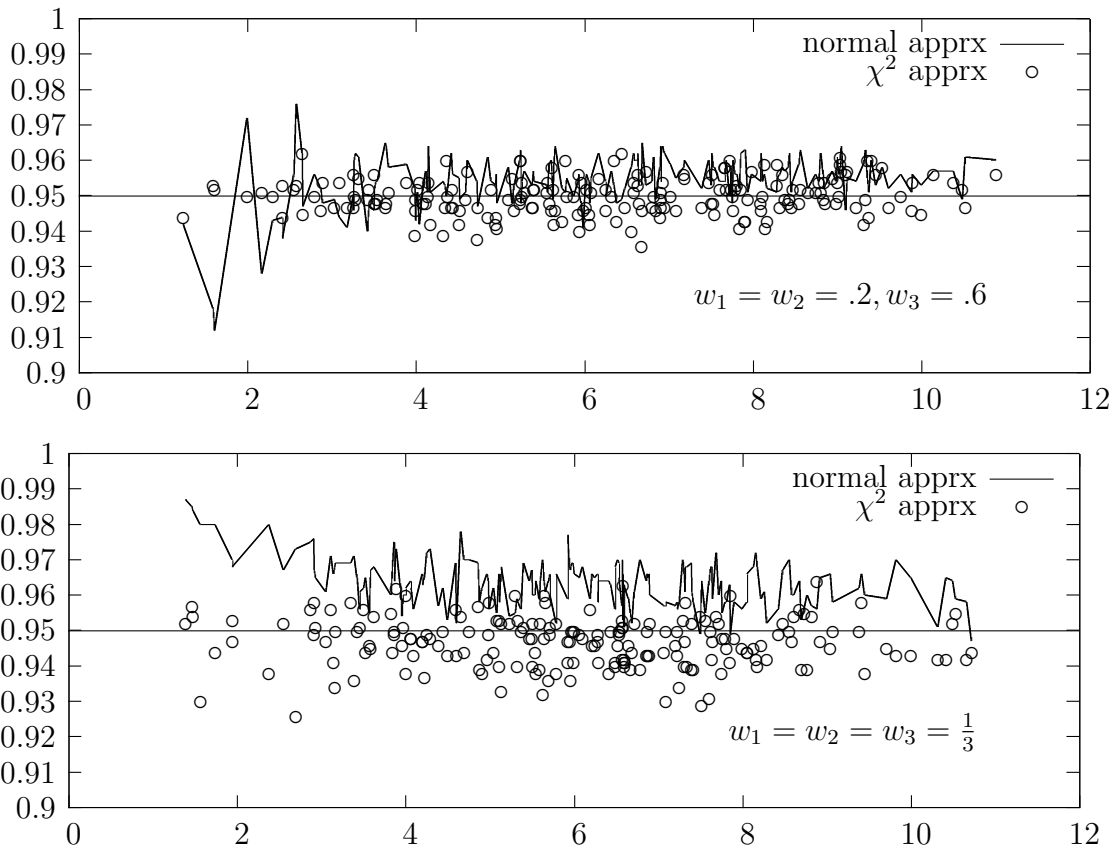


Figure 3. Coverage probabilities of 95% confidence intervals for $\mu = \sum_{i=1}^k w_i \lambda_i$ as a function of μ ; $n_1 = n_2 = n_3 = 1$

4.2d An Example for the Weighted Sum of Poisson Means

We shall illustrate the methods of calculating CIs for weighted sums of Poisson means using the incidence rates given in Table III of Dobson et al. (1991). The data, reported in (WHO MONICA project, 1988), were collected from an urban area (reporting unit 1) and a rural area (reporting unit 2) in Federal Republic of Germany. The 1986 incidence rates for non-fatal definite myocardial infarction in women aged 35-64 years stratified by 5-year age group and reporting unit. The incidence rates along with weights c_i for age groups

Table 2: Summary statistics of expected widths of the approximate and fiducial CIs for weighted sum of Poisson means based on 1,000 values of $(\lambda_1, \lambda_2, \lambda_3)$ generated from uniform(.5, 5) distribution

statistics	$w_1 = w_2 = w_3 = 1/3$		$w_1 = w_2 = .2, w_3 = .6$	
	CI (11)	CI (12)	CI (11)	CI (12)
mean	4.061	3.980	4.553	4.475
std dev	0.533	0.510	0.969	0.821
5th percentile	3.179	3.123	2.832	3.046
first quartile	3.749	3.644	3.796	3.818
median	4.106	4.006	4.688	4.567
third quartile	4.460	4.334	5.377	5.154
90th percentile	4.760	4.619	5.763	5.507
95th percentile	4.893	4.757	5.866	5.615

(corresponding to the Segi World Standard Population) are reproduced here in Table 3.

We are interested in estimating the age-standardized incidence rates per 10,000 person-years, $\mu = \sum_{i=1}^6 w_i \xi_i$ with $w_i = c_i/n_i$. The sample estimates for reporting units 1 and 2 are 2.75 and 1.41 respectively. The calculated 95% CIs for μ based on the normal approximate method and fiducial approach with simulation consisting of 10,000 runs, and the one in (12) are given in Table 3. The CI on the basis of the chi-square approximation and the one based on the simulation are practically the same for both reporting units. The CIs based on the normal approximation (11) are shifted to the left of the corresponding fiducial intervals. We also note that the normal based approximate method produced slightly shorter intervals for both reporting units.

Table 3: 95% Incidence rates for myocardial infarction in women by age and reporting unit

Age (years)	c_i	Reporting unit 1		Reporting unit 2	
		Person-years, n_i	Events, Y_i	Person-years, n_i	Events, Y_i
35-39	6/31	7,971	0	10,276	0
40-44	6/31	7,084	0	9,365	1
45-49	6/31	9,291	1	11,623	0
50-54	5/31	7,743	2	8,684	4
55-59	4/31	7,798	4	7,926	0
60-64	4/31	8,809	10	8,375	3
Age standardized rate					
per 10,000		2.75		1.41	
95% CI from (11)		(1.59, 4.42)		(0.61, 2.79)	
95% fiducial CI		(2.05, 5.05)		(0.96, 3.26)	
95% CI based on χ^2 approx.		(2.04, 5.04)		(0.97, 3.26)	

5. Fiducial tests

In general, following the generalized variable approach by Tsui and Weerahandi (1989), a *fiducial test variable* for testing a parameter is obtained as the fiducial quantity minus the parameter. For example, the fiducial test variable for testing a binomial success probability p is given by $Q_t = B_{k+1/2, n-k+1/2} - p$. Because, for a given k , Q_t is stochastically decreasing in p , the *fiducial p-value* for testing $H_0 : p \leq p_0$ vs. $H_a : p > p_0$ is given by $\sup_{H_0} P(Q_t \leq 0) = P(B_{k+1/2, n-k+1/2} - p_0 \leq 0)$. The fiducial test rejects H_0 if the fiducial p-value is less than or equal to a specified nominal level. On the basis of the coverage studies by Brown et al. (2001), the fiducial test should be comparable with the score test.

Fiducial tests for a function of parameters can be obtained similarly. For example, if one wants to test $\zeta = p_1/p_2$, then the fiducial p-value for testing hypotheses $H_0 : \zeta \geq \zeta_0$ vs. $H_a : \zeta < \zeta_0$ is given by $P(B_{k_1+1/2, n_1-k_1+1/2}/B_{k_2+1/2, n_2-k_2+1/2} \geq \zeta_0)$. Here k_1 and k_2 are the observed values of $X_1 \sim \text{binomial}(n_1, p_1)$ and $X_2 \sim \text{binomial}(n_2, p_2)$, respectively. The properties of the tests should be similar to those of the CIs studied in earlier sections.

6. Application to other problems

We shall now briefly explain other problems where the fiducial approach produces satisfactory results. More details with examples and coverage studies are given in Lee (2009).

As mentioned in the introduction, Fleiss (1973) introduced the relative difference, defined by $RD = \frac{p_2 - p_1}{1 - p_1}$, as the index of benefit when patients who respond positively to treatment 1 are also expected respond positively to treatment 2. The FQ for the RD can be obtained by replacing the parameters by their fiducial quantities. Appropriate percentiles of Q_{RD} form a $1 - \alpha$ CI for the relative difference. The problem of estimating a product of binomial probabilities or of Poisson means is considered in Buehler (1957), Madansky (1965) and Harris (1971). For estimating the product of two binomial success probabilities, a FQ is given by $Q_{p_1 p_2} = B_{k_1+1/2, n_1-k_1+1/2} B_{k_2+1/2, n_2-k_2+1/2}$. The percentiles of $Q_{p_1 p_2}$ can be estimated using Monte Carlo simulation or can be approximated by a beta(c, d), where c and d are to be determined by matching the first two moments. The approximation is satisfactory, and produced results that are comparable with those of Madansky's (1965) likelihood method and randomized limits given in Harris (1971). It should be noted that the methods provided in these articles just cited are asymptotic and computationally intensive whereas the fiducial CIs can be easily computed. More details and coverage studies can be found in Lee (2009).

There are other problems for which the fiducial method can be readily applied to find approximate solutions. For example, approximate CIs for the ratio of two products of binomial success probabilities or of Poisson means (Buehler, 1957 and Harris, 1971) can be readily obtained using the fiducial method described in this paper. Finally, we note that the proposed approach may work for single parameter distributions such as binomial, Poisson,

negative binomial and logarithmic series. A fiducial distribution for the negative binomial can be obtained using its relation to the beta distribution (Hannig, 2009). For the logarithmic series, fiducial distribution can not be expressed explicitly, and so the proposed approach is difficult to use (see Wani, 1975).

7. Concluding remarks

In this article we showed that the fiducial approach is not only simple but is also useful to obtain satisfactory solutions to several important problems involving estimation of a function of binomial or of Poisson parameters. In many situations, the fiducial approach produced results that are comparable to or better than those of some existing methods. In addition, the accuracy of the fiducial approach for a specific problem can be evaluated using Monte Carlo simulation. Even though we made no claim about small sample properties of the fiducial approach, our coverage studies for all the problems considered show that the fiducial method is satisfactory when the sample sizes are not too small.

In general, Fisher's fiducial argument is faulted when we attempt to extend this argument to mutliparameter case. For examples, fiducial approaches for testing equality of several Poisson means or for testing equality of several binomial proportions are not clear. As noted in Section 2, it is possible to find more than one fiducial distributions for a single parameter. This inconsistency is severe in multiparameter cases. For example, Dempster (1963) has shown that one could arrive at different joint fiducial distributions for normal mean μ and variance σ^2 using Fisher's own step-by-step argument. We again note that these inconsistencies do not pose serious problems in developing approximate or exact inferential procedures for some real-valued functions of μ and σ^2 such as normal quantile or log-normal mean (Krishnamoorthy and Mathew, 2003). As noted in the introduction, the fiducial approach, in the name of *generalized variable* approach, has been used in numerous articles to find CIs for a function of parameters such as quantile in random effects model and mixed models, to analyze interlaboratory trials (Krishnamoorthy and Mathew, 2004 and Iyer, Wang and Mathew, 2004) to develop tests for establishing bioequivalence (McNally, Iyer and Mathew, 2003), and to find CIs for various correlation coefficients (Krishnamoorthy and Xia, 2007).

Finally, we note that the fiducial approach produced satisfactory results for making inferences on several popular summary indices that can be expressed as a function of binomial parameters or of Poisson parameters. However, we caution that there is no guarantee that the fiducial approach works for any given real-valued function of parameters. In general, the procedures should be evaluated at least numerically before recommending for applications. Furthermore, the numerical and Monte Carlo studies of the coverage probabilities indicate that the fiducial CIs are comparable, and in some cases better than the existing asymptotic CIs, and so they share the properties of available asymptotic methods. For the one-sample problems, the theoretical coverage properties of the CIs have been studied by various authors

(e.g., see Cai (2005) and the references therein). However, we are unable to establish any asymptotic properties theoretically because the fiducial CIs are not in closed form, and the methods for the one-sample problems can not be easily extended to study the asymptotic properties of the fiducial CIs for the odds ratio or for the relative risk.

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