A Simple Approximate Procedure for Constructing Binomial and Poisson Tolerance Intervals

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The problems of constructing tolerance intervals for the binomial and Poisson distributions are considered. Closed-form approximate equal-tailed tolerance intervals (that control percentages in both tails) are proposed for both distributions. Exact coverage probabilities and expected widths are evaluated for the proposed equal-tailed tolerance intervals and the existing intervals . Furthermore, an adjustment to the nominal confidence level is suggested so that an equal-tailed tolerance interval can be used as a tolerance interval which includes a specified proportion of the population, but does not necessarily control percentages in both tails. Comparison of such coverage-adjusted tolerance intervals with respect to coverage probabilities and expected widths indicates that the closed-form approximate tolerance intervals are comparable with others, and less conservative, with minimum coverage probabilities close to the nominal level in most cases. The approximate tolerance intervals are simple and easy to compute using a calculator, and they can be recommended for practical applications. The methods are illustrated using two practical examples.

Keywords Coverage probability; Clopper-Pearson confidence interval; Expected widths; Score confidence interval; Two-sided tolerance interval.

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1. Introduction

A confidence interval (CI) based on a random sample is used to estimate an unknown scalar population parameter, such as the population mean or variance. A prediction interval based on a random sample is used to predict the value of a future observation from the sampled population. However, there are many practical situations where one is interested in the characteristics of a majority of the units in the population, rather than a single unit or the overall mean. A tolerance interval (TI) can be used for this purpose, and is constructed so that it would contain at least a specified proportion, say, p of the population with a specified confidence, say, $1-\alpha$. Such an interval is usually referred to as the p-content – $(1-\alpha)$ coverage TI, or simply a $(p, 1-\alpha)$ TI. A $(p, 1-\alpha)$ one-sided lower tolerance limit is constructed so that at least a proportion p of the population falls above the limit while a $(p, 1-\alpha)$ one-sided upper tolerance limit is constructed so that at least a proportion p of the population falls below the limit. The probability that a TI would include at least a proportion p of the population is referred to as the coverage probability. For an exact $(p, 1-\alpha)$ TI, this coverage probability should be greater than or equal to the nominal level $1-\alpha$ for all values of the population parameter.

A two-sided TI can be used to find a conservative estimate of the proportion of the population that falls between two specified values, and one-sided tolerance limits are useful to judge the proportion of the population that falls above or below a threshold value. For example, engineering products are usually tested to check if they meet certain tolerance specifications. In this case, one wants to find the actual proportion of the components that meets the specifications. If a $(p, 1-\alpha)$ TI based on a sample of components falls within the specifications, then we can conclude that at least a proportion p of the components meets the specifications with confidence $1-\alpha$. In some situations, population units are required to meet a lower or upper specification only. For example, in workplace exposure monitoring, one is interested in finding the proportion of exposure measurements that fall below an occupational exposure limit (OEL, usually set by the Occupational Safety and Health Administration). In this case, if a $(p, 1-\alpha)$ one-sided upper tolerance limit is less than the OEL, then it can be concluded that at least a proportion p of the exposure measurements is less than the OEL with confidence $1-\alpha$. For continuous distributions, the theory of TI is well developed, specifically for univariate normal distributions, and univariate linear regression models under normality. Early work on the construction of normal tolerance limits is due to Wilks (1941, 1942), Wald (1943) and Wald and Wolfowitz (1946). Normal based tolerance limits are used in acceptance sampling plan, in specifying tolerance limits for engineering products, and to assess pollution levels at a workplace. For more details and numerous applications, see the book by Guttman (1970), and the recent book by Krishnamoorthy and Mathew (2009).

The problem of constructing TIs for a discrete distribution has not received much attention in the literature. TIs for discrete models are useful to assess the magnitude of discrete quality characteristics of a product, for example, the number of defective components in a system. Zacks (1970, 1971), Hahn and Chandra (1981), Hahn and Meeker (1991), and Zaslavsky (2007) have considered the problems of constructing TIs for the binomial and Poisson distributions. Hahn and Chandra have provided an example where an upper tolerance limit for a Poisson distribution is required to assess the number of unscheduled system shutdowns in a year, and have also outlined a situation where one needs to construct an upper tolerance limit for a binomial distribution. Zaslavsky (2007) noted that TIs are frequently used in the validation and process control of medical products. Recently, Wang and Tsung (2009) provided an example where it is desired to find a TI to assess the number of defective chips in a wafer.

To define a $(p, 1-\alpha)$ TI for a discrete distribution, let $\boldsymbol{X}$ be a sample from a discrete distribution, and let X follow the same distribution independently of $\boldsymbol{X}$. A $(p, 1-\alpha)$ TI $[L(\boldsymbol{X}), U(\boldsymbol{X})]$ is constructed so that

$$P_{\boldsymbol{X}} \left\{ P_X \left(L(\boldsymbol{X}) \le X \le U(\boldsymbol{X}) | \boldsymbol{X} \right) \ge p \right\} \ge 1-\alpha. \tag{1}$$

Because of the discrete nature of the problem, the coverage probability requirement is at least $1-\alpha$, rather than exactly equal to $1-\alpha$. A $(p, 1-\alpha)$ upper tolerance limit $U_1(\boldsymbol{X})$ can be similarly defined, and satisfies

$$P_{\boldsymbol{X}} \left\{ P_X \left(X \le U_1(\boldsymbol{X}) | \boldsymbol{X} \right) \ge p \right\} \ge 1-\alpha.$$

If k_p is the p quantile of the population, then the $(p, 1-\alpha)$ one-sided upper tolerance limit is also a $1-\alpha$ upper confidence limit for k_p. That is, $P_{\boldsymbol{X}}(k_p \le U_1(\boldsymbol{X})) \ge 1-\alpha$. Similarly, if k_{1-p} is the $1-p$ quantile of the population, then a $1-\alpha$ lower confidence limit is the $(p, 1-\alpha)$ lower tolerance limit. It is important to note that the computation of $L(\boldsymbol{X})$ and $U(\boldsymbol{X})$ that satisfy (1) does not reduce to the computation of confidence limits for certain percentiles.

A $(p, 1-\alpha)$ tolerance interval $[L_e(\boldsymbol{X}), U_e(\boldsymbol{X})]$ controlling percentages in both tails is determined by

$$P_{\boldsymbol{X}} \left(L_e(\boldsymbol{X}) \le k_{\frac{1-p}{2}} \text{ and } k_{\frac{1+p}{2}} \le U_e(\boldsymbol{X}) \right) \ge 1-\alpha. \tag{2}$$

Note that the interval $[L_e(\boldsymbol{X}), U_e(\boldsymbol{X})]$ is constructed so that no more than a proportion $(1-p)/2$ of the population is less than $L_e(\boldsymbol{X})$ and no more than a proportion $(1-p)/2$ of the population is greater than $U_e(\boldsymbol{X})$. Notice that the interval $[L_e(\boldsymbol{X}), U_e(\boldsymbol{X})]$ that satisfies (2) also satisfies (1), but the interval $[L(\boldsymbol{X}), U(\boldsymbol{X})]$ that satisfies (1) does not necessarily satisfy (2). Owen (1964) proposed the criterion (2) for constructing TIs for a normal distribution, and noted the difference between the TIs on the basis of (1) and (2). The TI satisfying (2) (assuming a continuous distribution) is used in clinical studies to capture the central $100p\%$ of the population, and is referred to as the reference interval (see, Harris and Boyd (1995) and Trost (2006)). Krishnamoorthy (2006) and Krishnamoorthy and Mathew (2009) refer to the TI satisfying (2) as the "equal-tailed" TI, noting that the maximum possible contents in the tails of the TI should be equal. In the sequel, we shall refer the interval satisfying (1) by simply tolerance interval and the one satisfying (2) by equal-tailed TI.

For the binomial and Poisson cases, one-sided tolerance limits and the equal-tailed TIs are determined using confidence intervals (CIs) for the appropriate quantiles (see the section below). As a consequence, a better CI for the parameter could yield a better equal-tailed TI. For a binomial probability π, the Clopper-Pearson (1934) one-sided CIs are known to be uniformly most accurate; thus the one-sided tolerance limits based on such CIs are also uniformly most accurate (see Zacks (1971) and Zaslavsky (2007)). Hahn and Chandra (1981) used the same exact CIs for constructing equal-tailed TIs for a binomial distribution, and they also used Garwood's (1936) exact CIs for a Poisson mean λ for constructing equal-tailed TIs for a Poisson distribution. These TIs are exact in the sense that the minimum coverage probability is at least the nominal level $1-\alpha$. Recently, Wang and Tsung (2009) showed via numerical studies that these exact TIs are too conservative while the ones based on the Wald intervals are too liberal in some situations. These authors have provided an exact method of computing the minimum and average coverage probabilities of a TI, and suggested a way to choose a confidence level (smaller than intended nominal level) so that the TI based on the exact CI becomes less conservative.

It is now well understood that the exact CIs for the binomial probability π and for the Poisson mean λ are too conservative. Many authors (see Agresti and Coull, 1998; Brown, Cai and Das Gupta, 2001) recommend approximate CIs for applications. There are several approximate CIs available for a binomial π (see Brown, Cai and Das Gupta, 2001, and Cai, 2005, and the references therein). Among the approximate CIs, the score CIs seem to be popular. The score CIs seem to be not only simple to compute, but are also shorter than the exact CIs, with good coverage properties (see Agresti and Coull, 1998). Therefore, it is of interest to study the properties of the TIs based on the exact CIs and those based on the score CIs.

The rest of the article is organized as follows. In the next section, we describe the methods of constructing one-sided TIs and equal-tailed TIs in a general setup, following Zacks (1970) and Hahn and Chandra (1981). In Section 3, we outline methods for constructing CIs and TIs for a binomial distribution. We also propose equal-tailed TIs based on the normal approximation to a binomial distribution. These approximate TIs are in closed-form and they are as simple as the Wald CIs for a binomial proportion or for a Poisson mean. The coverage probabilities and expected widths of the proposed tolerance intervals are evaluated using an exact method. On the basis of extensive numerical studies, we find that a $1-2\alpha$ confidence interval for the binomial success probability π can be used to obtain a TI that includes at least a proportion p of the population with the minimum coverage probability close to the intended nominal level $1-\alpha$. Furthermore, the proposed approximate TIs are comparable with the existing ones in terms of coverage probabilities and expected widths. We illustrate the results using an example where it is of interest to assess the number of defective chips in a wafer. The results are extended to the Poisson case in Section 4. The Poisson TIs are illustrated using an example where it is desired to estimate the number of surface defects in steel plates. Some concluding remarks and available software packages to compute TIs are given in Section 5.

2. Construction of tolerance intervals in a general setup

Consider a distribution $F_X(x|\theta)$ that depends only on a single parameter θ, and is stochastically increasing in θ. That is, for every t, $P(X>t|\theta_1)\geq P(X>t|\theta_2)$ for all $\theta_1>\theta_2$. Then the p quantile $k_p(\theta)$, defined as the smallest value for which $P(X\leq k_p(\theta)|\theta)\geq p$, is a nondecreasing function of θ. As a consequence, if θ_u is a $1-\alpha$ upper confidence limit for θ, then $k_p(\theta_u)$, defined as the smallest value so that

$$P\left(X\leq k_p(\theta_u)|\theta_u\right)\geq p,$$

is a $1-\alpha$ upper confidence limit for $k_p(\theta)$, and so it is a $(p, 1-\alpha)$ upper tolerance limit for the distribution. Similarly, if θ_l is a $1-\alpha$ lower confidence limit for θ, then $k_{1-p}(\theta_l)$, defined as the largest integer so that

$$P\left(X \geq k_{1-p}(\theta_l)|\theta_l\right) \geq p,$$

is a $(p, 1-\alpha)$ lower tolerance limit for the distribution. If (θ_l, θ_u) is a $1-\alpha$ CI for θ, then simultaneously the inequalities $k_{\frac{1-p}{2}}(\theta_l) \leq k_{\frac{1-p}{2}}(\theta)$ and $k_{\frac{1+p}{2}}(\theta) \leq k_{\frac{1+p}{2}}(\theta_u)$ hold with probability $1-\alpha$. Therefore,

$$\left[k_{\frac{1-p}{2}}(\theta_l),\ k_{\frac{1+p}{2}}(\theta_u)\right]$$

is a $(p, 1-\alpha)$ equal-tailed TI.

As the binomial and Poisson distributions are stochastically increasing in their respective parameters, the above procedures can be used to obtain equal-tailed TIs. Also, the above procedures with the exact CIs for θ produce exact TIs. For the binomial and Poisson distributions, Zacks (1970) used the above procedures with the exact CIs for computing one-sided tolerance limits, and Hahn and Chandra (1981) used the same for computing equal-tailed TIs.

3. Binomial distribution

Let $X \sim$ binomial(n, π) independently of $Y \sim$ binomial(m, π). We like to find TIs for the binomial(m, π) distribution based on a realization k of X. Specifically, we estimate π using k and n, and use the estimated π to construct TIs for the binomial(m, π) distribution. Towards that, we first describe the exact and score interval estimation procedures for π.

3.1 Confidence Intervals for π

Exact Confidence Intervals for π

The Clopper-Pearson approach for obtaining an exact confidence interval for a binomial proportion π is as follows. For a given sample size n and an observed number of successes k, let π_l be the solution to the equation $\sum_{i=k}^{n} \binom{n}{i} \pi_l^i (1-\pi_l)^{n-i} = \alpha/2$, and let π_u be the solution to the equation $\sum_{i=0}^{k} \binom{n}{i} \pi_u^i (1-\pi_u)^{n-i} = \alpha/2$. Then, (π_l, π_u) is a $1-\alpha$ CI for π. Using a relation between the binomial and beta distributions it can be shown that

$$(\pi_l, \pi_u) = \left(B_{k,n-k+1;\alpha/2}, B_{k+1,n-k;1-\alpha/2}\right),\ k = 0, 1, ..., n, \tag{3}$$

where $B_{a,b;q}$ denotes the q quantile of a beta distribution with parameters $(a,\ b)$. The above CI should be used with the convention that $B_{0,b;\alpha/2} = 0$ and $B_{a,0;1-\alpha/2} = 1$. The interval (π_l, π_u) is an exact $1-\alpha$ CI for π, in the sense that the coverage probability is always greater than or equal to the specified confidence level $1-\alpha$. One-sided confidence limits for π can be obtained by replacing $\alpha/2$ in (3) by α.

Score Confidence Intervals for π

Let $\widehat{\pi} = \frac{X}{n}$, where $X \sim$ binomial(n, π). The score CI is based on the asymptotic result that

$$T(\widehat{\pi}, \pi) = \frac{\widehat{\pi} - \pi}{\sqrt{\pi(1-\pi)/n}} \sim N(0,1).$$

Let $c = z_{1-\alpha/2}$ be the $1-\alpha/2$ quantile of the standard normal distribution. The endpoints of the $1-\alpha$ score CI are the roots of the quadratic equation $T^2(\widehat{\pi}, \pi) = c^2$, and they are given by

$$(\pi_l, \pi_u) = \left(\frac{\widehat{\pi} + \frac{c^2}{2n}}{1 + \frac{c^2}{n}} \right) \pm \frac{\frac{c}{\sqrt{n}} \sqrt{\widehat{\pi}(1-\widehat{\pi}) + c^2/(4n)}}{1 + \frac{c^2}{n}}. \tag{4}$$

3.2 Binomial Tolerance Intervals

Let X be a binomial(n, π) random variable so that

$$P(X \le x | n, \pi) = \sum_{i=0}^{x} \binom{n}{i} \pi^i (1-\pi)^{n-i}, \ x = 0, 1, ..., n.$$

Let k be an observed value of X. Furthermore, let Y be a binomial(m, π) random variable independent of X. The $(p, 1-\alpha)$ upper tolerance limit for the binomial(m, π) distribution is the smallest integer $k_p(\pi_u, m)$ for which

$$P(Y \le k_p(\pi_u, m) | m, \pi_u) \ge p, \tag{5}$$

where π_u is a $1-\alpha$ upper confidence limit for π based on k. Similarly, the largest number $k_{1-p}(\pi_l)$ for which

$$P(Y \ge k_{1-p}(\pi_l, m) | m, \pi_l) \ge p, \tag{6}$$

where π_l is a $1-\alpha$ lower confidence limit for π based on k, is a $(p, 1-\alpha)$ lower tolerance limit for the binomial(m, π) distribution. The $(p, 1-\alpha)$ equal-tailed TI is given by

$$\left[k_{\frac{1-p}{2}}(\pi_l, m), k_{\frac{1+p}{2}}(\pi_u, m) \right], \tag{7}$$

where (π_l, π_u) is a $1-\alpha$ two-sided CI for π based on k.

We shall now describe a numerical procedure to compute tolerance limits. To find a $(p, 1-\alpha)$ upper tolerance limit, we compute right-tail probabilities $P(Y = m | m, \pi_u), \ P(Y = m-1 | m, \pi_u), ...$ until the sum of these probabilities is greater than or equal to $1-p$. If this happens at $Y = m-j$, that is, $\sum_{i=m-j}^{m} P(Y = i | m, \pi_u) \ge 1-p$ and $\sum_{i=m-j+1}^{m} P(Y = i | m, \pi_u) < 1-p$, then $m-j$ is the $(p, 1-\alpha)$ upper tolerance limit. Furthermore, it can be checked that $P(Y \le m-j | m, \pi_u) \ge p$. Similarly, the $(p, 1-\alpha)$ lower tolerance limit based on a $1-\alpha$ lower confidence limit π_l can be computed by computing left-tail probabilities.

3.3 Binomial Tolerance Intervals Based on Approximate Quantiles

Recall that one-sided tolerance limits are essentially confidence bounds on appropriate quantiles. On the basis of the normal approximation to the quantity $\frac{Y-m\pi}{\sqrt{m\pi(1-\pi)}}$, the p quantile of a binomial(m,π) distribution is given by $k_p(\pi,m) \simeq m\pi + z_p\sqrt{m\pi(1-\pi)}$. Noting that the above quantile is an increasing function of π, an approximate $(p, 1-\alpha)$ upper tolerance limit for the binomial(m,π) distribution can be obtained by replacing the π in the above expression by a $1-\alpha$ upper confidence limit π_u. More specifically,

$$k_p(\pi_u, m) \simeq [m\pi_u + z_p\sqrt{m\pi_u(1-\pi_u)}], \tag{8}$$

where $[x]$ is the integer nearest to x. Similarly, an approximate $(p, 1-\alpha)$ lower tolerance limit can be obtained as

$$k_p(\pi_l, m) \simeq [m\pi_l - z_p\sqrt{m\pi_l(1-\pi_l)}]. \tag{9}$$

If (π_l, π_u) is a $1-\alpha$ confidence interval for π, then

$$\left[k_{\frac{1-p}{2}}(\pi_l, m), k_{\frac{1+p}{2}}(\pi_u, m)\right] \simeq \left[\left[m\pi_l - z_{\frac{1+p}{2}}\sqrt{m\pi_l(1-\pi_l)}\right], \left[m\pi_u + z_{\frac{1+p}{2}}\sqrt{m\pi_u(1-\pi_u)}\right]\right] \tag{10}$$

is an approximate $(p, 1-\alpha)$ equal-tailed TI.

We refer to the equal-tailed TI (7) with an exact CI for π as the exact TI, while the ones with the score CI as the score TI. The approximate equal-tailed TIs in (8), (9) and (10) can be obtained using either the exact CI or the score CI for π. In general, these approximate TIs based on the exact CIs are as conservative as the exact TIs, and so we consider only the approximate TIs (10) based on the score CIs for π in the sequel; we also refer to these TIs by approximate-score TIs.

3.4 Coverage probabilities of binomial tolerance intervals

It is not necessary to carry out coverage studies for one-sided tolerance limits since one-sided tolerance limits are also one-sided confidence limits for appropriate quantiles. In fact, coverage properties of one-sided tolerance limits are similar to those of the CIs that were used to derive them. On the other hand, coverage probabilities of equal-tailed tolerance intervals should be studied to judge their degree of conservatism.

The coverage probability of a TI is the probability that the TI includes at least a proportion p of the sampled population. It follows from (1) that the coverage probability of a $(p, 1-\alpha)$ TI $[L(X;\alpha,p), U(X;\alpha,p)]$ is given by

$$\begin{aligned} & P_X\left\{P\left(L(X;\alpha,p) \le Y \le U(X;\alpha,p) \ge p | X\right)\right\} \\ = & \sum_{k=0}^{n} \binom{n}{k} \pi^k (1-\pi)^{n-k} \mathrm{I}\left\{\sum_{y=L(k;\alpha,p)}^{U(k;\alpha,p)} \binom{m}{y} \pi^y (1-\pi)^{m-y} \ge p\right\}, \end{aligned} \tag{11}$$

where $X \sim$ binomial(n,π) independently of $Y \sim$ binomial(m,π), and I$\{.\}$ is the indicator function. For the case of $n = m$, Wang and Tsung (2009) evaluated coverage probabilities of the exact equal-tailed TIs using the expression (11). However, it should be noted that the expression (11) is not the appropriate one to evaluate the coverage probability of an equal-tailed TI. Recall that an equal-tailed TI not only should cover at least a proportion p of the distribution, but should also meet the requirement that the proportions falling in the tails should not be more than $(1-p)/2$. As the equal-tailed TIs are more stringent than two-sided TIs that satisfy (1), the coverage probability (11) of an equal-tailed TI must be larger than the true coverage probability. Indeed, to evaluate the coverage probability of a $(p, 1-\alpha)$ equal-tailed TI, the following expression must be used:

$$\sum_{k=0}^{n} \binom{n}{k} \pi^k (1-\pi)^{n-k} \mathrm{I}\left\{k_{\frac{1-p}{2}}(\pi_l, m) \le k_{\frac{1-p}{2}}(\pi, m) \text{ and } k_{\frac{1+p}{2}}(\pi, m) \le k_{\frac{1+p}{2}}(\pi_u, m)\right\}, \tag{12}$$

where (π_l, π_u) is a $1-\alpha$ CI for π based on k and n, and $\left[k_{\frac{1-p}{2}}(\pi_l, m), k_{\frac{1+p}{2}}(\pi_u, m)\right]$ is a tolerance interval as defined in (7). Note that the above expression is the probability (with respect to the binomial(n, π) distribution) that the equal-tailed TI $\left[k_{\frac{1-p}{2}}(\pi_l, m), k_{\frac{1+p}{2}}(\pi_u, m)\right]$ includes the interval $\left[k_{\frac{1-p}{2}}(\pi, m), k_{\frac{1+p}{2}}(\pi, m)\right]$.

We evaluated coverage probabilities of the following equal-tailed TIs using (12) for (a) the interval in (7) with the exact CI for π (referred to as the exact TI), (b) the interval in (7) with the score CI for π (referred to as the score TI), and (c) the approximate TI (10) with the score CI for π (referred to as the approximate-score TI). The summary statistics of the coverage probabilities are given in Table 1 for some sample sizes n, m, and $(1-\alpha, p)$. These summary statistics are based on the exact coverage probabilities for 1,000 randomly generated π's from a uniform(0, 1) distribution. It is clear from Table 1 that the score TIs and the approximate-score TIs have minimum coverage probabilities close to the nominal level. The exact TI is too conservative, especially for small samples, and its minimum coverage probabilities are appreciably higher than the nominal level. The score TI and the approximate-score TI exhibit similar performance, and between them, the approximate-score TI is preferable for its simplicity.

Table 1: Summary statistics of coverage probabilities (12) for equal-tailed binomial TIs
(a) exact TI, (b) score TI, (c) approx.-score TI

	$n=m=20$			$n=30,m=20$			$n=50,m=30$			$n=100,m=60$		
	(a)	(b)	(c)	(a)	(b)	(c)	(a)	(b)	(c)	(a)	(b)	(c)
(.90, .90) TIs												
mean	.971	.953	.953	.974	.962	.958	.966	.953	.954	.955	.939	.945
sd	.017	.030	.030	.017	.023	.028	.020	.025	.024	.018	.025	.022
min	.942	.884	.884	.934	.900	.900	.919	.893	.893	.924	.887	.902
median	.961	.948	.949	.973	.958	.958	.961	.893	.951	.953	.938	.941
(.90, .95) TIs												
mean	.986	.979	.983	.989	.980	.983	.985	.978	.979	.979	.972	.974
sd	.011	.014	.013	.008	.013	.012	.009	.012	.012	.009	.012	.012
min	.959	.944	.956	.961	.951	.957	.961	.949	.952	.956	.948	.953
med	.987	.980	.982	.990	.981	.983	.984	.977	.977	.978	.971	.973
(.90, .99) TIs												
mean	.999	.996	.997	.998	.996	.997	.998	.996	.997	.996	.995	.996
sd	.001	.003	.003	.002	.003	.002	.002	.003	.002	.002	.002	.002
min	.994	.985	.988	.992	.987	.991	.992	.987	.991	.991	.989	.991
med	.999	.997	.998	.998	.996	.997	.998	.996	.997	.996	.995	.995

3.5 Two-sided tolerance intervals

In practical applications, especially in discrete quality assessment, the requirement of "controlling both tails" is unnecessary; a two-sided TI that would include at least a proportion p of the population with coverage probability $1-\alpha$ would serve the intended purpose. A way to find such a two-sided TI is to determine the value of α^* so that $(p, 1-\alpha^*)$ equal-tailed TI will have the minimum coverage probability (11) close to the intended nominal level $1-\alpha$. The resulting TI may not control the percentages in the tails, but it will include at least a proportion p of the distribution with the minimum coverage probability close to the nominal level $1-\alpha$.

For the case of $n=m$, Wang and Tsung (2009) provided a numerical approach to find the value of α^* so that an exact $(p, 1-\alpha^*)$ equal-tailed TI will have the minimum coverage probability (11) close to $1-\alpha$. We have reported the corresponding minimum and average coverage probabilities (based on (11)) in Table 2. The values of α^* for the exact equal-tailed TIs are taken from Table 3 of Wang and Tsung's (2009) paper. For instance, to construct a (0.90, 0.95) tolerance interval when $n=20$, the value of α^* is 0.16. That is, an 84% exact CI for π can produce a TI with content at least 0.90 and minimum coverage probability (based on (11)) 0.945. Note that the values of α^* suggested by Wang and Tsung (2009) depend on n, and they were obtained numerically. Based on extensive coverage studies, we found that a $(p, 1-2\alpha)$ equal-tailed score TI, or a $(p, 1-2\alpha)$ equal-tailed approximate-score TI, had minimum coverage probability (11) close to the nominal level $1-\alpha$. Note that a $(p, 1-2\alpha)$ equal-tailed TI is the same as the one formed by $\left(\frac{1+p}{2}, 1-\alpha\right)$ one-sided lower and upper tolerance limits. It is clear from the coverage probabilities in Table 2 that a 90% score CI for π produces a TI with the minimum coverage probability (11) very close to 0.95.

In Table 3, we give the minimum values and the 5th percentiles of the coverage probabilities

Table 2: The values of α^* so that a $(.90, 1-\alpha^*)$ equal-tailed binomial TI has the minimum coverage probability (11) close to 0.95

	exact TI			score TI			approx.-score TI	
$n = m$	α^*	minimum coverage	average coverage	α^*	minimum coverage	average coverage	minimum coverage	average coverage
10	.25	.949	.984	.10	.949	.984	.949	.987
15	.17	.959	.985	.10	.960	.985	.960	.985
20	.16	.945	.980	.10	.956	.980	.945	.980
25	.16	.955	.979	.10	.955	.979	.955	.980
30	.15	.950	.978	.10	.950	.978	.950	.980
35	.13	.951	.979	.10	.946	.976	.946	.978
40	.12	.958	.982	.10	.944	.974	.944	.974
45	.12	.957	.979	.10	.949	.975	.954	.976
50	.12	.956	.978	.10	.946	.974	.952	.974

(11) of $(p, 1-2\alpha)$ equal-tailed score and approximate-score TIs for some practical values of p and α, and for some values of $n = m$. Examination of the 5th percentiles of coverage probabilities indicates that, in most cases, a $(p, 1-2\alpha)$ equal-tailed TI can be safely used as a $(p, 1-\alpha)$ TI. In particular, we observe from the coverage probabilities under the heading $(p, 0.90)$, $p = 0.90, 0.95, 0.99$ that the 5th percentiles of coverage probabilities are close to the intended nominal level $1 - \alpha = 0.95$ in most cases. Overall, these coverage studies indicate that a $(p, 1-2\alpha)$ equal-tailed TI can be used as a two-sided TI for most practical choices of content and coverage levels.

Table 3: The minimums and the 5th percentiles of coverage probabilities (11) of binomial equal-tailed tolerance intervals

	(.90,.80) TI				(.90,.90) TI				(.90,.98) TI			
	score		approx.-score		score		approx.-score		score		approx.-score	
$n = m$	min	5th	min	5th	min	5th	min	5th	min	5th	min	5th
10	.850	.899	.850	.906	.948	.959	.949	.959	.993	.995	.993	.995
15	.904	.913	.852	.903	.960	.963	.960	.963	.994	.995	.994	.995
25	.891	.897	.891	.897	.956	.959	.945	.958	.993	.995	.993	.995
30	.896	.908	.901	.911	.950	.959	.950	.964	.992	.994	.992	.994
40	.887	.911	.887	.912	.944	.959	.944	.957	.990	.993	.992	.993
50	.881	.905	.881	.904	.946	.959	.952	.959	.993	.993	.993	.994
$n = m$	(.95,.80) TI				(.95,.90) TI				(.95,.98) TI			
10	.862	.869	.890	.920	.972	.978	.939	.961	.996	.998	.998	.999
15	.880	.885	.880	.885	.955	.962	.958	.965	.995	.998	.998	.999
25	.871	.890	.878	.893	.946	.954	.943	.956	.995	.997	.996	.998
30	.868	.880	.869	.887	.944	.952	.938	.956	.994	.997	.997	.997
40	.862	.882	.873	.890	.943	.945	.943	.946	.996	.996	.996	.996
50	.861	.881	.876	.882	.940	.955	.946	.954	.995	.997	.996	.997
$n = m$	(.99,.80) TI				(.99,.90) TI				(.99,.98) TI			
10	.891	.932	.805	.844	.945	.956	.915	.938	.998	.999	.998	.999
15	.853	.863	.776	.850	.947	.957	.949	.958	.995	.996	.998	.998
25	.848	.889	.845	.878	.929	.932	.932	.945	.995	.997	.996	.997
30	.855	.858	.837	.873	.933	.935	.926	.948	.996	.997	.997	.997
40	.854	.873	.854	.882	.938	.943	.943	.944	.995	.996	.996	.996
50	.847	.875	.848	.889	.940	.943	.936	.947	.995	.996	.996	.996

Finally, we note that if n and m are not drastically different, then a $(p, 1-2\alpha)$ equal-tailed TI can be used as a $(p, 1-\alpha)$ TI. If n and m are quite different, then it is not clear as to the choice of α^*. Our coverage studies (not reported here) indicated that if n is much larger than m, then the minimum coverage probabilities (11) of $(p, 1-2\alpha)$ equal-tailed TIs are larger than $1-\alpha$. On the other hand, if n is much smaller than m, then the minimum coverage probabilities are smaller than $1-\alpha$. At present, we do not know any satisfactory method for constructing two-sided TIs when and n and m are quite different.

3.6 Expected Widths

In general, TIs are compared with respect to coverage probabilities and expected widths. As the exact equal-tailed TIs are too conservative, they are expected to be wider than the score or the approximate-score equal-tailed TIs. Therefore, we carried out limited numerical studies to compare the equal-tailed TIs in terms of expected widths. In Table 4, we report summary statistics of expected widths of binomial TIs for $n = m = 20, 30, 50$ and 100. The reported expected widths clearly indicate that the score and the approximate-score equal-tailed TIs are shorter than the exact equal-tailed TIs. We also see that the score and the approximate-score equal-tailed TIs are comparable with respect to expected widths.

Table 4: Summary statistics of expected widths of $(0.90, 0.90)$ equal-tailed binomial TIs; based on 1,000 values of π generated from uniform$(0, 1)$

	$n = 20$			$n = 30$		
statistics	exact	score	approx.-score	exact	score	approx.-score
mean	11.24	10.67	10.82	14.20	13.49	13.78
sd	2.61	2.28	2.34	3.25	3.34	3.32
min	5.04	5.04	5.04	6.06	5.06	5.06
median	12.20	11.44	11.52	15.31	14.67	15.05
	$n = 50$			$n = 100$		
mean	18.52	17.78	17.91	26.36	25.37	25.42
sd	4.73	4.61	4.67	7.16	6.81	6.79
min	6.10	5.15	5.10	6.20	5.30	5.30
median	20.25	19.35	19.51	28.89	27.98	28.03

3.7 An example for binomial tolerance intervals

To illustrate the methods for constructing binomial tolerance intervals, we shall use the example given in Wang and Tsung (2009). The data are given on the NIST webpage[1], and they represent fractions of defective chips in a sample of wafers. A chip in a wafer is considered to be defective whenever a misregistration, in terms of horizontal and/or vertical distances from the center, is recorded. On each wafer, locations of 50 chips were measured and the proportion of defective chips was recorded. As the original data (based on 30 wafers) was over-dispersed, Wang and Tsung (2009) used a part of the data consisting of 21 wafers, for illustration purpose. In order to compare our results with those in Wang and Tsung's paper, we shall use data on the same 21 wafers, and

[1]http://www.itl.nist.gov/div898/handbook/pmc/section3/pmc332.htm

the data are reproduced in Table 5. To begin with, we tested the equality of the proportions of defective chips across the 21 wafers using a chi-square statistic $\sum_{i=1}^{21} \frac{n_i(\widehat{\pi}_i - \widehat{\pi})^2}{\widehat{\pi}(1-\widehat{\pi})} = 19.58$ yielding the p-value $P(\chi^2_{20} > 19.58) = 0.4842$. Here the n_i's are all equal to 50, $\widehat{\pi}_i$'s are the sample fractions of defective given in Table 5, and the overall proportion of defective $\widehat{\pi} = \frac{\sum_{i=1}^{21} n_i \widehat{\pi}_i}{\sum_{i=1}^{21} n_i} = \frac{196}{1050} = 0.1867$. So, we can ignore the wafer-to-wafer variability, and combine the sample fractions to estimate the π, the true proportion of defective chips in a wafer.

We shall compute (0.90, 0.95) one-sided as well as (0.90, 0.95) two-sided TIs using the approaches given in the preceding sections. Towards this, we note that $n = \sum_{i=1}^{21} n_i = 1050$, $k =$ the total number of defective chips, which is 196, and $m = 50$. We want to construct TIs for the binomial(m, π) distribution, where π is the unknown proportion of defective chips in a wafer. Using (3), we computed 95% exact lower confidence limit as $\pi_l = B_{196,855;.05} = 0.1671$ and the 95% exact upper confidence limit as $\pi_u = B_{197,854;.95} = 0.2076$. Using these confidence limits, it can be readily checked that

$$\sum_{i=5}^{50} \binom{50}{i} \pi_l^i (1-\pi_l)^{50-i} = 0.9366 \quad \text{and} \quad \sum_{i=0}^{14} \binom{50}{i} \pi_u^i (1-\pi_u)^{50-i} = 0.9205. \tag{13}$$

Furthermore, it can be checked that 5 is the largest integer for which the first sum is at least 0.90, and 14 is the smallest integer for which the second sum is at least 0.90. So, a (0.90, 0.95) lower tolerance limit is equal to 5, and a (0.90, 0.95) upper tolerance limit is equal to 14. Thus, with 95% confidence, we can say that at least 90% of wafers have five or more defective chips. Furthermore, at least 90% of wafers have 14 or fewer defective chips, with confidence 95%. On the basis of our earlier coverage studies, we shall use a 90% CI for π to find a (0.90, 0.95) two-sided tolerance interval. Note that the above 95% one-sided confidence limits form a 90% two-sided CI for π. Also, we see that

$$\sum_{i=4}^{50} \binom{50}{i} \pi_l^i (1-\pi_l)^{50-i} = 0.9766 \quad \text{and} \quad \sum_{i=0}^{15} \binom{50}{i} \pi_u^i (1-\pi_u)^{50-i} = 0.9580. \tag{14}$$

We also note that 4 is the largest integer so that the first sum in (14) is at least $\frac{1+p}{2} = 0.95$, and 15 is the smallest integer so that the second sum in (14) is at least 0.95. Thus a (0.90, 0.95) two-sided tolerance interval is the interval [4, 15]. This means that at least 90% of the wafers have 4 to 15 defective chips with 95% confidence.

To find the TIs (5), (6) and (7) using the score CIs for π, the necessary normal percentiles are $z_{.90} = 1.282$ and $z_{.95} = 1.645$. The 90% score CI (using $\widehat{\pi} = 0.1867$ and $n = 1050$ in (4)) is (0.1677, 0.2072). Note that the endpoints are 95% one-sided confidence limits for π. Using these limits, and proceeding as above, we get the (0.90, 0.95) tolerance interval to be the interval [4, 15]. Furthermore, 5 is the one-sided lower tolerance limit and 14 is the one-sided upper tolerance limit. Thus, the score and the exact tolerance TIs are the same for this example.

To find the TIs using the approximate quantiles, we can use the score CIs. Using the score

Table 5: Fractions of defective chips in a sample of 21 wafers

Sample Number	Fraction of Defective, $\widehat{\pi}_i$	Sample Number	Fraction Defective, $\widehat{\pi}_i$	Sample Number	Fraction of Defective, $\widehat{\pi}_i$
1	.24	11	.10	21	.22
2	.16	12	.12	22	.18
3	.20	13	.24	23	.24
4	.14	14	.16	24	.14
5	.18	15	.20	25	.26
6	.28	16	.10	26	.18
7	.20	17	.26	27	.12

confidence limit $\pi_u = 0.2072$ in (8) with $p = 0.90$, we get $14.06 \simeq 14$; using $\pi_l = 0.1677$ in (9), we get $4.99 \simeq 5$. Thus, 5 and 14 are one-sided tolerance limits. Similarly, using these confidence limits in (10) with $p = 0.90$, we get [4, 15] as a (0.90, 0.95) two-sided tolerance interval.

Thus, we see that all the methods produced results that are in complete agreement. It should be noted that the reported (0.90, 0.95) tolerance interval in Wang and Tsung's (2009) paper is $[1, 21]$, which is in error. In fact, their (0.90, 0.95) TI is based on the 88% (choosing $\alpha^* = .12$; see the value of α^* for $n = 50$ in Table 2) exact CI for π, which is (0.1681, 0.2064). The equal-tailed TI based on this CI is also [4, 15].

4. Poisson distribution

One-sided as well as equal-tailed TIs for a Poisson distribution can be obtained using the methods for the binomial case. Let $X_1, ..., X_n$ be a sample from a Poisson(λ) distribution. Let $S = \sum_{i=1}^n X_i$ so that $S \sim$ Poisson($n\lambda$). The maximum likelihood estimator of λ is given by $\widehat{\lambda} = \frac{S}{n}$.

4.1 The exact and score CIs for a Poisson mean

The $1 - \alpha$ Garwood (1936) exact CI for λ is given by

$$(\lambda_l, \lambda_u) = \left(\frac{1}{2n} \chi^2_{2s;\alpha/2}, \frac{1}{2n} \chi^2_{2s+2;1-\alpha/2} \right), \tag{15}$$

where s is an observed value of S, and $\chi^2_{m;\alpha}$ is the α quantile of a chi-square distribution with m degrees of freedom. The above CI should be used with the convention that $\chi^2_{0;\alpha} = 0$.

The score CI is on the basis of asymptotic normality of the test statistic $T(\widehat{\lambda}, \lambda) = \frac{\widehat{\lambda} - \lambda}{\sqrt{\lambda/n}}$, where $\widehat{\lambda} = \frac{S}{n}$. In particular, the endpoints of the $1 - \alpha$ score CI are the roots of the quadratic equation $T^2(\widehat{\lambda}, \lambda) = c^2$, where $c = z_{1-\alpha/2}$, and are given by

$$(\lambda_l, \lambda_u) = \widehat{\lambda} + \frac{c^2}{2n} \pm \frac{c}{\sqrt{n}} \sqrt{\widehat{\lambda} + \frac{c^2}{4n}}. \tag{16}$$

4.2 Tolerance intervals for a Poisson distribution

The $(p, 1-\alpha)$ upper tolerance limit is the smallest integer $k_p(\lambda_u)$ so that $P(X \le k_p(\lambda_u)|\lambda_u) \ge p$, where λ_u is the $1-\alpha$ upper confidence limit for λ based on an observed value s of S. Similarly, a $(p, 1-\alpha)$ lower tolerance limit is the largest integer $k_{1-p}(\lambda_l)$ so that $P(X \ge k_{1-p}(\lambda_l)|\lambda_l) \ge p$, where λ_l is the $1-\alpha$ lower confidence limit for λ. If (λ_l, λ_u) is a $1-\alpha$ confidence interval λ, then $\left[k_{\frac{1-p}{2}}(\lambda_l), k_{\frac{1+p}{2}}(\lambda_u)\right]$ is a $(p, 1-\alpha)$ equal-tailed TI.

As in the binomial case, we shall refer to the TIs based on the exact CIs as the exact TIs and the ones based on the score CIs as the score TIs.

The TIs based on the normal approximation to a Poisson quantile are as follows: Let λ_l and λ_u be a $1-\alpha$ one-sided lower and upper confidence limits for λ, respectively. Then $\lambda_u + z_p\sqrt{\lambda_u}$ is a $(p, 1-\alpha)$ upper tolerance limit, $\lambda_l - z_p\sqrt{\lambda_l}$ is a $(p, 1-\alpha)$ lower tolerance limit. If (λ_l, λ_u) is a $1-\alpha$ CI for λ, then $\left[\lambda_l - z_{\frac{1+p}{2}}\sqrt{\lambda_l},\ \lambda_u + z_{\frac{1+p}{2}}\sqrt{\lambda_u}\right]$ is a $(p, 1-\alpha)$ equal-tailed TI. If λ_l and λ_u are score confidence limits, then we refer to the corresponding TIs as the approximate-score TIs.

4.3 Two-sided tolerance intervals for a Poisson distribution

As in the binomial case, we suggest to use $1-2\alpha$ CI for λ so that $\left[k_{\frac{1-p}{2}}(\lambda_l), k_{\frac{1+p}{2}}(\lambda_u)\right]$ can be used as a two-sided TI with the minimum coverage probability close to the nominal level $1-\alpha$. We evaluated the coverage probabilities (17) of (0.90, 0.90) equal-tailed TIs using the expression

$$\sum_{k=0}^{\infty} \frac{e^{-\lambda}\lambda^k}{k!} I\left\{P\left(k_{\frac{1-p}{2}}(\lambda_l) \le X \le k_{\frac{1+p}{2}}(\lambda_u)\middle|\lambda_l, \lambda_u\right) \ge p\right\}, \quad (17)$$

where (λ_l, λ_u) is a $1-\alpha$ CI for λ based on k, and $\left[k_{\frac{1-p}{2}}(\lambda_l), k_{\frac{1+p}{2}}(\lambda_u)\right]$ is an equal-tailed TI based on (λ_l, λ_u). An equal-tailed TI is considered to be a satisfactory two-sided TI provided the above coverage probabilities are close to the nominal level $1-\alpha$ for all λ. The coverage probabilities were evaluated for 1,000 values of λ generated from different uniform distributions as indicated in Table 6. The minimum and average coverage probabilities are given in Table 6 for the exact, score and the approximate-score equal-tailed TIs. We observe from these reported values that the minimum coverage probabilities of the score and approximate-score TIs are closer to 0.95 than those of the exact TIs. This indicates that the score and the approximate-score TIs are less conservative than the exact TIs.

Table 6: Minimum and average coverage probabilities (17) of (.90, .90) equal-tailed Poisson TIs

$\lambda \sim$	exact		score		approx.-score	
	minimum coverage	average coverage	minimum coverage	average coverage	minimum coverage	average coverage
uniform(1, 4)	.979	.994	.970	.989	.956	.989
uniform(5,10)	.969	.986	.955	.972	.960	.976
uniform(10,15)	.966	.976	.949	.970	.953	.974
uniform(15,20)	.962	.972	.957	.969	.955	.968

We also did similar coverage studies for (0.90, 0.98) and (0.95, 0.90) equal-tailed TIs (these are not reported here). In general, we found that a $1-2\alpha$ CIs for λ produce TIs with minimum coverage probability close to $1-\alpha$. We once again observe that (see Table 6) the approximate-score TIs and the score CIs are comparable with respect to coverage probabilities. Furthermore, the minimum coverage probabilities of these two TIs are very close to 0.95 for large λ. These results indicate that a $(p, 1-2\alpha)$ equal-tailed TI can be used as an ordinary $(p, 1-\alpha)$ TI (that is, the one that does not necessarily control the percentages in tails).

4.4 An example for Poisson tolerance intervals

This example concerns the number of surface defects of steel plates. The data are given in Montgomery (1996), and Wang and Tsung (2009) used a part of the data for constructing Poisson tolerance intervals. We use the same set of data as given in Wang and Tsung (2009) so that we can compare their tolerance intervals with those based on the methods considered in this paper. The counts of surface defects on 21 steel plates are

$$1, 0, 4, 3, 1, 2, 0, 2, 1, 1, 0, 0, 2, 1, 3, 4, 3, 1, 0, 2, 4.$$

The maximum likelihood estimate $\widehat{\lambda} = \frac{35}{21} = 1.6667$. The one-sided as well as two-sided tolerance intervals based on the approaches given in the preceding section are given in Table 7. On the basis of our coverage studies, we used a 90% two-sided confidence interval for the Poisson mean to construct a (0.90, 0.95) two-sided tolerance interval. All approaches produced the same tolerance interval $[0, 5]$. Furthermore, we can conclude that the number of surface defects in 90% of steel plates is at most four with 95% confidence.

For this example, the reported (0.90, 0.95) tolerance interval in Wang and Tsung's (2009) paper is $[0, 12]$, which is also in error. The suggested value of α^* by these authors to find a $(0.90, 0.95)$ TI is 0.17. That is, TI should be constructed on the basis of an 83% two-sided exact CI of λ, which is (1.2957, 2.1188). It can be easily verified that the TI based on this CI is also [0, 5].

5. Concluding remarks

Since the publication of Agresti and Coull's (1998) paper, several authors have studied the properties of likelihood score-based approaches for problems involving binomial proportions or Poisson means. In general, it was observed that the exact approaches, such as the Clopper-Pearson method, produced confidence limits that were too conservative, yielding confidence intervals unnecessarily wide. In this article, we showed another situation where the exact method is less efficient than the approximate ones. For computing TIs, we note that Young (2009) provides a R-package that computes TIs for a binomial or Poisson distribution using different CIs (exact, score, Agresti-Coull, etc.) for the parameter. Zaslavsky (2009) gives details of computing Poisson TIs using *StatXact*. However, we note that no software is required to compute our closed-form approximate TIs; they can be computed easily using a calculator. Furthermore, we have shown via numerical studies that these approximate TIs are preferable to others for their simplicity and accuracy.

Table 7: (0.90, 0.95) Poisson TIs for the number of surface defects in a steel plate

Method	Confidence Intervals for λ	Tolerance Intervals	Method	Confidence Intervals for π	Tolerance Intervals
Exact	95% lower limit: $\frac{1}{42}\chi^2_{70;.05} = 1.2319$ 95% upper limit: $\frac{1}{42}\chi^2_{72;.95} = 2.2097$ 90% two-sided: (1.2319, 2.2097)	one-sided lower: 0 one-sided upper: 4 two-sided: [0, 5]	Score	one-sided lower: 1.2184 one-sided: upper: 2.1542 90% two-sided: (1.2184, 2.1542)	one-sided lower: 0 one-sided upper: 4 two-sided: [0, 5]
			Approx.-score		one-sided lower: 0 one-sided upper: 4 two-sided: $[0, 5]$

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