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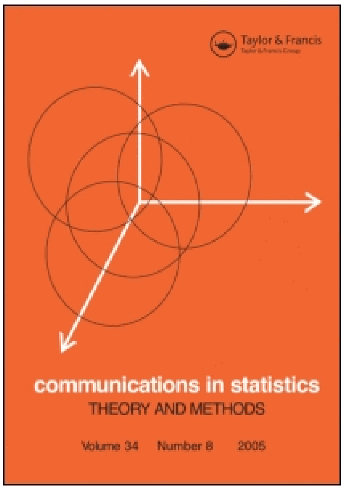

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Tolerance Intervals for the Distribution of the Difference Between Two Independent Normal Random Variables

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Tolerance Intervals for the Distribution of the Difference Between Two Independent Normal Random Variables

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The problem of constructing tolerance interval for the distribution of the difference between two independent normal random variables possibly with different variances is considered. Assuming that the variance ratio is known, an exact method and a simple approximate method for constructing tolerance factor are presented. Also, approximate methods for obtaining tolerance interval are provided when the variances are unknown and arbitrary. The coverage probabilities of the approximate methods and an available approximate method are evaluated via Monte Carlo simulation study. Simulation study indicates that the approximate tolerance intervals are quite satisfactory even when the sample sizes are small whereas the existing method is conservative for small to moderate samples. The proposed tolerance interval can be used safely for practical applications. The methods are illustrated using an example.

Keywords Coverage probability; Lognormal distribution; Moment approximation; Stress-strength reliability.

Mathematics Subject Classification 62F03; 62F25.

1. Introduction

In many practical situations, one wants to assess the proportion of data that fall within an interval. Typically, sample is drawn from a population of interest, and interval based on sample data is constructed so that it would contain at least a proportion p of the population with confidence γ. Such an interval is usually referred to as the p content–γ coverage tolerance interval. For example, engineering products must satisfy specification limits to work properly. To estimate the proportion of the

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products that are within the specification limits, a p content–γ coverage tolerance interval (TI) based on a sample of products may be constructed. If the interval falls within the specification limits, then it can be concluded that at least proportion p of products would function correctly. Tolerance intervals are also routinely used in acceptance sampling plan.

If the variable of interest follows a normal distribution, then methods for constructing TIs are available in the literature. Earlier work on construction of tolerance limits, due to Wald (1943) and Wald and Wolfowitz (1946), are all based on normality assumption. Exact methods for computing one-sided tolerance limits, two-sided TIs and two-sided TIs controlling both tails (Owen, 1964) are available for normal distribution. Factors for constructing TIs for normal distributions are tabulated for a wide range of sample sizes, and many software that compute tolerance factors are also available.

In this article, we consider the problem of constructing tolerance intervals for the distribution of the difference between two independent normal random variables X_1 and X_2. One-sided tolerance limits for the distribution of $X_1 - X_2$ have been used to assess the reliability in stress-strength models (Guo and Krishnamoorthy, 2004; Hall, 1984; Reiser and Guttman, 1986; Weerahandi and Johnson, 1992). Specifically, in classical stress-strength model X_1 represents the random strength of a component and X_2 represents the random stress to which the component is subjected. If $X_1 \le X_2$, then either the component fails or the system that uses the component may malfunction. It is desired to estimate the reliability $R = P(X_1 - X_2 > 0)$, that is, the probability that X_1 exceeds X_2. In this case, if a p content–γ coverage lower tolerance limit based on a sample of data is greater than zero, then the reliability is expected to be at least p. Assuming normal distributions, the aforementioned articles provide approximate one-sided tolerance limits for this stress-strength reliability.

We are here mainly concerned about TI for the distribution of $X_1 - X_2$, where X_1 and X_2 are independent normal random variables. Tolerance intervals for the distribution of $X_1 - X_2$ can be used to assess the range of the distribution of $X_1 - X_2$ or to check if two sets of measurements are significantly different. To setup the problem, let $X_1 \sim N(\mu_1, \sigma_1^2)$ independently of $X_2 \sim N(\mu_2, \sigma_2^2)$. Let $\bar{x}_i$ and s_i^2 denote, respectively, the mean and variance based on a sample of n_i observations from $N(\mu_i, \sigma_i^2)$, $i = 1, 2$. Let $X_d = X_1 - X_2$, $\mu_d = \mu_1 - \mu_2$ and $\bar{x}_d = \bar{x}_1 - \bar{x}_2$. The two-sided tolerance factor k is determined such that the interval $\bar{x}_d \pm kh(s_1^2, s_2^2)$ would contain at least $100p\%$ of the distribution of X_d with confidence $1 - \alpha$. That is, k is to be determined such that:

$$P_{\bar{x}_d, s_1^2, s_2^2}\left[P_{X_d}\left(\bar{x}_d - kh(s_1^2, s_2^2) \le X_d \le \bar{x}_d + kh(s_1^2, s_2^2)\right) > p \,\middle|\, \bar{x}_d, s_1^2, s_2^2\right] = 1 - \alpha. \quad (1.1)$$

It should be noted that the method that we will propose is also applicable to construct tolerance limits for the ratio of two independent lognormal random variables. Specifically, if Y_1 and Y_2 are independent lognormal random variables with parameters (μ_1, σ_1^2) and (μ_2, σ_2^2), then $X_i = \ln(Y_i) \sim N(\mu_i, \sigma_i^2)$, $i = 1, 2$. Therefore, if $\bar{x}_d \pm kh(s_1^2, s_2^2)$ is a tolerance interval for the distribution of $X_1 - X_2$, then $(\exp(\bar{x}_d - kh(s_1^2, s_2^2)), \exp(\bar{x}_d + kh(s_1^2, s_2^2))$ is a tolerance interval for the distribution of Y_1/Y_2.

Liao et al. (2005) considered the problem of setting TIs in a general setup where the variance of the normal distribution is assumed to be of the form $\sum_{i=1}^{g} c_i\sigma_i^2$. These authors provided a generalized variable approach that yields approximate TIs.

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We shall show in the sequel that their TIs for the present special case are too conservative when the sample sizes are small to moderate.

The rest of the article is organized as follows. In the following section, we outline an exact method and a convenient approximate method for computing tolerance factors assuming that the variance ratio is known. In Sec. 3, using the ideas of Hall (1984) and Guo and Krishnamoorthy (2004), we provide an approximate method when the variances are unknown and arbitrary. We also describe the generalized variable approach due to Liao et al. (2005), and develop a TI using the Satterthwaite approximation. In Sec. 4, we carry out Monte Carlo studies to evaluate the accuracy of the approximate method. Our Monte Carlo studies indicate that the proposed approximate TIs are satisfactory even for small samples whereas the generalized variable approach seems to be conservative. The methods are illustrated using an example in Sec. 5, and some concluding remarks are given in Sec. 6.

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2. Tolerance Intervals When the Variance Ratio is Known

In this section, assuming that the variance ratio is known, we shall outline an exact method and an approximate method of computing the tolerance factor that satisfies 1.1.

2.1. *An Exact Method*

Let $X_1 \sim N(\mu_1, \sigma_1^2)$ independently of $X_2 \sim N(\mu_2, \sigma_2^2)$. Let $\bar{x}_i$ and s_i^2 denote, respectively, the mean and variance based on a sample of n_i observations from $N(\mu_i, \sigma_i^2)$, $i = 1, 2$. Let $X_d = X_1 - X_2$, $\mu_d = \mu_1 - \mu_2$, $\bar{x}_d = \bar{x}_1 - \bar{x}_2$, $q_1 = \sigma_1^2/\sigma_2^2$ and

$$h(s_1^2, s_2^2) = s_p^2 = \frac{(1 + q_1^{-1})((n_1 - 1)s_1^2 + (n_2 - 1)q_1 s_2^2)}{n_1 + n_2 - 2}. \tag{2.1}$$

Let

$$\eta_1 = \sqrt{\frac{\sigma_1^2/n_1 + \sigma_2^2/n_2}{\sigma_1^2 + \sigma_2^2}} = \sqrt{\frac{q_1/n_1 + 1/n_2}{q_1 + 1}}, \quad \text{and} \quad Y \sim N(0, \eta_1^2). \tag{2.2}$$

An exact p content–$(1 - \alpha)$ coverage TI is given by

$$\bar{x}_d \pm k s_p, \tag{2.3}$$

where k is the solution of the integral equation

$$\frac{2}{\sqrt{2\pi}\eta_1} \int_0^\infty P\left(\chi^2_{n_1+n_2-2} > \frac{(n_1 + n_2 - 2)\chi^2_{1,p}(y^2)}{k^2}\right) e^{-\frac{y^2}{2\eta_1^2}} dy = 1 - \alpha, \tag{2.4}$$

where $\chi^2_{m,p}(\delta)$ denotes the pth quantile of a noncentral chi-square random variable with the degrees of freedom (df) $= m$ and noncentrality parameter δ. For derivation of (2.4), see the Appendix.

The integral equation (2.4) is similar to the one for finding the tolerance factor in the one-sample case (except that in the one-sample case $Y \sim N(0, 1/n)$, where n is the sample size, whereas the Y in (2.4) follows $N(0, \eta_1^2)$), and so available numerical method for the latter case (e.g., Eberhardt et al., 1989) can be used to compute the factor k.

2.2. *An Approximate Method*

Even though the tolerance factor k satisfying (2.4) can be computed using available numerical method, still it is computationally intensive. So, it is worthwhile to point out a highly accurate and simple approximate method. An approximation to the tolerance factor k is derived in the Appendix, and is given by

$$k \simeq \left(\frac{(n_1 + n_2 - 2)\chi^2_{1,p}(\eta_1^2)}{\chi^2_{n_1+n_2-2,\alpha}} \right)^{\frac{1}{2}}, \tag{2.5}$$

where $\chi^2_{m,p}(c)$ denotes the 100pth percentile of a noncentral chi-square distribution with df m and non centrality parameter c, and $\chi^2_{m,p}$ is the 100pth percentile of a chi-square distribution with df$= m$. It is interesting note that the tolerance factor k does not depend on any parameters when $n_1 = n_2 = n$, because in this case $\eta_1 = 1/n$.

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3. Tolerance Intervals When the Variances are Unknown and Arbitrary

If the variance ratio is unknown, then the TI can be constructed from the one in Sec. 2 with $q_1 = \sigma_1^2/\sigma_2^2$ replaced by its sample estimate. However, our preliminary numerical studies showed that the resulting TI is too liberal in the sense that the coverage probabilities are often much smaller than the nominal level. Therefore, we give an alternative approximate method following Hall (1984) and Guo and Krishnamoorthy (2004).

3.1. *An Approximate Method*

When the variances are unknown, we use $s_d^2 = s_1^2 + s_2^2$ as an estimate of $\mathrm{Var}(X_1 - X_2)$ instead of s_p^2 used in Sec. 2.1. Using the moment matching method, it can be shown that

$$\frac{s_d^2}{\sigma_1^2 + \sigma_2^2} \sim \frac{\chi^2_{f_1}}{f_1} \quad \text{approximately, where } f_1 = \frac{(n_1 - 1)(q_1 + 1)^2}{q_1^2 + (n_1 - 1)/(n_2 - 1)} \quad \text{and} \quad q_1 = \frac{\sigma_1^2}{\sigma_2^2}.$$

In the above f_1 is determined so that the mean and variance of $s_d^2/(\sigma_1^2 + \sigma_2^2)$ are matching with those of $\chi^2_{f_1}/f_1$. Using s_d^2 as a variance estimate, an approximate tolerance factor can be derived using the results in Sec. 2.1. The only difference is that the variance estimate s_p^2 in (2.1) is distributed as a constant times $\chi^2_{n_1+n_2-2}$ random variable whereas s_d^2 is approximately distributed as a constant times $\chi^2_{f_1}$ random variable. Therefore, an approximate tolerance factor can be obtained by replacing $n_1 + n_2 - 2$ in (2.5) by f_1. Thus, using the variance estimate s_d^2, we have an approximation to the tolerance factor as

$$k_* \simeq \left(\frac{f_1 \chi^2_{1,p}(\eta_1^2)}{\chi^2_{f_1,\alpha}} \right)^{\frac{1}{2}}, \tag{3.1}$$

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where η_1 is given in (2.2). The factor k_* can be easily computed using software that compute the percentiles of the central and noncentral chi-square distributions.

Notice that k_* in (3.1) depends on the unknown variance ratio q_1, which must be estimated. If q_1 is replaced by s_1^2/s_2^2, then resulting tolerance factor is not accurate, especially when the sample sizes are unequal. So, we decided to use the approach for constructing one-sided tolerance limits given in Guo and Krishnamoorthy (2004). Toward this, we note first that an unbiased estimate of the variance ratio depending on the labelling of the population variances. For example, if the variance ratio is defined as $q_1 = \sigma_1^2/\sigma_2^2$, then $\hat{q}_1 = (n_2-3)s_1^2/((n_2-1)s_2^2)$ is an unbiased estimate of q_1. On the other hand, if it is defined as $q_2 = \sigma_2^2/\sigma_1^2$, then $\hat{q}_2 = (n_1-3)s_2^2/((n_1-1)s_1^2)$ is an unbiased estimate of q_2. Using $\hat{q}_1$, we get the tolerance factor

$$k_1 = \left(\frac{\hat{f}_1\chi^2_{1,p}(\hat{\eta}_1^2)}{\chi^2_{\hat{f}_1,\alpha}}\right)^{\frac{1}{2}}, \tag{3.2}$$

where $\hat{\eta}_1^2$ is the estimate of η_1^2, which is obtained by replacing q_1 by $\hat{q}_1$ in the expression for η_1^2 in (2.2). The estimate $\hat{f}_1$ of f_1 is obtained similarly.

If the variance ratio is defined as $q_2 = \sigma_2^2/\sigma_1^2$, then the tolerance factor in (3.1) can be expressed as $\left(\frac{f_2\chi^2_{1,p}(\eta_2^2)}{\chi^2_{f_2,\alpha}}\right)^{\frac{1}{2}}$ with $f_2 = \frac{(n_2-1)(q_2+1)^2}{q_2^2+(n_2-1)/(n_1-1)}$ and $\eta_2^2 = \frac{(q_2 n_1/n_2+1)}{n_1(q_2+1)}$. Again replacing q_2 by $\hat{q}_2 = (n_1-3)s_2^2/((n_1-1)s_1^2)$, we get

$$k_2 = \left(\frac{\hat{f}_2\chi^2_{1,p}(\hat{\eta}_2^2)}{\chi^2_{\hat{f}_2,\alpha}}\right)^{\frac{1}{2}}, \tag{3.3}$$

where $(\hat{f}_2, \hat{\eta}_2)$ is (f_2, η_2) with q_2 replaced by $\hat{q}_2$. Because $\eta_1 = \eta_2 = 1/n$ when $n_1 = n_2 = n$, we should take $\hat{\eta}_1 = \hat{\eta}_2 = 1/n$ in the equal sample size case. Thus, the approximate tolerance interval based on k_1 is given by

$$(L_1, U_1) = \left(\bar{x}_d - k_1\sqrt{s_1^2+s_2^2},\ \bar{x}_d + k_1\sqrt{s_1^2+s_2^2}\right), \tag{3.4}$$

and the one based on k_2 is given by

$$(L_2, U_2) = \left(\bar{x}_d - k_2\sqrt{s_1^2+s_2^2},\ \bar{x}_d + k_2\sqrt{s_1^2+s_2^2}\right). \tag{3.5}$$

Our preliminary simulation studies of the coverage probabilities indicated that the coverage probabilities of TI with $\hat{q}_1$ are very close to the nominal level whenever $\sigma_1^2 < \sigma_2^2$ and $n_1 > n_2$ and the coverage probabilities of TI with $\hat{q}_2$ are very close to the nominal level whenever $\sigma_1^2 > \sigma_2^2$ and $n_1 < n_2$. In general, we found if one of the TIs is too liberal for some sample size and parameter configurations, then the other performs satisfactorily at the same configurations. These findings suggest that the TI defined by

$$\bar{x}_d \pm k_m s_d, \quad \text{with } k_m = \max\{k_1, k_2\}, \tag{3.6}$$

would be less liberal for all parameter and sample size configurations. Notice that the interval (3.6) is the wider of the intervals (3.4) and (3.5).

3.2. *Other Approximate Approaches*

We shall now provide some alternative approximate approaches following the Liao et al.'s (2005) method. Their approach involves determining a "margin of error statistic" D so that

$$P_{\bar{x}_d, s_1^2, s_2^2}\left[P_{X_d}\{\bar{x}_d - D \le X_d \le \bar{x}_d + D \,|\, \overline{X}_d, s_1^2, s_2^2\} \ge p\right] = \gamma, \tag{3.7}$$

where $X_d \sim N(\mu_d, \sigma_d^2)$ and $\sigma_d^2 = \sigma_1^2 + \sigma_2^2$. Once the D satisfying the above requirement is obtained, the TI is given by $\overline{X}_d \pm D$. Let $\sigma_{dn}^2 = \sigma_1^2/n_1 + \sigma_2^2/n_2$. After standardizing X_d, we can write

$$P_{X_d}\{\bar{x}_d - D \le X_d \le \bar{x}_d + D \,|\, \overline{X}_d, s_1^2, s_2^2\} = \Phi\left(Z\frac{\sigma_{dn}}{\sigma_d} + \frac{D}{\sigma_d}\right) - \Phi\left(Z\frac{\sigma_{dn}}{\sigma_d} - \frac{D}{\sigma_d}\right), \tag{3.8}$$

where $Z = (\bar{x}_d - \mu_d)/\sigma_{dn}$ is a standard normal random variable, and Φ is the standard normal distribution function. Thus, the probability requirement in (3.7) can be expressed as

$$P_{\bar{x}_d, s_1^2, s_2^2}\left[\Phi\left(\frac{\sigma_{dn}Z + D}{\sigma_d}\right) - \Phi\left(\frac{\sigma_{dn}Z - D}{\sigma_d}\right) \ge p\right] = \gamma. \tag{3.9}$$

Notice that, for a fixed u, $\Phi(u + x) - \Phi(u - x)$ is an increasing function x, and so $\Phi(u + x) - \Phi(u - x) \ge p$ if and only if $x \ge r$, where r is the root of the equation $\Phi(u + r) - \Phi(u - r) = p$. Applying this result with $u = \sigma_{dn}Z/\sigma_d$ and $x = D/\sigma_d$, we see that (3.9) holds if and only if

$$P_{\bar{x}_d, s_1^2, s_2^2}\left[D^2 \ge c^2\sigma_d^2\right] = \gamma, \tag{3.10}$$

where c is the root of the equation

$$\Phi\left(\frac{\sigma_{dn}}{\sigma_d}Z + c\right) - \Phi\left(\frac{\sigma_{dn}}{\sigma_d}Z - c\right) = p. \tag{3.11}$$

Noting that $\sigma_{dn}Z/\sigma_d \sim N(0, \sigma_{dn}^2/\sigma_d^2)$, it can be shown that (see (A.6) of the Appendix), for fixed Z, the c that satisfies (3.11) is given by

$$c^2 = \chi^2_{1;p}\left(Z^2\frac{\sigma_{dn}^2}{\sigma_d^2}\right). \tag{3.12}$$

Thus, we need to determine D^2 so that

$$P_{Z^2, D^2}\left\{D^2 \ge \sigma_d^2\chi^2_{1;p}\left(Z^2\frac{\sigma_{dn}^2}{\sigma_d^2}\right)\right\} = \gamma. \tag{3.13}$$

3.2.1. *The Liao–Lin–Iyer Tolerance Interval.* Howe (1969) provided a series expression for $\chi^2_{1;p}(\delta^2)$ in terms of a standard normal quantile as

$$\chi^2_{1;p}(\delta^2) = z^2_{\frac{1+p}{2}}\left[1 + \delta^2 + \frac{3 - z^2_{\frac{1+p}{2}}}{6}\delta^4 + \dots\right],$$

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where z_α denote the α quantile of a standard normal distribution. Replacing δ^2 with $E(Z^2\sigma_{dn}^2/\sigma_d^2)$, and keeping only the first two terms, Liao et al. (2005) obtained an approximation for (3.13) as

$$P_{D^2}\left\{D^2 \geq z^2_{\frac{1+p}{2}}(\sigma_d^2 + \sigma_{dn}^2)\right\} = \gamma. \tag{3.14}$$

The above equation implies that D^2 is a γ level upper confidence limit for $z^2_{\frac{1+p}{2}}(\sigma_d^2 + \sigma_{dn}^2)$. Thus, the problem simplifies to find a γ level upper confidence limit for $\sigma_d^2 + \sigma_{dn}^2 = \sigma_1^2(1 + 1/n_1) + \sigma_2^2(1 + 1/n_2)$. To find a confidence limit for this quantity, Liao et al. (2005) suggested a generalized confidence limit, which is the γ quantile of the *generalized pivotal quantity*

$$Q = \left(1 + \frac{1}{n_1}\right)\frac{s_{10}^2(n_1 - 1)}{\chi^2_{n_1-1}} + \left(1 + \frac{1}{n_2}\right)\frac{s_{20}^2(n_2 - 1)}{\chi^2_{n_2-1}}, \tag{3.15}$$

where (s_{10}^2, s_{20}^2) is an observed value of (s_1^2, s_2^2). For a given (s_{10}^2, s_{20}^2), the percentiles of Q can be estimated using Monte Carlo simulation. If Q_α denotes the α quantile of Q, then an approximate $(p, 1 - \alpha)$ TI is given by

$$\bar{x}_d \pm z_{\frac{1+p}{2}}\sqrt{Q_\gamma}. \tag{3.16}$$

3.2.2. *The Tolerance Interval Based on the Satterthwaite Approximation.* Notice that Liao et al's approach requires Monte Carlo simulation to find an upper confidence limit for $\sigma_d^2 + \sigma_{dn}^2 = a_1\sigma_1^2 + a_2\sigma_2^2$, where $a_i = (1 + 1/n_i)$, $i = 1, 2$. Instead, we can use the following confidence limit based on the Satterthwaite approximation that

$$\frac{a_1s_1^2 + a_2s_2^2}{a_1\sigma_1^2 + a_2\sigma_2^2} \sim \frac{\chi^2_{\hat{f}}}{\hat{f}}, \quad \text{with } \hat{f} = \frac{(a_1s_1^2 + a_2s_2^2)^2}{a_1^2s_1^4/(n_1 - 1) + a_2^2s_2^4/(n_2 - 1)}. \tag{3.17}$$

On the basis of the above approximation, an approximate γ level upper confidence limit is given by $\hat{f}(a_1s_1^2 + a_2s_2^2)/\chi^2_{\hat{f};1-\gamma}$. Using this upper limit, we get an approximate (p, γ) TI as

$$\bar{x}_d \pm z_{\frac{1+p}{2}}\sqrt{\frac{\hat{f}(a_1s_1^2 + a_2s_2^2)}{\chi^2_{\hat{f};1-\gamma}}}. \tag{3.18}$$

4. Monte Carlo Studies

To appraise the accuracies of the approximate TIs in the previous section, we estimate their coverage probabilities by Monte Carlo simulation. It is easy to check that the intervals are location-scale invariant, and so without of loss of generality, we can take $\mu_d = \mu_1 - \mu_2 = 0$ and $\sigma_1^2 = 1$. The coverage probabilities are estimated for different sample sizes, and plotted as a function σ_1^2/σ_2^2. These plots are given in Fig. 1a for some equal sample size cases, and in Fig. 1b for some unequal sample sizes. We first observe from Fig. 1a that when the sample sizes are equal to 5, the Liao–Lin–Iyer tolerance interval in (3.16) is conservative. The TI (3.18) could be liberal and conservative depending on the value of the

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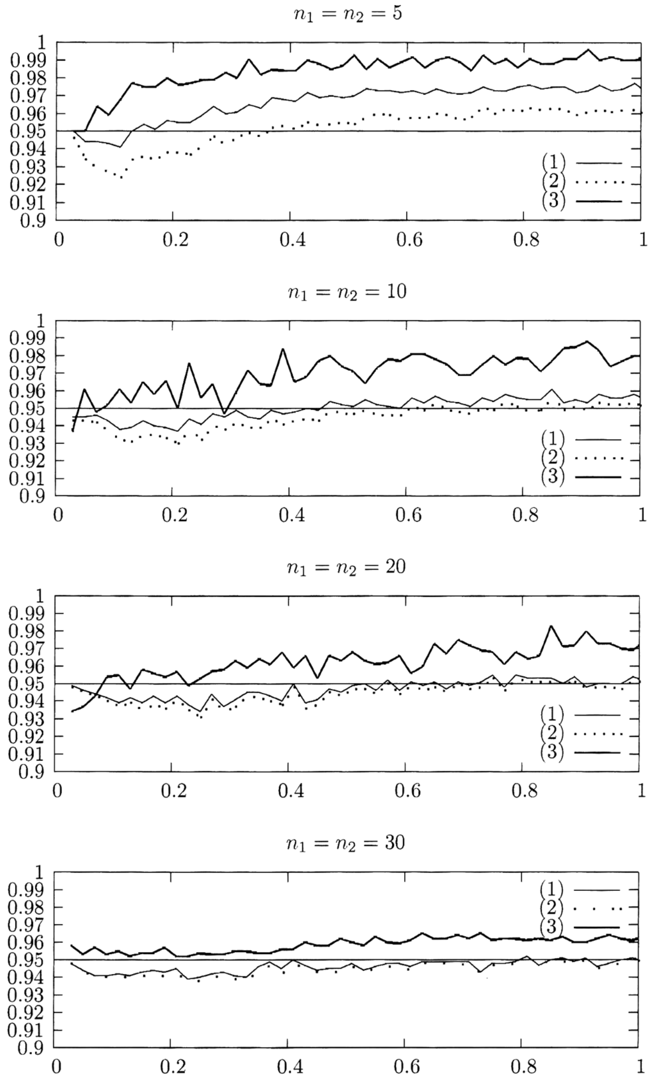

Figure 1a. Coverage probabilities of 0.90 content −0.95 coverage TIs when $n_1 = n_2$; (1) TI in (3.6), (2) TI in (3.18), and (3) TI in (3.16).

variance ratio; the approximate TI in (3.6) is in general slightly conservative but less conservative than the one due to Liao, Lin and Iyer in (3.16). For other equal sample sizes, the approximate TIs in (3.6) and (3.18) are in good agreement whereas the generalized TI in (3.16) is conservative except for the values of variance ratio are small. The plot for $n_1 = n_2 = 30$ indicates that the degree of conservativeness of the TI in (3.16) diminishes as the sample size increases.

The coverage probability plots in Fig. 1b for unequal sample sizes indicate similar performances of the methods as in the case of equal sample size. In particular, the approximate methods exhibit similar performance for all the cases considered while the generalized variable approach is conservative.

Overall, the TI (3.18) based on the Satterthwaite approximation followed by the one in (3.6) seem to be satisfactory in terms of simplicity and accuracy.

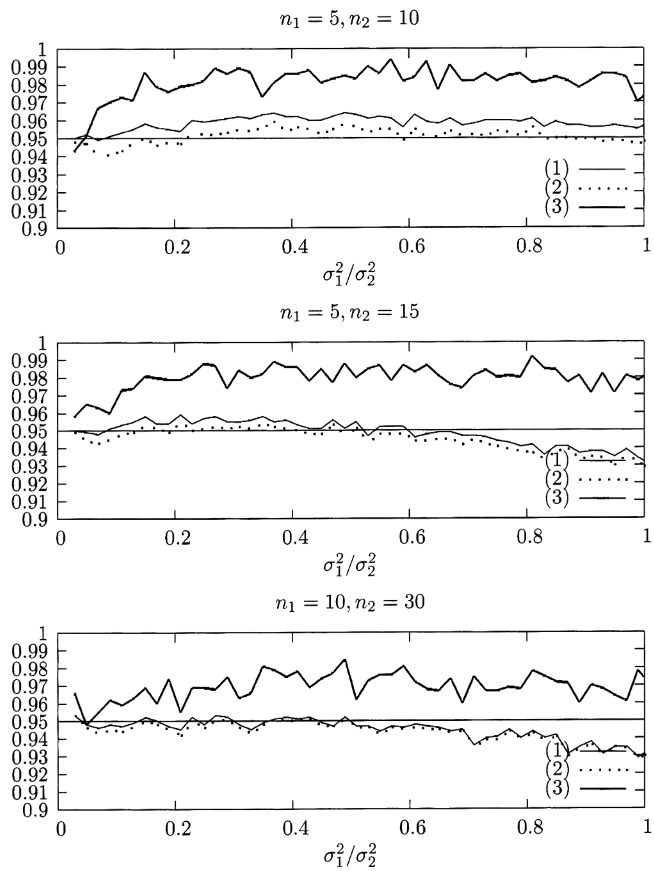

Figure 1b. Coverage probabilities of 0.90 content −0.95 coverage TIs when $n_1 \neq n_2$; (1) TI in (3.6), (2) TI in (3.18), and (3) TI in (3.16).

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Our extensive coverage studies (not all reported here) indicated that these approximate TIs performed very satisfactorily as long as both sample sizes are 10 or more, and they are not drastically different. So, we can recommend these approximate TIs for practical applications.

5. An Illustrative Example

To illustrate the methods that we proposed, we apply them to Example 16.5 given in Mendenhall (1983). An experiment was conducted to compare the strength of the two types of kraft papers, one is a standard kraft paper of specified weight and the other is treated with a chemical substance. The data are given below.

Standard (x_1)	1.21	1.43	1.35	1.51	1.39	1.17	1.48	1.42	1.29	1.40
Treated (x_2)	1.49	1.37	1.67	1.50	1.31	1.29	1.52	1.37	1.44	1.53

The summary statistics are: $\bar{x}_1 = 1.3650$, $s_1 = 0.1112$, $\bar{x}_2 = 1.4490$ and $s_2 = 0.1164$. Other statistics are computed as follows. $\hat{q}_1 = (7s_1^2)/(9s_2^2) = 0.7098$,

$\hat{q}_2 = (7s_2^2)/(9s_1^2) = 0.8522$,

$$\hat{f}_1 = \frac{9(\hat{q}_1 + 1)^2}{\hat{q}_1^2 + 1} = 17.496 \quad \text{and} \quad \hat{f}_2 = \frac{9(\hat{q}_2 + 1)^2}{\hat{q}_2^2 + 1} = 17.886.$$

Using these numbers, we have:

$$k_1 = \left(\frac{\hat{f}_1 \chi^2_{1,.99}(1/10)}{\chi^2_{\hat{f}_1,0.05}}\right)^{\frac{1}{2}} = \left(\frac{(17.496)(7.2604)}{9.0273}\right)^{\frac{1}{2}} = 3.7512.$$

Similarly, we compute

$$k_2 = \left(\frac{\hat{f}_2 \chi^2_{1,.99}(1/10)}{\chi^2_{\hat{f}_2,0.05}}\right)^{\frac{1}{2}} = \left(\frac{(17.886)(7.2604)}{9.3082}\right)^{\frac{1}{2}} = 3.7351.$$

The required tolerance factor is $k_m = \max\{k_1, k_2\} = 3.7512$, and the 0.99 content -0.95 coverage TI (3.6) for the distribution of $X_1 - X_2$ is

$$\bar{x}_d \pm k\sqrt{s_1^2 + s_2^2} = -0.084 \pm 0.604 = (-0.688, 0.520).$$

To compute the TI (3.18) based on the Satterthwaite approximation, we computed the quantities in (3.17), and they are $\hat{f} = 17.963$, $a_1 s_1^2 + a_2 s_2^2 = 0.02849$, $z_{.995} = 2.5758$ and $\chi^2_{\hat{f};.99} = 9.363$. Thus, the TI in (3.18) is

$$-0.084 \pm 2.5758 \times \sqrt{\frac{17.963 \times 0.02849}{9.363}} = (-0.686, 0.518).$$

To compute the Lin–Liao–Iyer TI, we estimated $Q_{.95}$ using simulation with 10,000 runs as 0.06432. Using these quantities, the TI in (3.16) is given by

$$-0.084 \pm 2.5758 \times \sqrt{0.06432} = (-0.737, 0.569).$$

It is interesting to note that the results reflect the properties of the TIs observed in Sec. 3 (see Fig. 1a). In particular, we note that the Liao–Lin–Iyer approach is conservative, and as a result it produces TI wider than those of other procedures. Thus, the approximate TIs (3.6) and (3.18) are in agreement while the Liao–Lin–Iyer tolerance interval is wider than the other two TIs.

All the above tolerance intervals suggest that the strength of standard articles are comparable with that of chemically treated articles.

6. Concluding Remarks

In this article, we provided simple approximate methods for constructing tolerance intervals for the distribution of difference between two independent normal random variables. As noted in the preceding sections, this tolerance interval problem is a special case of a more general setup considered in Liao et al. (2005). In particular, these authors provided a generalized variable approach for constructing tolerance intervals for a $N(\mu, \tau^2)$ distribution, where $\tau^2 = \sum_{i=1}^{g} c_i \sigma_i^2$ and

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c_i's are known constants. Their results are applicable for constructing tolerance intervals for balanced or unbalanced one-way random effects model and general mixed models. The generalized variable approach is applicable even if some of the c_i's are negative whereas the Satterthwaite approximation in Sec. 3.2.2 is not valid in this case. However, the Satterthwaite approximation provided a better solution than that of the generalized variable approach for the special case considered in this article. In view of this result, we expect that the Satterthwaite approximation may yield satisfactory solutions for the aforementioned general problem provided all the c_i's are positive. In situations where some of the c_i's are negative, one could use the modified large sample procedure (Graybill and Wang, 1980) to find confidence interval for τ^2, and thereby tolerance interval for the distribution $N(\mu, \tau^2)$. It should be noted that these approximate methods, unlike the generalized variable approach, do not require simulation. We are currently investigating these alternative approaches for setting tolerance intervals for $N(\mu, \tau^2)$, where $\tau^2 = \sum_{i=1}^{g} c_i\sigma_i^2$ and c_i's are known constants.

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Appendix

The tolerance factor k should be determined so that

$$P_{\bar{x}_d, s_p^2}\left[P_{X_d}\left(\bar{x}_d - ks_p \le X_d \le \bar{x}_d + ks_p\right) > p \,\middle|\, \bar{x}_d, s_p^2\right] = 1 - \alpha. \tag{A.1}$$

Let $Z_d = (X_d - \mu_d)/\sqrt{\sigma_1^2 + \sigma_2^2}$. Then, the inner probability in (A.1) can be written as

$$P_{X_d}\left(\frac{\bar{x}_d - \mu_d - ks_p}{\sqrt{\sigma_1^2 + \sigma_2^2}} \le Z_d \le \frac{\bar{x}_d - \mu_d + ks_p}{\sqrt{\sigma_1^2 + \sigma_2^2}}\right). \tag{A.2}$$

Let

$$\eta_1 = \sqrt{\frac{\sigma_1^2/n_1 + \sigma_2^2/n_2}{\sigma_1^2 + \sigma_2^2}}, \quad Z = \frac{\bar{x}_d - \mu_d}{\sqrt{\sigma_1^2/n_1 + \sigma_2^2/n_2}} \quad \text{and} \quad Y = Z\eta_1.$$

Since $Z_d \sim N(0, 1)$, we can write (A.2) as

$$\Phi\left(Y + k\frac{s_p}{\sqrt{\sigma_1^2 + \sigma_2^2}}\right) - \Phi\left(Y - k\frac{s_p}{\sqrt{\sigma_1^2 + \sigma_2^2}}\right) = g(Y, s_p), \quad \text{say}, \tag{A.3}$$

where $Y \sim N(0, \eta_1)$. Thus, (A.1) can be expressed as

$$P_{Y, s_p^2}[g(Y, s_p) > p] = 1 - \alpha. \tag{A.4}$$

As argued in Wald and Wolfowitz (1946) approach for finding tolerance factor in the one-sample case, for fixed Y, $\Phi(Y + c) - \Phi(Y - c)$ is an increasing function of c, and so

$$g(Y, s_p) > p \quad \text{if and only if} \quad k\frac{s_p}{\sqrt{\sigma_1^2 + \sigma_2^2}} \ge r \text{ or } \frac{s_p^2}{\sigma_1^2 + \sigma_2^2} \ge \frac{r^2}{k^2}, \tag{A.5}$$

where r is the solution of the equation

$$\Phi(Y + r) - \Phi(Y - r) = p, \tag{A.6}$$

and $Y \sim N(0, \eta_1^2)$. Equation (A.6) can be written as $P((Z - Y)^2 \leq r^2) = p$, where Z is a standard normal random variable. Since $(Z - Y)^2$ is distributed as the non central chi-square random variable with $df = 1$ and non centrality parameter Y^2, we get $r = \sqrt{\chi^2_{1,p}(Y^2)}$, where $\chi^2_{m,p}(\delta)$ denotes the pth quantile of a non central chi-square distribution with df $= m$ and non centrality parameter δ. Thus, using (A.5), we can write (A.4) as

$$\begin{aligned} P_{Y,s_p^2}\left(\frac{s_p^2}{\sigma_1^2 + \sigma_2^2} \geq \frac{r^2}{k^2}\right) &= E_Y P_{s_p^2}\left(\frac{s_p^2}{\sigma_1^2 + \sigma_2^2} \geq \frac{\chi^2_{1,p}(Y^2)}{k^2}\middle| Y\right) \\ &= E_Y P_{\chi^2_{n_1+n_2-2}}\left(\frac{\chi^2_{n_1+n_2-2}}{n_1 + n_2 - 2} \geq \frac{\chi^2_{1,p}(Y^2)}{k^2}\middle| Y\right) \\ &= 1 - \alpha. \end{aligned} \tag{A.7}$$

To get the second equation in (A.7), we used the fact that s_p^2 in (2.1) is defined so that $s_p^2/(\sigma_1^2 + \sigma_2^2) \sim \chi^2_{n_1+n_2-2}/(n_1 + n_2 - 2)$. Thus, k is the solution of (A.7), or equivalently, k is the solution of the integral equation in (2.4).

Following Wald and Wolfowitz (1946), we can replace the Y^2 in (A.7) by its expectation η_1^2 to get an approximation for k. After replacing, we see that (A.7) becomes

$$P_{\chi^2_{n_1+n_2-2}}\left(\frac{\chi^2_{n_1+n_2-2}}{n_1 + n_2 - 2} \geq \frac{\chi^2_{1,p}(\eta_1^2)}{k^2}\right) = \gamma,$$

from which it is easy to check that the approximation for k is given in (2.5).

References

Eberhardt, K. R., Mee, R. W., Reeve, C. P. (1989). Computing factors for exact two-sided tolerance limits for a normal distribution. *Commun. Statist. Simul. Computat.* 18:397–413.

Graybill, F. A., Wang, C. M. (1980). Confidence intervals on nonnegative linear combinations of variances. *J. Amer. Statist. Assoc.* 75:869–873.

Guo, H., Krishnamoorthy, K. (2004). New approximate inferential methods for the reliability parameter in a stress-strength model: The normal case. *Commun. Statist. Theor. Meth.* 33:1715–1731.

Hall, I. J. (1984). Approximate one-sided tolerance limits for the difference or sum of two independent normal variates. *J. Qual. Technol.* 16:15–19.

Howe, W. G. (1969). Two-sided tolerance limits for normal populations – some improvements. *J. Amer. Statist. Assoc.* 64:610–620.

Liao, C. T., Lin, T. Y., Iyer, H. K. (2005). One- and two-sided tolerance intervals for general balanced mixed models and unbalanced one-way random models. *Technometrics* 47:323–335.

Mendenhall, W. (1983). *Introduction to Probability and Statistics*. London: Duxbury Press.

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Owen, D. B. (1964). Control of percentages in both tails of the normal distribution (Corr: V8 p. 570). *Technometrics* 6:377–387.

Reiser, B. J., Guttman, I. (1986). Statistical inference for $\Pr(Y < X)$: the normal case. *Technometrics* 28:253–257.

Wald, A. (1943). An extension of Wilks' method for setting tolerance limits. *Ann. Mathemat. Statist.* 14:45–55.

Wald, A., Wolfowitz, J. (1946). Tolerance limits for a normal distribution. *Ann. Mathemat. Statist.* 17:208–215.

Weerahandi, S., Johnson, R. A. (1992). Testing reliability in a stress-strength model when X and Y are normally distributed. *Technometrics* 34:83–91.

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