Communications in Statistics—Simulation and Computation®, 35: 461–478, 2006

ISSN: 0361-0918 print/1532-4141 online
DOI: 10.1080/03610910600591883

Multivariate Analysis

Improved Tolerance Factors for Multivariate Normal Distributions

K. KRISHNAMOORTHY AND SUMONA MONDAL

Department of Mathematics, University of Louisiana at Lafayette, Lafayette, Louisiana, USA

In this article, an improved method of computing tolerance factors for multivariate normal distributions is proposed. The method involves an approximation and simulation, and is more accurate than the several approximate methods considered in Krishnamoorthy and Mathew (1999). The accuracies of the tolerance regions are evaluated using Monte Carlo simulation. Simulation study shows that the new approach is very satisfactory even for small samples. Tolerance factors based on the proposed approach are tabulated for the dimension of the normal distribution $p = 2(1)10$, *and various sample sizes ranging from* $2p + 1$ *to* 1000.

Keywords Confidence; Content; Coverage probability; Non central chi-square approximation; Wishart distribution.

Mathematics Subject Classification 62H99; 62F25.

1. Introduction

In many practical situations one wants to assess the proportion of the data fall in an interval or a region. For example, engineering products are usually required to satisfy certain tolerance specifications. The proportion of the products that are within the specifications can be assessed by constructing a suitable tolerance region based on a sample of products. Applications and detailed treatment of theory for the univariate normal tolerance limits can be found in the book by Guttman (1970a). A Fortran program for computing tolerance factors for the univariate normal case is given in Eberhardt et al. (1989). For the multivariate normal case, John (1963) provided theoretical formulation for the problem of constructing tolerance region and gave a few approximate methods. Since then many authors proposed approximate methods to construct tolerance regions for

Received August 24, 2005; Accepted November 11, 2005
Address correspondence to K. Krishnamoorthy, Department of Mathematics, University of Louisiana at Lafayette, Lafayette, LA 70504, USA; E-mail: krishna@louisiana.edu

multivariate normal distributions. Among others, see Siotani (1964), Chew (1966), Guttman (1970b), and Hall and Sheldon (1979). Fuchs and Kenett (1987, 1988) have argued that many statistical process control problems are multivariate in nature, and the need for multivariate tolerance region. These authors also discussed two practical examples. Hall and Sheldon (1979) used bivariate tolerance region to assess ballistic miss distances. Recently, Krishnamoorthy and Mathew (1999) developed an approximate method and compared it with several approximate methods given in the aforementioned papers via Monte Carlo simulation. Comparison studies by Krishnamoorthy and Mathew showed that their approximate method is, in general, more accurate than the other methods even though it is not satisfactory for all situations. For this reason, these authors tabulated Monte Carlo estimates of the tolerance factors for the dimension of the distribution $p = 2(1)9$ and various sample sizes.

To describe the multivariate tolerance region formally, let $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ be a random sample from a p-variate normal distribution with mean vector μ and variance-covariance matrix Σ, say, $N_p(\mu, \Sigma)$. Assume that $n > p$ and μ and Σ are unknown. The sample mean vector $\bar{\mathbf{x}}$ and the sums of squares and cross-product matrix A are defined by

$$\bar{\mathbf{x}} = \frac{1}{n}\sum_{i=1}^{n}\mathbf{x}_i, \quad A = \sum_{i=1}^{n}(\mathbf{x}_i - \bar{\mathbf{x}})(\mathbf{x}_i - \bar{\mathbf{x}})'. \tag{1}$$

Then $\bar{\mathbf{x}}$ and $\frac{1}{n-1}A$ are the usual estimators of μ and Σ, respectively. Furthermore,

$$\bar{\mathbf{x}} \sim N_p(\mu, \Sigma) \quad \text{independently of } A \sim W_p(n-1, \Sigma),$$

where $W_p(m, \Omega)$ denote the Wishart distribution with degrees of freedom m and scale parameter matrix Ω.

Let $0 < \beta < 1$ and $0 < \gamma < 1$. A β content–γ coverage tolerance region based on $\bar{\mathbf{x}}$ and $\frac{1}{n-1}A$ is constructed so that it would contain at least proportion β of the data from $N_p(\mu, \Sigma)$ distribution with confidence γ. In notation, we write the tolerance region as

$$\{\mathbf{x} : (n-1)(\mathbf{x} - \bar{\mathbf{x}})'A^{-1}(\mathbf{x} - \bar{\mathbf{x}}) \le c\}, \tag{2}$$

where c is the tolerance factor to be determined subject to the condition that

$$\mathrm{P}_{\bar{\mathbf{x}},A}\left[P_{\mathbf{x}}\left\{(n-1)(\mathbf{x} - \bar{\mathbf{x}})'A^{-1}(\mathbf{x} - \bar{\mathbf{x}}) \le c \,\middle|\, \bar{\mathbf{x}}, A\right\} \ge \beta\right] = \gamma, \tag{3}$$

and $\mathbf{x} \sim N_p(\mu, \Sigma)$ independently of $(\bar{\mathbf{x}}, A)$.

To see that the tolerance factor that satisfies the probability requirement in (3) does not depend on any unknown parameters, we write (3) in terms of the following transformed variables. Let $\mathbf{y} = \Sigma^{-\frac{1}{2}}(\mathbf{x} - \mu)$, $\mathbf{u} = \Sigma^{-\frac{1}{2}}(\bar{\mathbf{x}} - \mu)$, and $V = \Sigma^{-\frac{1}{2}}A\Sigma^{-\frac{1}{2}}$. Note that $\mathbf{y}$, $\mathbf{u}$, and V are independent with

$$\mathbf{y} \sim N_p(0, I_p), \quad \mathbf{u} \sim N_p\left(0, \frac{1}{n}I_p\right), \quad \text{and} \quad V \sim W_p(n-1, I_p).$$

In terms of these variables, we can write (3) as

$$P_{\mathbf{u},V}\big[P_{\mathbf{y}}[(n-1)(\mathbf{y}-\mathbf{u})'V^{-1}(\mathbf{y}-\mathbf{u}) \le c \,\big|\, \mathbf{u}, V] \ge \beta\big] = \gamma. \tag{4}$$

Because exact method of computing c is difficult to obtain, many authors proposed approximate methods. As pointed out earlier, none of the available approximate methods is satisfactory for all sample size, content, and confidence level configurations, and the only satisfactory method is Monte Carlo simulation. For easy reference and to explain the difficulties involved in Monte Carlo method based on (4), we describe the simulation in the following algorithm.

Algorithm 1. For given n, p, β, and γ:

```
For i = 1, m_1
    generate a u ~ N_p(0, (1/n) I_p) and a V ~ W_p(n − 1, I_p)
    For j = 1, m_2
        generate a y ~ N_p(0, I_p)
        compute Q_j = (n − 1)(y − u)' V^{-1} (y − u)
    end j loop
    compute T_i = 100β th percentile of the Q_j's
end i loop
```

The 100γth percentile of the T_i's is a Monte Carlo estimate of the tolerance factor c.

Even though the above Monte Carlo method is simple to implement, it poses some problems while computing the tolerance factor. First, it needs a total of $m_1 \times m_2$ simulation runs. For example, Hall and Sheldon (1979) and Krishnamoorthy and Mathew (1999) used $m_1 = m_2 = 1200$, which amounts to a total of 1,440,000 runs. Even with these many runs, the final estimate of the tolerance factor depends on the initial seed used for the random number generators. In other words, for a given (n, p, β, γ), two different seeds may produce tolerance factors that are appreciably different. This instability is severe when the sample sizes are small relative to the dimension. As a consequence, as noted in Krishnamoorthy and Mathew (1999), the coverage probabilities of the tolerance regions based on Monte Carlo estimates of the tolerance factors are sometimes quite different from the nominal level γ.

To overcome the above shortcomings, in the following section we approximate the inner probability in (4) using a chi-square approximation, and then use simulation to compute the probability with respect to the joint distribution of $(\mathbf{u}, V)$. Because the new approach use simulation with only one "do loop", its accuracy can be increased by raising the number of runs. Using this approach we computed tolerance factors for $p = 2(1)10$, and for various sample sizes ranging from $2p + 1$ to 1000. In Sec. 3, we evaluate accuracies of the tolerance factors based on the new approach. Our simulation studies show that the proposed approach is very satisfactory even for small samples.

2. The New Method

We note first that the distributions of $\mathbf{u}$, $\mathbf{y}$, and V, and the quadratic form in (4) are orthogonal invariant. Let $\Gamma'\text{diag}(l_1, \ldots, l_p)\Gamma$, where $\Gamma'\Gamma = I_p$ and $l_1 > \cdots > l_p$ are the eigenvalues of V, be the spectral decomposition of V. Let $\mathbf{z} = \Gamma'\mathbf{y}$ and $\mathbf{q} = \Gamma'\mathbf{u}$.

Then, we can write (4) as

$$P_{\mathbf{q},\mathbf{l}}\left[P_{\mathbf{z}}[(n-1)\sum_{i=1}^{p}(z_i-q_i)^2/l_i \leq c \,\middle|\, \mathbf{q},\mathbf{l}] \geq \beta\right] = \gamma, \tag{5}$$

where $\mathbf{z} = (z_1, \ldots, z_p)'$, $\mathbf{q} = (q_1, \ldots, q_p)'$, and $\mathbf{l} = (l_1, l_2, \ldots, l_p)'$. The conditional distribution of $\mathbf{z}$, given V and $\mathbf{q}$, is $N_p(0, I_p)$, and hence conditionally given V and $\mathbf{q}$, $(z_i - q_i)^2 \sim \chi_1^2(q_i^2)$ for $i = 1, \ldots, p$, where $\chi_m^2(\delta)$ denotes the non central chi-square random variable with df $= m$ and non centrality parameter δ. We also note that $\chi_1^2(q_i^2)$'s are independent when $q_1^2, \ldots, q_p^2$ are fixed, and $q_1^2, \ldots, q_p^2$ are independent χ_1^2/n random variables.

To approximate the inner probability in (5), we need the following lemma due to Imhof (1961).

Lemma 2.1. *Let $Q = \sum_{i=1}^{k} \lambda_i \chi_{m_i}^2(\delta_i)$, where $\lambda_i > 0$ for $i = 1, \ldots, k$. Then*

$$P(Q > x) \simeq P\big(\chi_a^2 > y\big),$$

where

$$y = \sqrt{\frac{a}{c_2}}(x - c_1) + a, \quad a = \frac{c_2^3}{c_3^2}, \quad \text{and} \quad c_j = \sum_{i=1}^{k} \lambda_i^j (m_i + j\delta_i), \;\; j = 1, 2, 3. \tag{6}$$

Furthermore, for $0 < \alpha < 1$, let $\chi_{a,\alpha}^2$ denote the αth *quantile of χ_a^2. Then the αth quantile of Q is approximated by*

$$x_\alpha = \sqrt{\frac{c_2}{a}}(\chi_{a,\alpha}^2 - a) + c_1. \tag{7}$$

The approximation in the above lemma is obtained by "moments matching" method. In particular, letting $R = \frac{\chi_a^2 - a}{\sqrt{2a}}\sqrt{\mathrm{Var(Q)}} + E(Q)$, we see that the first three central moments of Q are equal to those of R when a is as defined in (6). Even though there are other approximations available for the non central chi-square distribution (e.g., Cox and Reid, 1987; Patnaik, 1949), the approximations to the distribution of Q based on them are not satisfactory. The recent article by Zhang (2005) mentioned that the approximation in Lemma 2.1 is in general accurate for approximating probabilities of Q. But, as pointed out by Imhof (1961), and based on our own numerical investigations, we found that the approximation is remarkably accurate only for computing the right-tail probabilities of Q. Also, see Krishnamoorthy and Mondal (2005).

To approximate the inner probability in (5), we note that, for fixed $\mathbf{q}$ and $\mathbf{l}$, $W = \sum_{i=1}^{p}(z_i - q_i)^2/l_i$ is distributed as $\sum_{i=1}^{p} l_i^{-1}\chi_1^2(q_i^2)$. Therefore, the results of Lemma 2.1 can be readily applied with $\lambda_i = l_i^{-1}$, $m_i = 1$ and $\delta_i = q_i^2$, $i = 1, \ldots, p$. In this case, we see that

$$c_j = \sum_{i=1}^{p} \frac{1 + jq_i^2}{l_i^j}, \quad j = 1, 2, 3 \quad \text{and} \quad a = \frac{c_2^3}{c_3^2}. \tag{8}$$

Thus, it follows from (6) that

$$\sum_{i=1}^{p} \frac{(z_i - q_i)^2}{l_i} \sim \sqrt{\frac{c_2}{a}}(\chi_a^2 - a) + c_1 \quad \text{approximately.} \tag{9}$$

Using this approximation, we see that the inner probability inequality in (5) is equivalent to

$$P\left[(n-1)\left(\sqrt{\frac{c_2}{a}}(\chi_a^2 - a) + c_1\right) \le c \,\middle|\, \mathbf{q}, \mathbf{l}\right] \ge \beta, \tag{10}$$

where a, c_1, and c_2 are given in (8). The probability inequality in (10) holds if and only if $c \ge (n-1)\left(\sqrt{\frac{c_2}{a}}(\chi_{a,\beta}^2 - a) + c_1\right)$, where $\chi_{a,\beta}^2$ denotes the βth quantile of the χ_a^2 distribution, and hence it follows from (5) that an approximation to the tolerance factor c satisfies

$$P_{\mathbf{q},\mathbf{l}}\left((n-1)\left(\sqrt{\frac{c_2}{a}}(\chi_a^2 - a) + c_1\right) \le c\right) = \gamma. \tag{11}$$

Noticing that q_i's are independent with $q_i^2 \sim \chi_1^2/n$ and l_i's are the eigenvalues of V, the following algorithm can be used to compute the tolerance factors.

Algorithm 2. For given n, p, β, and γ:

For $i = 1, m_1$
  generate $q_j \sim \chi_1^2/n$, $j = 1, \ldots, p$ and $V \sim W_p(n-1, I_p)$
  compute the eigenvalues $l_1, \ldots, l_p$ of V
  compute c_1, c_2, c_3 and a using (8)
  set $T_i = (n-1)\left(\sqrt{\frac{c_2}{a}}(\chi_a^2 - a) + c_1\right)$
end i loop

The 100γth percentile of the T_i's is an approximate tolerance factor c.

We used the above algorithm with $m_1 = 100,000$ runs to compute the tolerance factors. The chi-square random numbers are generated using IMSL subroutine RNNCHI, the chi-square percentiles are computed using IMSL subroutine CHIIN and Wishart random matrices are generated using an algorithm similar to the one due to Smith and Hocking (1972). The eigenvalues of the Wishart matrices are computed using IMSL routine EVLSF. The tolerance factors are presented in Table 1 for $p = 2(1)10$, $\beta = 0.90, 0.95, 0.99$, $\gamma = 0.90, 0.95, 0.99$, and various sample sizes ranging from $2p + 1$ to 1000. The tolerance factors for omitted sample sizes can be computed using the following interpolation: Suppose that the tolerance factor for the sample size n_2 is not listed in the table, and the factors a_1 for n_1 and a_3 for n_3 are listed, where $n_1 < n_2 < n_3$. Then, the factor a_2 for n_2 can be interpolated by $(n_3a_1 + n_1a_3)/(n_1 + n_3)$ if $n_2 - n_1 \le n_3 - n_2$; by $(n_1a_1 + n_3a_3)/(n_1 + n_3)$ otherwise. As an example, when $p = 2$, $\beta = 0.90$, and $\gamma = 0.95$, the reported value for $n = 37$ is 6.99. The interpolated value for $n = 37$ using the tolerance factors for $n = 35$ and $n = 40$ is given by $(40 \times 7.09 + 35 \times 6.84)/(40 + 35) = 6.97$ which is very close to the reported value 6.99 (see Table 1, $p = 2$).

Table 1
Tolerance factors

	β								
	$\gamma = 0.90$			$\gamma = 0.95$			$\gamma = 0.99$		
n	0.90	0.95	0.99	0.90	0.95	0.99	0.90	0.95	0.99
					$p = 2$				
5	41.61	57.10	95.62	66.81	93.49	155.4	199.6	289.4	476.9
6	26.00	36.18	60.57	38.33	53.36	89.96	90.53	124.8	209.2
7	19.88	27.23	45.17	27.27	37.59	62.77	54.45	76.82	125.1
8	16.31	22.36	36.89	21.35	29.46	48.77	37.93	52.76	87.86
9	14.16	19.36	31.56	17.94	24.93	40.53	30.06	41.70	69.66
10	12.69	17.20	28.30	15.78	21.40	34.93	24.51	34.26	57.48
11	11.63	15.67	25.73	14.12	19.15	31.65	21.13	29.49	48.88
12	10.82	14.61	23.82	12.88	17.52	28.85	18.77	25.87	43.43
13	10.19	13.75	22.28	12.01	16.31	26.54	17.09	23.44	38.56
14	9.70	13.02	21.09	11.35	15.27	24.83	15.53	21.55	35.25
15	9.30	12.45	20.18	10.77	14.53	23.63	14.52	20.02	32.92
16	8.91	12.03	19.24	10.23	13.86	22.35	13.55	18.52	30.51
17	8.63	11.56	18.57	9.89	13.27	21.44	12.90	17.56	28.50
18	8.41	11.22	17.95	9.53	12.75	20.61	12.31	16.79	27.25
19	8.17	10.92	17.48	9.24	12.37	19.90	11.72	16.04	26.30
20	7.99	10.67	17.02	8.98	12.08	19.33	11.33	15.35	25.18
21	7.81	10.43	16.62	8.71	11.72	18.76	11.03	14.82	24.13
22	7.66	10.21	16.33	8.55	11.38	18.30	10.71	14.34	23.22
23	7.54	10.01	15.94	8.34	11.17	17.82	10.31	13.93	22.50
24	7.40	9.86	15.65	8.20	10.91	17.45	10.04	13.55	21.89
25	7.29	9.72	15.39	8.02	10.70	17.13	9.74	13.17	21.03
26	7.20	9.56	15.11	7.92	10.56	16.78	9.61	12.81	20.72
27	7.11	9.43	14.94	7.81	10.38	16.50	9.35	12.60	20.13
28	7.03	9.31	14.75	7.67	10.22	16.25	9.18	12.38	19.72
30	6.88	9.09	14.37	7.49	9.96	15.77	8.89	11.84	19.07
32	6.74	8.92	14.10	7.32	9.70	15.36	8.59	11.43	18.34
35	6.58	8.71	13.70	7.09	9.40	14.84	8.29	10.99	17.61
37	6.48	8.57	13.50	6.99	9.25	14.58	8.11	10.66	17.14
40	6.36	8.40	13.18	6.84	9.00	14.26	7.84	10.38	16.55
45	6.20	8.16	12.79	6.63	8.72	13.68	7.50	9.96	15.71
50	6.07	7.99	12.47	6.45	8.49	13.33	7.27	9.59	15.11
60	5.87	7.72	12.03	6.21	8.14	12.74	6.89	9.08	14.27
70	5.74	7.52	11.69	6.02	7.90	12.31	6.64	8.72	13.66
80	5.62	7.37	11.47	5.89	7.72	11.99	6.42	8.44	13.14
90	5.55	7.26	11.25	5.79	7.58	11.77	6.29	8.23	12.83
100	5.48	7.16	11.10	5.70	7.46	11.60	6.15	8.07	12.56
150	5.27	6.87	10.62	5.43	7.10	10.98	5.77	7.55	11.68
200	5.15	6.72	10.37	5.30	6.91	10.67	5.57	7.29	11.25
300	5.03	6.56	10.10	5.14	6.71	10.33	5.36	6.99	10.77
500	4.92	6.41	9.86	5.00	6.51	10.03	5.17	6.72	10.35
10^3	4.81	6.27	9.64	4.87	6.34	9.75	4.98	6.49	9.98
∞	4.61	5.99	9.21	4.61	5.99	9.21	4.61	5.99	9.21

Table 1
Continued.

	β								
	$\gamma = 0.90$			$\gamma = 0.95$			$\gamma = 0.99$		
n	0.90	0.95	0.99	0.90	0.95	0.99	0.90	0.95	0.99
					$p = 3$				
7	44.05	60.13	99.03	62.21	87.59	146.6	143.4	200.5	345.3
8	31.36	42.92	70.71	42.31	58.47	97.34	84.29	114.9	196.1
9	25.27	34.44	55.51	32.75	44.62	73.80	58.13	80.07	134.7
10	21.53	28.98	46.75	26.76	36.77	59.42	43.66	60.54	99.65
11	18.94	25.33	40.49	23.11	31.06	50.45	35.71	49.24	81.87
12	17.07	22.72	36.37	20.49	27.50	44.04	30.16	41.42	68.30
13	15.75	20.87	33.13	18.64	24.78	39.68	26.43	35.62	58.05
14	14.71	19.38	30.49	17.03	22.79	36.29	23.60	31.87	52.30
15	13.90	18.29	28.70	15.88	21.25	33.67	21.50	29.06	47.21
16	13.19	17.34	27.05	15.12	19.88	31.40	20.06	26.78	42.99
17	12.70	16.59	25.74	14.29	18.92	29.79	18.68	24.67	39.85
18	12.19	15.91	24.75	13.77	17.98	28.29	17.50	23.17	37.69
19	11.81	15.35	23.76	13.16	17.29	27.03	16.72	22.04	34.92
20	11.44	14.89	22.93	12.76	16.61	25.92	15.80	21.02	33.10
21	11.15	14.53	22.22	12.35	16.16	24.93	15.16	20.13	31.72
22	10.90	14.12	21.68	12.01	15.62	24.19	14.61	19.31	30.58
23	10.65	13.75	21.14	11.68	15.23	23.42	14.12	18.57	29.34
24	10.46	13.44	20.60	11.46	14.87	22.82	13.70	18.00	28.10
25	10.28	13.20	20.16	11.20	14.49	22.24	13.36	17.44	27.23
26	10.07	13.01	19.75	10.95	14.17	21.68	12.95	17.16	26.56
27	9.91	12.77	19.42	10.79	13.91	21.32	12.75	16.58	25.70
28	9.78	12.62	19.08	10.60	13.65	20.90	12.43	16.18	25.10
29	9.65	12.39	18.74	10.43	13.43	20.48	12.17	15.83	24.62
30	9.54	12.24	18.56	10.29	13.26	20.08	11.92	15.51	23.97
32	9.33	11.93	18.08	10.03	12.85	19.60	11.49	14.88	23.07
34	9.13	11.73	17.59	9.79	12.55	19.05	11.20	14.47	22.24
35	9.07	11.58	17.40	9.68	12.43	18.78	11.06	14.28	21.84
37	8.93	11.38	17.09	9.48	12.18	18.32	10.72	13.88	21.12
40	8.73	11.08	16.63	9.23	11.84	17.83	10.42	13.42	20.43
45	8.47	10.77	16.04	8.94	11.39	17.10	9.99	12.79	19.35
50	8.26	10.49	15.61	8.70	11.07	16.52	9.63	12.25	18.48
60	7.95	10.07	14.93	8.32	10.59	15.70	9.10	11.59	17.27
70	7.75	9.81	14.49	8.08	10.22	15.13	8.75	11.11	16.46
80	7.60	9.58	14.12	7.90	9.98	14.71	8.48	10.77	15.92
90	7.48	9.42	13.87	7.75	9.77	14.39	8.30	10.46	15.48
100	7.39	9.29	13.65	7.64	9.62	14.13	8.14	10.27	15.13
150	7.09	8.90	13.02	7.28	9.14	13.37	7.66	9.62	14.09
200	6.93	8.70	12.70	7.08	8.89	12.99	7.40	9.28	13.57
300	6.77	8.48	12.36	6.89	8.64	12.59	7.14	8.94	13.02
500	6.63	8.29	12.07	6.72	8.40	12.24	6.90	8.64	12.55
10^3	6.50	8.13	11.81	6.56	8.21	11.92	6.68	8.36	12.15
∞	6.25	7.81	11.34	6.25	7.81	11.34	6.25	7.81	11.34

(continued)

Table 1
Continued.

	β								
	γ = 0.90			γ = 0.95			γ = 0.99		
n	0.90	0.95	0.99	0.90	0.95	0.99	0.90	0.95	0.99
					p = 4				
9	45.66	61.37	100.4	60.55	82.66	137.3	116.9	162.0	277.9
10	35.42	47.98	76.80	45.78	61.74	100.8	79.56	110.6	182.6
11	29.91	39.58	63.01	36.95	49.83	80.79	59.47	81.04	135.4
12	25.78	34.14	53.89	31.34	41.86	67.02	47.50	63.83	106.3
13	23.14	30.48	47.95	27.49	36.41	57.85	39.95	54.29	87.76
14	21.16	27.61	43.16	24.79	32.63	51.58	34.42	46.79	76.57
15	19.63	25.52	39.46	22.52	29.59	46.44	30.48	41.23	65.74
16	18.36	23.89	36.67	20.93	27.40	42.71	27.76	37.54	59.81
17	17.38	22.45	34.46	19.69	25.53	39.87	25.44	34.01	53.95
18	16.59	21.35	32.70	18.58	24.10	37.31	23.75	31.28	49.67
19	15.91	20.47	31.06	17.71	23.01	35.27	22.33	29.06	46.70
20	15.33	19.67	29.70	17.00	21.95	33.41	21.09	27.55	43.45
21	14.88	18.92	28.60	16.33	21.15	31.99	20.09	26.32	41.14
22	14.42	18.40	27.53	15.85	20.33	31.01	19.14	24.93	38.69
23	14.04	17.86	26.68	15.30	19.64	29.65	18.39	24.11	37.17
24	13.70	17.37	26.02	14.92	19.06	28.74	17.64	22.92	35.44
25	13.39	16.98	25.30	14.54	18.55	27.83	17.21	22.21	34.01
26	13.13	16.65	24.70	14.22	18.10	27.05	16.79	21.46	33.01
27	12.89	16.34	24.13	13.91	17.69	26.43	16.27	20.91	31.85
28	12.67	16.04	23.66	13.66	17.33	25.86	15.86	20.36	30.97
29	12.48	15.74	23.25	13.41	16.92	25.21	15.54	19.73	30.15
30	12.30	15.47	22.81	13.19	16.65	24.71	15.19	19.44	29.29
31	12.14	15.26	22.42	12.99	16.44	24.28	14.86	18.91	28.69
32	11.96	15.07	22.13	12.77	16.17	23.85	14.65	18.59	28.00
34	11.70	14.66	21.47	12.48	15.69	23.12	14.07	17.93	26.78
35	11.58	14.52	21.28	12.30	15.44	22.81	13.92	17.51	26.24
37	11.35	14.21	20.72	12.06	15.13	22.19	13.47	17.07	25.36
39	11.17	13.95	20.33	11.80	14.78	21.59	13.24	16.71	24.71
40	11.08	13.81	20.07	11.71	14.65	21.38	13.02	16.36	24.28
45	10.71	13.32	19.27	11.26	14.07	20.42	12.40	15.63	22.87
50	10.41	12.94	18.66	10.91	13.58	19.62	11.96	14.98	21.83
60	9.99	12.37	17.75	10.41	12.91	18.56	11.26	14.02	20.27
70	9.71	11.99	17.14	10.07	12.45	17.81	10.82	13.43	19.28
80	9.49	11.71	16.68	9.81	12.11	17.28	10.47	12.94	18.60
90	9.33	11.49	16.32	9.61	11.86	16.87	10.20	12.61	18.01
100	9.19	11.31	16.07	9.47	11.65	16.55	10.02	12.35	17.59
150	8.81	10.80	15.26	9.01	11.05	15.62	9.41	11.54	16.36
200	8.61	10.54	14.84	8.78	10.75	15.15	9.11	11.16	15.76
300	8.40	10.27	14.43	8.53	10.43	14.67	8.78	10.73	15.10
500	8.22	10.04	14.08	8.32	10.15	14.25	8.50	10.39	14.58
10^3	8.06	9.84	13.79	8.13	9.92	13.90	8.26	10.07	14.12
∞	7.78	9.49	13.28	7.78	9.49	13.28	7.78	9.49	13.28

Table 1
Continued.

	β								
	$\gamma = 0.90$			$\gamma = 0.95$			$\gamma = 0.99$		
n	0.90	0.95	0.99	0.90	0.95	0.99	0.90	0.95	0.99
					$p = 5$				
11	47.24	62.95	101.9	59.98	81.75	133.2	104.1	143.8	239.0
12	38.87	51.63	81.56	48.00	64.25	104.7	76.16	107.5	172.3
13	33.46	44.15	69.42	40.30	53.53	85.88	59.84	82.14	134.6
14	29.75	38.87	60.59	35.22	45.86	73.40	50.51	67.97	109.4
15	27.00	35.10	53.95	31.37	41.01	64.45	43.32	57.63	93.16
16	24.90	32.08	49.30	28.49	37.16	57.62	38.61	50.67	81.16
17	23.19	29.85	45.33	26.22	34.22	52.84	34.67	45.64	72.26
18	21.89	28.01	42.20	24.50	31.74	48.56	31.50	41.58	65.68
19	20.77	26.55	39.94	23.24	29.87	45.23	29.40	38.17	59.65
20	19.88	25.18	37.82	21.95	28.27	42.65	27.43	35.77	55.86
21	19.12	24.18	36.01	20.97	26.81	40.43	25.74	33.51	51.46
22	18.43	23.27	34.57	20.20	25.65	38.38	24.56	31.71	48.99
23	17.82	22.44	33.19	19.47	24.67	37.01	23.28	30.03	46.07
24	17.34	21.78	32.05	18.87	23.76	35.48	22.43	28.71	43.80
25	16.91	21.15	31.07	18.28	23.02	34.22	21.68	27.49	41.76
26	16.48	20.60	30.15	17.76	22.42	33.07	20.84	26.71	40.13
27	16.12	20.17	29.41	17.35	21.78	32.07	20.26	25.67	38.60
28	15.80	19.72	28.64	16.99	21.22	31.22	19.61	24.85	37.39
29	15.51	19.33	28.01	16.57	20.81	30.42	19.13	24.14	36.33
30	15.26	18.99	27.46	16.26	20.32	29.62	18.63	23.59	35.00
31	15.00	18.67	26.93	16.00	19.94	29.11	18.29	22.96	34.05
32	14.81	18.32	26.49	15.77	19.64	28.50	17.84	22.53	33.24
34	14.38	17.80	25.60	15.26	18.97	27.49	17.16	21.62	31.66
35	14.23	17.57	25.20	15.07	18.68	26.99	16.87	21.20	31.14
37	13.91	17.15	24.55	14.66	18.19	26.14	16.41	20.49	29.94
39	13.61	16.81	23.93	14.37	17.74	25.45	15.97	19.88	28.88
40	13.49	16.65	23.64	14.23	17.55	25.08	15.73	19.59	28.38
41	13.40	16.48	23.43	14.10	17.36	24.86	15.57	19.35	27.96
43	13.16	16.19	22.97	13.84	17.03	24.27	15.25	18.83	27.15
45	12.99	15.94	22.59	13.62	16.76	23.80	14.93	18.43	26.56
47	12.82	15.72	22.20	13.42	16.47	23.38	14.67	18.14	25.99
50	12.61	15.41	21.71	13.13	16.12	22.84	14.34	17.64	25.17
60	12.03	14.67	20.54	12.49	15.24	21.44	13.41	16.47	23.32
70	11.65	14.16	19.74	12.02	14.65	20.49	12.83	15.69	22.09
80	11.35	13.77	19.17	11.70	14.23	19.83	12.44	15.11	21.16
90	11.14	13.50	18.73	11.46	13.89	19.32	12.12	14.71	20.52
100	10.98	13.29	18.38	11.27	13.64	18.90	11.87	14.39	19.96
150	10.47	12.61	17.38	10.69	12.89	17.75	11.13	13.42	18.50
200	10.22	12.30	16.89	10.40	12.52	17.20	10.75	12.94	17.80
300	9.96	11.97	16.40	10.10	12.14	16.63	10.37	12.48	17.10
500	9.74	11.69	15.98	9.84	11.82	16.15	10.04	12.07	16.49
10^3	9.56	11.46	15.64	9.63	11.55	15.76	9.76	11.71	15.98
∞	9.24	11.07	15.09	9.24	11.07	15.09	9.24	11.07	15.09

(continued)

Table 1
Continued.

	β								
	$\gamma = 0.90$			$\gamma = 0.95$			$\gamma = 0.99$		
n	0.90	0.95	0.99	0.90	0.95	0.99	0.90	0.95	0.99
					$p = 6$				
13	49.14	65.00	102.9	60.62	80.55	129.9	93.69	132.5	211.6
14	42.00	55.04	85.79	50.67	66.82	105.9	74.33	103.3	167.2
15	37.14	48.01	74.38	43.65	57.03	89.89	61.79	82.15	135.4
16	33.37	42.97	65.80	38.75	50.31	78.30	53.33	70.23	112.3
17	30.59	39.27	59.46	34.89	45.17	69.41	46.81	61.10	98.40
18	28.41	36.28	54.78	31.96	41.34	63.03	41.63	55.03	86.47
19	26.59	33.97	50.80	29.83	38.19	58.20	37.68	49.80	78.08
20	25.21	31.84	47.62	28.04	35.65	53.96	35.11	45.75	70.81
21	24.09	30.35	44.96	26.41	33.63	50.57	32.89	42.35	65.01
22	23.03	28.91	42.66	25.28	31.98	47.42	30.83	39.54	60.51
23	22.13	27.70	40.62	24.24	30.45	45.19	29.08	37.32	57.11
24	21.45	26.75	39.08	23.33	29.33	43.20	27.81	35.69	53.53
25	20.80	25.88	37.56	22.47	28.24	41.42	26.62	33.77	50.78
26	20.20	25.10	36.26	21.79	27.25	39.88	25.62	32.28	48.51
27	19.69	24.38	35.15	21.19	26.33	38.38	24.49	31.03	46.16
28	19.24	23.77	34.18	20.61	25.62	37.23	23.73	30.01	44.50
29	18.81	23.22	33.33	20.08	25.01	36.16	23.01	29.08	43.17
30	18.45	22.75	32.57	19.69	24.34	35.22	22.52	28.13	41.47
31	18.11	22.26	31.75	19.25	23.89	34.30	21.96	27.36	40.40
32	17.81	21.87	31.19	18.90	23.37	33.56	21.32	26.71	39.09
33	17.51	21.49	30.54	18.53	22.89	32.69	20.95	26.00	37.89
34	17.24	21.15	29.94	18.31	22.48	32.18	20.46	25.52	37.11
35	17.03	20.85	29.45	17.99	22.08	31.50	20.10	24.96	36.14
36	16.78	20.52	29.00	17.71	21.74	31.00	19.79	24.44	35.38
37	16.58	20.24	28.57	17.43	21.40	30.37	19.48	24.09	34.75
39	16.22	19.75	27.81	17.04	20.89	29.43	18.85	23.22	33.32
40	16.04	19.53	27.43	16.86	20.60	29.07	18.59	22.86	32.76
41	15.90	19.33	27.09	16.66	20.36	28.59	18.36	22.64	32.31
43	15.60	18.99	26.54	16.35	19.93	28.00	17.91	21.96	31.35
45	15.36	18.63	26.00	16.05	19.56	27.36	17.51	21.41	30.39
47	15.13	18.32	25.50	15.78	19.17	26.84	17.12	20.93	29.67
50	14.83	17.94	24.86	15.46	18.75	26.07	16.74	20.35	28.63
60	14.08	16.98	23.38	14.58	17.61	24.33	15.62	18.90	26.34
70	13.57	16.32	22.35	14.00	16.86	23.15	14.86	17.95	24.83
80	13.22	15.85	21.65	13.61	16.32	22.34	14.36	17.26	23.79
90	12.95	15.51	21.12	13.30	15.92	21.73	13.98	16.77	22.95
100	12.73	15.23	20.68	13.04	15.61	21.23	13.69	16.39	22.35
150	12.10	14.42	19.46	12.34	14.70	19.84	12.79	15.25	20.61
200	11.79	14.02	18.87	11.98	14.25	19.19	12.35	14.70	19.82
300	11.49	13.62	18.29	11.63	13.80	18.53	11.91	14.14	18.99
500	11.22	13.29	17.81	11.33	13.42	17.98	11.54	13.68	18.31
10^3	11.00	13.03	17.41	11.08	13.12	17.54	11.22	13.28	17.77
∞	10.64	13.03	16.81	10.64	13.03	16.81	10.64	13.03	16.81

Table 1
Continued.

	β								
	γ = 0.90			γ = 0.95			γ = 0.99		
n	0.90	0.95	0.99	0.90	0.95	0.99	0.90	0.95	0.99
					p = 7				
15	51.23	66.65	103.8	61.40	80.43	128.3	90.29	122.9	203.0
16	44.97	58.13	89.60	52.54	68.46	106.8	74.08	100.3	161.6
17	40.20	51.78	78.75	46.46	59.84	93.27	63.19	84.16	135.3
18	36.84	46.93	70.77	41.70	53.98	82.53	55.89	73.53	116.3
19	34.06	43.07	64.51	38.16	49.04	74.68	49.41	65.17	101.4
20	31.77	40.10	59.81	35.42	45.23	68.24	44.43	58.53	91.20
21	29.99	37.66	55.80	33.07	42.22	62.69	41.20	53.23	82.34
22	28.47	35.69	52.25	31.40	39.58	58.71	38.49	49.32	76.21
23	27.23	34.00	49.74	29.73	37.42	55.32	35.98	46.22	70.11
24	26.10	32.57	47.29	28.42	35.63	52.45	33.84	43.32	65.76
25	25.20	31.26	45.03	27.25	34.12	49.90	32.28	40.96	61.93
26	24.37	30.19	43.41	26.37	32.73	47.68	30.76	39.15	58.21
27	23.67	29.26	41.77	25.41	31.58	45.88	29.52	37.24	55.39
28	23.07	28.35	40.47	24.64	30.51	44.03	28.63	35.69	52.57
29	22.42	27.60	39.28	24.03	29.67	42.58	27.56	34.46	50.58
30	21.94	26.86	38.18	23.39	28.78	41.31	26.77	33.40	48.85
31	21.47	26.29	37.16	22.82	28.11	40.15	25.93	32.33	47.06
32	21.06	25.72	36.31	22.36	27.43	39.03	25.22	31.44	45.56
33	20.68	25.23	35.49	21.91	26.87	38.09	24.66	30.56	44.13
34	20.34	24.74	34.75	21.50	26.30	37.13	24.16	29.87	42.84
35	20.04	24.31	34.02	21.15	25.81	36.44	23.62	29.08	41.70
36	19.69	23.92	33.44	20.79	25.32	35.75	23.07	28.50	40.75
37	19.46	23.55	32.90	20.46	24.90	35.02	22.73	27.96	39.81
38	19.20	23.22	32.37	20.18	24.57	34.39	22.28	27.31	39.19
39	18.95	22.88	31.88	19.93	24.18	33.83	21.96	26.87	38.19
40	18.74	22.66	31.40	19.65	23.83	33.28	21.57	26.34	37.35
41	18.52	22.36	30.98	19.40	23.50	32.76	21.32	26.03	36.79
42	18.32	22.11	30.58	19.19	23.24	32.32	21.07	25.63	36.16
43	18.15	21.86	30.22	18.97	22.97	31.86	20.69	25.30	35.54
45	17.83	21.47	29.55	18.58	22.46	31.09	20.26	24.61	34.54
47	17.53	21.05	28.96	18.24	21.97	30.40	19.82	24.00	33.52
50	17.15	20.56	28.16	17.84	21.43	29.49	19.25	23.26	32.20
60	16.19	19.33	26.25	16.73	20.03	27.32	17.91	21.44	29.52
70	15.55	18.51	25.02	16.04	19.08	25.90	16.96	20.30	27.70
80	15.10	17.93	24.13	15.49	18.45	24.86	16.35	19.47	26.35
90	14.75	17.49	23.49	15.15	17.95	24.11	15.85	18.84	25.43
100	14.50	17.16	22.96	14.84	17.57	23.55	15.50	18.40	24.67
150	13.73	16.18	21.50	13.98	16.47	21.91	14.44	17.05	22.73
200	13.35	15.71	20.80	13.55	15.95	21.14	13.94	16.41	21.76
300	12.98	15.25	20.13	13.13	15.42	20.37	13.43	15.78	20.86
500	12.66	14.85	19.58	12.78	14.99	19.76	13.00	15.25	20.11
10^3	12.41	14.55	19.13	12.49	14.64	19.26	12.64	14.80	19.48
∞	12.02	14.07	18.48	12.02	14.07	18.48	12.02	14.07	18.48

(continued)

Table 1
Continued.

	β								
	$\gamma = 0.90$			$\gamma = 0.95$			$\gamma = 0.99$		
n	0.90	0.95	0.99	0.90	0.95	0.99	0.90	0.95	0.99
					$p = 8$				
17	53.56	68.78	105.4	62.26	81.00	126.8	88.26	117.3	188.4
18	47.72	60.86	92.02	54.69	70.45	109.1	74.26	100.1	157.9
19	43.32	54.88	82.67	49.40	62.91	96.63	64.52	84.73	133.5
20	39.98	50.30	75.07	44.87	57.24	86.43	57.52	75.15	119.2
21	37.30	46.86	69.02	41.28	52.55	78.55	52.09	67.36	105.0
22	35.09	43.79	64.29	38.69	48.85	72.38	47.63	61.25	95.05
23	33.23	41.39	60.05	36.43	45.63	67.41	44.53	56.56	86.84
24	31.67	39.22	56.96	34.43	43.09	63.19	41.50	52.77	80.32
25	30.31	37.50	54.05	32.90	41.03	59.83	39.24	49.52	74.41
26	29.18	35.93	51.52	31.57	39.18	56.91	36.99	46.91	69.88
27	28.20	34.62	49.43	30.31	37.54	54.13	35.32	44.45	65.71
28	27.28	33.43	47.58	29.31	36.18	51.97	33.83	42.63	62.72
30	25.83	31.46	44.41	27.54	33.81	48.19	31.29	39.22	57.23
31	25.24	30.69	43.14	26.83	32.80	46.64	30.38	37.62	54.81
32	24.64	29.96	42.00	26.13	31.97	45.29	29.70	36.59	52.92
33	24.15	29.26	40.86	25.61	31.12	43.95	28.86	35.37	51.16
34	23.70	28.64	39.92	25.05	30.47	42.76	28.05	34.53	49.58
35	23.29	28.08	39.14	24.55	29.77	41.82	27.39	33.72	47.86
36	22.85	27.65	38.30	24.10	29.18	40.73	26.79	32.72	46.77
37	22.50	27.13	37.55	23.69	28.65	39.95	26.22	32.11	45.77
38	22.17	26.72	36.86	23.32	28.20	39.06	25.70	31.37	44.42
39	21.85	26.33	36.22	22.95	27.65	38.54	25.19	30.73	43.21
40	21.59	25.93	35.69	22.63	27.28	37.77	24.78	30.06	42.52
43	20.85	24.98	34.09	21.76	26.13	36.00	23.72	28.83	40.09
45	20.42	24.40	33.30	21.25	25.56	35.03	23.13	27.92	38.71
47	20.03	23.95	32.57	20.87	24.99	34.11	22.56	27.18	37.70
50	19.54	23.29	31.57	20.29	24.24	33.06	21.92	26.27	36.14
55	18.89	22.43	30.28	19.56	23.29	31.56	20.91	25.04	34.22
60	18.35	21.76	29.22	18.94	22.50	30.33	20.19	24.04	32.76
65	17.93	21.20	28.41	18.48	21.89	29.42	19.61	23.28	31.47
70	17.57	20.75	27.71	18.08	21.37	28.63	19.11	22.65	30.54
75	17.26	20.37	27.15	17.74	20.96	27.98	18.69	22.15	29.71
80	17.00	20.04	26.64	17.45	20.57	27.42	18.36	21.65	28.98
90	16.59	19.51	25.86	17.00	19.99	26.56	17.77	20.95	27.91
100	16.26	19.09	25.25	16.62	19.53	25.86	17.36	20.42	27.07
150	15.34	17.94	23.54	15.60	18.24	23.96	16.10	18.86	24.79
200	14.89	17.38	22.73	15.10	17.63	23.05	15.51	18.12	23.72
300	14.45	16.83	21.94	14.61	17.02	22.19	14.93	17.40	22.68
500	14.09	16.38	21.30	14.21	16.53	21.48	14.44	16.79	21.84
10^3	13.80	16.03	20.80	13.88	16.13	20.93	14.03	16.31	21.16
∞	13.36	15.51	20.09	13.36	15.51	20.09	13.36	15.51	20.09

Table 1
Continued.

	β								
	$\gamma = 0.90$			$\gamma = 0.95$			$\gamma = 0.99$		
n	0.90	0.95	0.99	0.90	0.95	0.99	0.90	0.95	0.99
					$p = 9$				
19	55.67	70.96	106.8	63.69	82.04	125.8	86.61	114.4	180.6
20	50.24	63.78	95.05	56.96	73.48	110.6	75.73	97.05	154.4
21	46.36	58.07	86.46	51.65	65.86	98.97	66.83	86.55	134.7
22	42.98	53.99	78.89	47.77	60.58	89.66	59.96	77.31	120.5
23	40.37	50.38	73.29	44.48	55.82	82.70	54.64	70.10	107.3
24	38.22	47.28	68.60	41.67	52.48	76.86	50.42	64.76	97.92
25	36.27	44.88	64.33	39.55	49.00	71.96	47.30	59.88	90.97
26	34.74	42.74	61.19	37.58	46.73	67.61	44.44	56.33	83.92
27	33.31	40.97	58.11	35.96	44.43	63.92	42.21	53.11	79.11
28	32.24	39.28	55.79	34.47	42.57	61.20	40.00	50.23	73.83
30	30.22	36.75	51.55	32.17	39.49	56.10	36.92	45.73	66.73
31	29.38	35.61	49.89	31.18	38.10	53.98	35.58	43.91	63.68
32	28.59	34.61	48.32	30.38	36.98	52.33	34.62	42.32	61.02
33	27.97	33.78	47.05	29.58	35.92	50.46	33.33	41.05	58.62
34	27.35	32.99	45.67	28.90	35.06	48.99	32.47	39.81	56.91
35	26.77	32.26	44.61	28.28	34.12	47.70	31.53	38.71	54.71
36	26.30	31.60	43.50	27.69	33.42	46.36	30.76	37.67	53.28
37	25.81	31.02	42.60	27.10	32.77	45.37	30.10	36.67	51.75
38	25.40	30.46	41.83	26.63	32.14	44.40	29.48	35.85	50.25
39	25.00	29.95	40.97	26.20	31.56	43.43	28.92	35.07	49.07
40	24.62	29.46	40.29	25.78	30.97	42.61	28.35	34.32	47.93
43	23.68	28.24	38.37	24.71	29.55	40.46	26.92	32.48	44.90
45	23.14	27.57	37.30	24.11	28.80	39.12	26.09	31.41	43.45
47	22.66	26.93	36.37	23.57	28.08	38.12	25.48	30.55	41.99
50	22.08	26.15	35.16	22.91	27.20	36.76	24.62	29.46	40.33
53	21.55	25.50	34.13	22.32	26.48	35.61	23.92	28.42	38.76
55	21.24	25.07	33.60	21.94	26.04	34.93	23.51	27.92	37.79
57	20.97	24.76	33.05	21.65	25.60	34.36	23.06	27.42	37.06
60	20.58	24.27	32.31	21.24	25.06	33.48	22.55	26.76	36.11
65	20.06	23.59	31.27	20.67	24.32	32.39	21.87	25.80	34.63
70	19.62	23.03	30.50	20.17	23.71	31.46	21.22	25.04	33.40
75	19.27	22.57	29.81	19.77	23.18	30.70	20.80	24.45	32.49
80	18.95	22.19	29.21	19.41	22.77	30.01	20.39	23.93	31.72
90	18.44	21.55	28.30	18.86	22.06	28.98	19.72	23.10	30.49
100	18.07	21.05	27.57	18.45	21.53	28.21	19.20	22.43	29.49
150	16.95	19.68	25.55	17.23	20.01	26.00	17.76	20.64	26.89
200	16.43	19.04	24.62	16.65	19.30	24.98	17.10	19.79	25.65
300	15.91	18.40	23.72	16.08	18.61	23.97	16.41	18.98	24.46
500	15.50	17.89	23.00	15.62	18.04	23.18	15.86	18.31	23.54
10^3	15.17	17.49	22.44	15.25	17.59	22.57	15.40	17.78	22.80
∞	14.68	16.92	21.67	14.68	16.92	21.67	14.68	16.92	21.67

(continued)

Table 1
Continued.

	β								
	$\gamma = 0.90$			$\gamma = 0.95$			$\gamma = 0.99$		
n	0.90	0.95	0.99	0.90	0.95	0.99	0.90	0.95	0.99
					$p = 10$				
21	57.86	72.95	108.9	65.22	83.56	127.5	84.98	111.5	176.3
22	53.09	66.50	98.19	59.10	74.95	112.5	75.72	98.22	151.6
23	49.24	61.46	89.83	54.51	68.75	102.1	68.13	87.52	135.3
24	46.06	57.17	82.67	50.67	63.55	93.82	62.31	79.21	121.5
25	43.42	53.57	77.21	47.31	59.19	86.36	57.51	72.93	109.7
26	41.24	50.75	72.42	44.72	55.61	80.65	53.74	67.53	101.3
27	39.34	48.26	68.67	42.52	52.61	75.81	50.40	62.77	94.31
28	37.69	46.13	65.21	40.64	49.96	71.61	47.62	59.54	87.26
30	35.07	42.62	59.78	37.52	45.89	64.96	43.03	53.44	77.57
31	33.98	41.18	57.64	36.23	44.13	62.34	41.17	51.44	73.88
32	33.06	39.94	55.43	35.07	42.61	60.21	39.95	48.87	70.83
33	32.13	38.76	53.67	34.07	41.39	57.67	38.58	47.12	67.81
34	31.35	37.81	52.20	33.20	40.18	55.91	37.24	45.75	65.22
35	30.68	36.80	50.72	32.38	39.05	54.25	36.13	44.15	62.70
36	29.99	35.94	49.48	31.60	38.06	52.86	35.16	43.01	60.35
37	29.47	35.20	48.18	30.90	37.20	51.39	34.33	41.78	58.66
38	28.87	34.50	47.10	30.34	36.37	50.24	33.41	40.69	57.05
39	28.35	33.88	46.14	29.75	35.69	48.94	32.72	39.68	55.19
40	27.92	33.26	45.22	29.20	34.89	47.92	32.03	38.69	53.68
43	26.71	31.77	42.92	27.89	33.22	45.17	30.43	36.43	50.30
45	26.05	30.88	41.57	27.13	32.26	43.69	29.38	35.21	48.29
47	25.47	30.13	40.47	26.46	31.38	42.37	28.62	34.00	46.69
50	24.71	29.16	38.97	25.60	30.33	40.69	27.50	32.76	44.46
53	24.05	28.35	37.70	24.91	29.39	39.29	26.59	31.65	42.70
55	23.70	27.86	37.01	24.51	28.82	38.55	26.12	30.98	41.61
57	23.35	27.45	36.34	24.13	28.39	37.77	25.68	30.36	40.77
60	22.91	26.87	35.50	23.63	27.74	36.77	25.07	29.69	39.61
63	22.50	26.36	34.77	23.16	27.19	36.00	24.52	28.83	38.49
65	22.26	26.03	34.30	22.90	26.81	35.46	24.23	28.52	37.89
67	22.06	25.77	33.92	22.66	26.51	35.00	23.92	28.10	37.32
70	21.74	25.39	33.35	22.31	26.09	34.34	23.51	27.58	36.51
75	21.31	24.84	32.51	21.85	25.52	33.44	22.89	26.83	35.48
80	20.94	24.38	31.82	21.44	24.97	32.70	22.45	26.24	34.53
90	20.33	23.61	30.74	20.78	24.16	31.49	21.66	25.27	33.02
100	19.87	23.06	29.91	20.26	23.54	30.54	21.06	24.50	31.91
150	18.56	21.44	27.58	18.85	21.79	28.04	19.42	22.43	28.91
200	17.96	20.69	26.49	18.20	20.96	26.85	18.63	21.49	27.54
300	17.36	19.96	25.47	17.54	20.16	25.74	17.88	20.56	26.26
500	16.89	19.39	24.67	17.02	19.53	24.86	17.26	19.82	25.22
10^3	16.52	18.94	24.05	16.60	19.03	24.18	16.76	19.23	24.42
∞	15.99	18.31	23.21	15.99	18.31	23.21	15.99	18.31	23.21

3. Accuracy Studies

To appraise the accuracies of the tolerance factors computed using Algorithm 2, we shall compare them with the exact ones (e.g., see Odeh, 1978) for the univariate case. For $3 \leq n \leq 10$, the exact tolerance factors and the approximate ones are given in Table 2. Comparison of these values clearly indicate that the tolerance factors based on our new approximation method practically coincide with the exact ones for sample sizes as small as three.

To appraise the validity of the approximate tolerance factors for $p \geq 2$, we estimated their coverage probabilities using the following algorithm.

Algorithm 3. For given n, p, β and γ, let c be the tolerance factor computed using Algorithm 2.

For $i = 1, m_1$
  generate a $\mathbf{u} \sim N_p\left(0, \frac{1}{n}I_p\right)$ and a $V \sim W_p(n-1, I_p)$
  For $j = 1, m_2$
    generate a $\mathbf{y} \sim N_p(0, I_p)$
    compute $Q_j = (n-1)(\mathbf{y}-\mathbf{u})'V^{-1}(\mathbf{y}-\mathbf{u})$
  end j loop
  Let P_i be the proportion of the Q_j's that are greater than c
  If $P_i \geq \beta$ set $K_i = 1$; else set $K_i = 0$
end i loop

$\hat{\gamma} = \frac{1}{m_1}\sum_{i=1}^{m_1} K_i$ is a Monte Carlo estimate of γ.

The accuracy of the tolerance factor is determined by the closeness of $\hat{\gamma}$ to the actual value γ. To understand the validity of the tolerance factors in Table 1, we estimated coverage probabilities of some arbitrarily selected tolerance factors using Algorithm 3 with $m_1 = m_2 = 5000$. The estimated coverage probabilities are rounded to two decimals and they are reported in Table 3. We observe from Table 3 that the estimated coverage probabilities are in complete agreement with the corresponding confidence levels for all the cases considered.

Table 2
Approximate and exact tolerance factors for $p = 1$

γ	0.90						0.95					
β	0.9		0.95		0.99		0.9		0.95		0.99	
n	App.	Exact	App.	Exact	App.	Exact	App.	Exact	App.	Exact	App.	Exact
3	5.82	5.79	6.81	6.82	8.85	8.82	8.33	8.31	9.82	9.79	12.56	12.65
4	4.17	4.16	4.89	4.91	6.36	6.37	5.36	5.37	6.36	6.34	8.17	8.22
5	3.48	3.50	4.15	4.14	5.42	5.39	4.32	4.29	5.03	5.08	6.61	6.60
6	3.14	3.14	3.72	3.72	4.87	4.85	3.72	3.73	4.41	4.42	5.76	5.76
7	2.90	2.91	3.45	3.46	4.52	4.51	3.38	3.39	4.03	4.02	5.25	5.24
8	2.75	2.75	3.26	3.27	4.28	4.27	3.15	3.16	3.74	3.75	4.87	4.89
9	2.63	2.64	3.13	3.13	4.10	4.09	2.98	2.99	3.55	3.55	4.62	4.63
10	2.55	2.55	3.02	3.03	3.96	3.96	2.85	2.86	3.39	3.39	4.43	4.44

Table 3
Monte Carlo estimate $\hat{\gamma}$ of the coverage probability of the tolerance factor c

(n, γ, β, c)	$\hat{\gamma}$	(n, γ, β, c)	$\hat{\gamma}$	(n, γ, β, c)	$\hat{\gamma}$
		$p = 2$			
(5, .90, .90, 41.61)	.90	(9, .90, .95, 19.36)	.90	(8, .99, .95, 52.76)	.99
(6, .99, .90, 90.53)	.99	(11, .95, .99, 31.65)	.95	(7, .95, .90, 27.27)	.95
(7, .95, .95, 37.59)	.95	(12, .99, .90, 18.77)	.99	(9, .95, .95, 24.93)	.95
		$p = 3$			
(7, .95, .90, 62.21)	.95	(11, .99, .99, 81.87)	.99	(12, .90, .90, 17.07)	.90
(17, .90, .95, 16.59)	.90	(16, .99, .90, 20.06)	.99	(13, .99, .99, 58.05)	.99
(32, .90, .90, 9.33)	.90	(45, .95, .95, 11.39)	.95	(90, .99, .90, 8.30)	.99
		$p = 6$			
(13, .90, .90, 49.14)	.90	(18, .99, .99, 86.47)	.99	(20, .99, .95, 45.75)	.99
(28, .90, .95, 23.77)	.90	(15, .99, .95, 82.15)	.99	(16, .90, .99, 65.80)	.90
(39, .90, .90, 16.22)	.90	(25, .99, .95, 33.77)	.99	(21, .95, .95, 33.63)	.95
		$p = 8$			
(17, .99, .90, 88.26)	.99	(19, .95, .95, 62.91)	.95	(21, .90, .90, 37.30)	.90
(23, .90, .99, 60.05)	.90	(23, .99, .99, 86.84)	.99	(25, .95, .90, 32.90)	.95
(35, .95, .90, 24.55)	.95	(100, .99, .90, 17.36)	.99	(65, .99, .99, 31.47)	.99
		$p = 10$			
(21, .95, .90, 65.22)	.95	(23, .90, .90, 49.24)	.90	(25, .99, .99, 109.7)	.99
(27, .90, .95, 48.26)	.90	(30, .95, .99, 64.96)	.95	(28, .95, .95, 49.96)	.95
(53, .95, .90, 24.91)	.95	(55, .99, .90, 26.12)	.99	(150, .95, .95, 21.79)	.95

To understand the errors of the estimates of the tolerance factor, we computed variance estimate using 50 repetitions. That is, for a given (n, p, β, γ), we computed tolerance factors using 50 repetitions, each consisting of 100,000 runs. The standard deviation (SD) of the 50 estimated tolerance factors is used to understand the variation in the estimates. The standard deviations of the estimates based on Algorithms 1 and 2 are presented in Table 4. For a given (n, p, β, γ), we also present the corresponding tolerance factor reported in Table 1, and the one given in Krishnamoorthy and Mathew (1999). It is clear from Table 4 that, for all the cases considered, the SDs of the new estimates are smaller than the SDs of the estimates based on Algorithm 1.

4. Concluding Remarks

Although the proposed approach for computing tolerance factors for a multivariate normal distribution involves both approximation and simulation, our simulation studies along with Krishnamoorthy and Mathew's (1999) comparison studies showed that this new approach seems to be the best among all the available methods. In general, we found that the tolerance factors based on our approach are very satisfactory for any combination of β and γ provided $n \geq 5p$; for $(\beta, \gamma) = (0.90, 0.95)$ or $(\beta, \gamma) = (0.95, 0.90)$, a sample size of at least $4p$ will guarantee accurate tolerance factors. We also tabulated tolerance factors for practical content-confidence levels for application purposes. For other cases, Algorithm 2 can

Table 4
Averages and standard errors of the Monte Carlo estimates of the tolerance factors

				Algorithm 2			Algorithm 1		
p	n	γ	β	Estimate[1]	Mean[2]	SE[2]	Estimate[3]	Mean[4]	SE[4]
2	10	0.99	0.95	34.26	34.20	0.344	—	33.35	2.498
2	16	0.90	0.99	19.24	19.25	0.036	19.05	19.06	0.523
2	30	0.99	0.99	19.07	19.05	0.057	—	19.14	0.609
2	40	0.90	0.95	8.40	8.40	0.011	8.29	8.31	0.075
3	10	0.99	0.90	43.66	43.83	0.367	—	42.78	3.770
3	20	0.90	0.90	11.44	11.46	0.020	11.24	11.23	0.124
3	30	0.95	0.95	13.26	13.28	0.019	12.94	13.04	0.146
3	90	0.90	0.99	13.87	13.87	0.005	13.97	13.99	0.081
5	20	0.90	0.99	37.82	37.79	0.057	37.10	37.40	0.605
5	30	0.95	0.95	20.32	20.34	0.024	19.86	20.01	0.187
5	38	0.99	0.90	16.18	16.16	0.031	—	15.90	0.226
5	60	0.99	0.90	13.41	13.43	0.017	—	13.34	0.145
7	15	0.90	0.99	103.8	103.9	0.284	—	103.6	2.378
7	28	0.90	0.95	28.35	28.33	0.031	27.87	27.91	0.358
7	35	0.95	0.99	36.44	36.41	0.048	36.84	36.36	0.467
7	120	0.95	0.95	17.00	17.01	0.006	—	17.03	0.071
10	25	0.90	0.95	53.57	53.64	0.083	—	52.43	0.523
10	35	0.99	0.90	36.13	36.16	0.052	—	35.58	0.575
10	60	0.95	0.99	36.77	36.79	0.024	—	37.10	0.191
10	100	0.99	0.99	31.91	31.91	0.022	—	32.62	0.385

[1]Reported in Table 1.

[2]The average and standard error are computed using 50 repetitions of simulations, each consisting of 100,000 runs.

[3]Reported in Krishnamoorthy and Mathew, 1999.

[4]The average and standard error are computed using 50 repetitions of simulations, each consisting of 1200 by 1200 runs.

be coded easily in Fortran with IMSL libraries or in SAS to compute tolerance factors.

Finally, we like to note that Lee and Mathew (2004) considered the problem of constructing tolerance regions in a multivariate linear regression (MLR) setup. These authors used the approach by Krishnamoorthy and Mathew (1999) for computing tolerance factors. Even though the covariance matrix in a MLR setup involves the design matrix $\mathbf{X}$, it is plausible that our approach can be used to find more accurate tolerance factors. We are currently investigating this problem, and plan to submit the work elsewhere.

Acknowledgments

The authors are grateful to a reviewer for providing some useful suggestions and comments.

References

Chew, V. (1966). Confidence, prediction, and tolerance regions for the multivariate normal distribution. *J. Amer. Statist. Assoc.* 61:605–617.

Cox, D. R., Reid, N. (1987). Approximations to noncentral distributions. *Can. J. Statist.* 15:105–114.

Eberhardt, K. R., Mee, R. W., Reeve, C. P. (1989). Computing factors for exact two-sided tolerance intervals for a normal distribution. *Commun. Statist. Simul. Computat.* 18:397–413.

Fuchs, C., Kenett, R. S. (1987). Multivariate tolerance regions and F-tests. *J. Qual. Technol.* 19:122–131.

Fuchs, C., Kenett, R. S. (1988). Appraisal of ceramic substrates by multivariate tolerance regions. *The Statistician* 37:401–411.

Guttman, I. (1970a). *Statistical Tolerance Regions, Classical, and Bayesian*. London: Griffin's Statistical Monographs.

Guttman, I. (1970b). Construction of β-content tolerance regions at confidence level γ for large samples from the k-variate normal distribution. *Ann. Mathemat. Statist.* 41:376–400.

Hall, I. J., Sheldon, D. D. (1979). Improved bivariate normal tolerance regions with some applications. *J. Qual. Technol.* 11:13–19.

Imhof, J. P. (1961). Computing the distribution of quadratic forms in normal variables. *Biometrika* 48:419–426.

John, S. (1963). A tolerance region for multivariate normal distributions. *Sankhya, Ser. A* 25:363–368.

Krishnamoorthy, K., Mathew, T. (1999). Comparison of approximation methods for computing tolerance factors for a multivariate normal population. *Technometrics* 41: 234–249.

Krishnamoorthy, K., Mondal, S. (2005). Letters to the editor. *J. Amer. Statist. Assoc.* Submitted.

Lee, Y.-T., Mathew, T. (2004). Tolerance regions in multivariate linear regression. *J. Statist. Plan. Infer.* 26:253–271.

Odeh, R. E. (1978). Tables of two-sided tolerance factors for a normal distribution. *Commun. Statist. Simul. Computat.* 7:183–201.

Patnaik, P. B. (1949). The noncentral chi-square and F-distributions and their applications. *Biometrika* 36:202–232.

Siotani, M. (1964). Tolerance regions for a multivariate normal population. *Ann. Instit. Statist. Math.* 16:135–153.

Smith, W. B., Hocking, R. R. (1972). Wishart variates generator (Algorithm AS 53). *Appl. Statist.* 21:341–345.

Zhang, J. (2005). Approximate and asymptotic distributions of chi-squared-type mixtures with applications. *J. Amer. Statist. Assoc.* 100:273–285.