Modified Large Sample Confidence Intervals for Poisson Distributions: Ratio, Weighted Average and Product of Means

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Interval estimation procedures for the ratio of two Poisson means, weighted average of Poisson rates, and product of powers of Poisson means are considered. We propose a CI based on the modified large sample approach (MLS), and compare with the Cox CIs and the Wilson-score CIs. Our numerical studies indicate that the MLS CI is comparable with the Cox CI, and offers improvement in some cases. We also provide MLS CI for a weighted sum of Poisson means and compare with the available approximate CIs. The methods are illustrated using two examples.

Keywords Conditional method; Fiducial approach; Precision; Rate ratio; Wald method; Weighted average

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1 Introduction

Confidence intervals (CIs) for some real-valued functions of Poisson parameters have applications in many areas of science ranging from health statistics to physics. For instance, the ratio of two independent Poisson means is used to compare the incident rates of a disease in a treatment group and control group, where the incident rate is defined as the number of events observed divided by the time at risk during the observed period. Many other applications of CIs for the ratio of Poisson means can be found, for example, Cousins (1998) and the references therein. A weighted sum of independent Poisson parameters is commonly used to assess the standardized mortality rates (Dobson et al., 1991). Confidence intervals (CIs) for a product of powers of Poisson parameters are also used to assess the reliability of a parallel system (Harris, 1971 and Kim, 2006). In particular, Kim (2006) has noted that estimation of the geometric mean of Poisson rates arises in environmental applications (EPA, 1986), and in economic applications.

The standardized mortality rates are expressed as weighted sums of Poisson rate parameters. Dobson et al. (1991) have also noted the need for a CI for weighted sums of Poisson rates arises in meta-analysis requiring aggregation of outcome rates from several different studies, using weights related to the numbers of subjects in each study. For finding CIs for a weighted average, Dobson et al. (1991) have proposed a method based on normal approximation. Various closed-form approximate CIs are also proposed in the literature; see Fay and Feuer (1997), Swift (1995, 2010) and Tiwari, Clegg, and Zou (2006). Recently, Krishnamoorthy and Lee (2010) proposed an improved closed-from approximate CIs based on fiducial approach. Their simulation studies indicated that the fiducial CIs are satisfactory and good enough for practical use.

The problem of estimating a product of Poisson rates or a product of powers of Poisson rates has received somewhat very limited attention. Harris (1971) has considered some asymptotic solutions, and Kim (2006) has developed some Bayesian credible intervals. It should be noted that the merits of the proposed methods in Harris (1971) are unknown, and the coverage probabilities of these methods are difficult to evaluate. The Bayesian credible interval developed by Kim is not in closed-from, and a weighted Monte Carlo simulation is required to find it. Furthermore, both Harris (1971) and Kim (2006) have noted that the solutions to the Poisson models can be used as approximation to the reliability problems involving product of powers of binomial probabilities. However, solutions to the problem of estimating the product of binomial probabilities can be found in a relatively easier manner directly under binomial models using the fiducial approach by Krishnamoorthy and Lee (2010).

Some of the aforementioned problems are special cases of the problem of estimating a product of powers of Poisson rates. To describe the problem formally, let $X_1, ..., X_g$ be independent Poisson random variables with $X_i \sim \text{Poisson}(n_i\lambda_i)$, $i = 1, ..., g$. Consider estimating $\Lambda = \prod_{i=1}^{g} \lambda_i^{a_i}$, where a_i's are known constants. When $a_i = 1/g$, then Λ is the geometric mean of g Poisson rates, and the estimation of Λ arises in environmental applications and in economic applications (Kenneth et al., 1998). When $g = 2$, $a_1 = 1$ and $a_2 = -1$, Λ is the ratio of two Poisson means. For suitable values of a_i's, Λ represents a quotient of two products of Poisson means, which is the parameter of interest in the problem of exponential series system availability (Martz and Waller, 1982).

In this article, we propose modified large sample (MLS) method first proposed by Graybill and Wang (1980) to find a CI for a linear combination of variances, and later for more general problems by Zou et al. (2009) and Zou and Donner (2008), Zou, Taleban and Huo (2009) and Zou, Huo and Taleban (2009). This approach essentially combines the individual CIs for λ_i's to obtain a CI for a linear combination of λ_i's. This approach is simple and has produced closed-form satisfactory solutions for finding tolerance limits in various linear models (see Chapters 5 and 6 of Krishnamoorthy and Mathew, 2009, and Krishnamoorthy and Lian, 2011), and for finding CIs for a lognormal mean and difference between two lognormal means (Zou, Huo and Taleban, 2009 and Zou, Taleban and Huo, 2009), and for estimating the difference between two Poisson rates (Li et al. 2011). It appears that this MLS approach is quite useful to find simple and accurate solutions for many complex problems. In general, one can arrive at two different CIs based on the MLS approach, and its accuracy depends on the individual CIs for λ_i's. Furthermore, the MLS method is based on an asymptotic theory, and the merits of the results should be evaluated numerically or by Monte Carlo simulation.

The rest of the article is organized as follows. In the following section, we outline the MLS method for a linear combination of parameters. In Section 3, we address the problem of estimating the ratio of two Poisson rates. We show that some of the available CIs are identical, and the Cox CI based on F percentiles can be obtained using Jeffreys CI for the conditional binomial parameter. We describe the MLS CI for the ratio of Poisson rates and compare it with the Cox CI and the likelihood-score CI with respect to coverage probability and precision. In Section 4, we develop a MLS CI for a weighted average of Poisson rates, and compare it with the approximate ones proposed by Swift (1995), Fay and Feuer (1997), Tiwari, Clegg and Zou (2006), and Krishnamoorthy and Lee (2010). In Section 5, we propose a MLS CI for $\Lambda = \prod_{i=1}^{g} \lambda_i^{a_i}$, and evaluate its coverage probabilities for some special cases. Illustrative examples for estimating the ratio of means and for estimating the weighted average of Poisson rates are provided in Section 6. Some concluding remarks and discussions are given in Section 7.

2 The Modified Large Sample Method

Graybill and Wang (1980) proposed a large sample approach to obtain CIs for a linear combination of variances $\sum_{i=1}^{g} c_i\theta_i$, where c_i's are known constants, based on individual confidence limits of θ_i. Let (l_i, u_i) be a $1-\alpha$ CI for θ_i, $i = 1, ..., g$. The MLS CI (L_{WG}, U_{WG}) for $\sum_{i=1}^{g} c_i\theta_i$ is expressed as

$$L_{WG} = \sum_{i=1}^{g} c_i\widehat{\theta}_i - \sqrt{\sum_{i=1}^{g} c_i \left(\widehat{\theta}_i - l_i^*\right)^2}, \quad \text{with} \quad l_i^* = \begin{cases} l_i & \text{if } c_i > 0, \\ u_i & \text{if } c_i < 0, \end{cases} \tag{1}$$

and

$$U_{WG} = \sum_{i=1}^{g} c_i\widehat{\theta}_i + \sqrt{\sum_{i=1}^{g} c_i^2 \left(\widehat{\theta}_i - u_i^*\right)^2}, \quad \text{with} \quad u_i^* = \begin{cases} u_i & \text{if } c_i > 0, \\ l_i & \text{if } c_i < 0. \end{cases} \tag{2}$$

Graybill and Wang (1980) have obtained the above CI by modifying the usual large sample Wald CI $\sum_{i=1}^{g} c_i\widehat{\theta}_i \pm z_{1-\alpha/2}\sqrt{\sum_{i=1}^{g} c_i^2 \widehat{\text{var}}(\widehat{\theta}_i)}$ so that the modified CI has better coverage probability than the one based on the Satterthwaite approximation. For more details on the MLS approach and numerical results, see the book by Burdick and Graybill (1992).

Zou and Donner (2008) and Zou et al. (2009a, 2009b) provided alternative arguments to arrive at CIs in (1) and (2). To outline their approach, let $\widehat{\theta}_1$ and $\widehat{\theta}_2$ be independent estimates of θ_1 and θ_2, respectively. Then the upper CL for $\theta_1 + \theta_2$ based on the central limit theorem is

$$\widehat{\theta}_1 + \widehat{\theta}_2 + z_{1-\alpha/2}\sqrt{\mathrm{var}(\widehat{\theta}_1) + \mathrm{var}(\widehat{\theta}_2)}, \tag{3}$$

where an estimate of $\mathrm{var}(\widehat{\theta}_i)$ is obtained on the basis of individual confidence limits (l_i, u_i), $i = 1, 2$. To obtain the left endpoint of a $1-\alpha$ CI for $\theta_1 + \theta_2$, the variances are estimated by $\widehat{\mathrm{var}}(\widehat{\theta}_i) = (\widehat{\theta}_i - l_i)^2/z^2_{1-\alpha/2}$ and to obtain the right endpoint of a $1-\alpha$ CI, they are estimated by $\widehat{\mathrm{var}}(\widehat{\theta}_i) = (u_i - \widehat{\theta}_i)^2/z^2_{1-\alpha/2}$. Substituting these variance estimates in (3), the CI (L, U) for $\theta_1 + \theta_2$ can be expressed as

$$L_{\theta_1+\theta_2} = \widehat{\theta}_1 + \widehat{\theta}_2 - \sqrt{(\widehat{\theta}_1 - l_1)^2 + (\widehat{\theta}_2 - l_2)^2} \ \text{ and } \ U_{\theta_1+\theta_2} = \widehat{\theta}_1 + \widehat{\theta}_2 - \sqrt{(\widehat{\theta}_1 - u_1)^2 + (\widehat{\theta}_2 - u_2)^2}.$$

To obtain a CI for $\theta_1 - \theta_2 = \theta_1 + (-\theta_2)$, note that $(-u_2, -l_2)$ is a $1-\alpha$ CI for $-\theta_2$. Using these limits in the above expressions, we obtain

$$L_{\theta_1-\theta_2} = \widehat{\theta}_1 - \widehat{\theta}_2 - \sqrt{(\widehat{\theta}_1 - l_1)^2 + (u_2 - \widehat{\theta}_2)^2} \ \text{ and } \ U_{\theta_1-\theta_2} = \widehat{\theta}_1 - \widehat{\theta}_2 + \sqrt{(\widehat{\theta}_1 - u_1)^2 + (\widehat{\theta}_2 - l_2)^2}, \tag{4}$$

which are similar to the ones in (1) and (2) developed by Graybill and Wang (1980). However, note that the above approach is useful to find a CI for a linear combination of parameters (not just variances) provided valid CIs for individual parameters are available. As both Graybill-Wang approach and the above approach due to Zou and co-authors are essentially obtained by modifying the usual large sample Wald CIs, we shall refer to these procedures as the modified large sample method as referred in Graybill and Wang's (1980) paper.

3 Confidence Intervals for the Ratio of Means

Let $X_1 \sim \mathrm{Poisson}(n_1\lambda_1)$ independently of $X_2 \sim \mathrm{Poisson}(n_2\lambda_2)$. Let $\widehat{\lambda}_i = X_i/n_i$, $i = 1, 2$. In the following, we shall describe some available methods and the MLS method for obtaining CIs for the ratio λ_1/λ_2. Further, we show that some different methods produce identical CIs.

3.1 CIs based on the Conditional Approach

Let $\eta = \lambda_1/\lambda_2$. As the conditional distribution of X_1 given that $X_1 + X_2 = m > 0$ is binomial with "success" probability

$$p_\eta = \frac{n_1\eta/n_2}{n_1\eta/n_2 + 1} \tag{5}$$

and the number of trials m, a CI for η can be readily obtained from the one for p_η. Note that any procedure available for a binomial proportion can be used to find a confidence limit or test for the ratio η. Among all available CIs, the Wilson score CI and the Jeffrey CI are commonly recommended for applications (see Agresti and Coull, 1993, and Brown, Cai and DasGupta, 2001).

Cox Confidence Interval: Suppose we choose Jeffrey's confidence limits for p_η (Brown, Cai, DasGupta, 2001) given by

$$\left(B_{X_1+\frac{1}{2},X_2+\frac{1}{2};\alpha/2}, B_{X_1+\frac{1}{2},X_2+\frac{1}{2};1-\alpha/2}\right),$$

where $B_{a,b;\alpha}$ denotes the α quantile of a beta distribution with parameters a and b. This CI for p_η yields CI for η as

$$\frac{n_2}{n_1}\left(\frac{B_{X_1+\frac{1}{2},X_2+\frac{1}{2};\alpha/2}}{1-B_{X_1+\frac{1}{2},X_2+\frac{1}{2};\alpha/2}}, \frac{B_{X_1+\frac{1}{2},X_2+\frac{1}{2};1-\alpha/2}}{1-B_{X_1+\frac{1}{2},X_2+\frac{1}{2};1-\alpha/2}}\right). \tag{6}$$

Using the relation between the beta and the F distributions that $B_{m,n} \sim (1+nF_{2n,2m}/m)^{-1}$ and the relation that $F_{mn;\alpha} = 1/F_{n,m;1-\alpha}$, we can express the above CI in terms of F percentiles as

$$\left(\frac{n_2(2X_1+1)}{n_1(2X_2+1)}F_{2X_1+1,2X_2+1;\alpha/2}, \frac{n_2(2X_1+1)}{n_1(2X_2+1)}F_{2X_1+1,2X_2+1;1-\alpha/2}\right), \tag{7}$$

where $F_{m,n,q}$ denotes the q quantile of an $F_{m,n}$ distribution. The above CI is referred to as the Cox (1953) CI, and Cox has derived it using somewhat fiducial argument.

Likelihood-score and the Wilson CIs: We shall now find the CI for η that can be deduced from the Wilson score CI for p_η. Let $\widehat{p} = X_1/m$, where $m = X_1 + X_2$. The Wilson score CI for p_η is given by

$$(p_L, p_U) = \left(\frac{\hat{p}+\frac{c^2}{2m}}{1+\frac{c^2}{m}}\right) \mp \frac{\frac{c}{\sqrt{m}}\sqrt{\hat{p}(1-\hat{p})+c^2/(4m)}}{1+\frac{c^2}{m}}, \tag{8}$$

where $c = z_{1-\alpha/2}$ is the $1-\alpha/2$ quantile of the standard normal distribution. When $(X_1, X_2) \neq (0,0)$, a $1-\alpha$ CI for η is given by $(p_L/(1-p_L), p_U/(1-p_U))$, which can expressed in terms of X_1 and X_2 as

$$\frac{n_2}{n_1}\left(\frac{2X_1+c^2-\sqrt{4c^2X_1X_2/m+c^4}}{2X_2+c^2+\sqrt{4c^2X_1X_2/m+c^4}}, \frac{2X_1+c^2+\sqrt{4c^2X_1X_2/m+c^4}}{2X_2+c^2-\sqrt{4c^2X_1X_2/m+c^4}}\right). \tag{9}$$

We shall refer to the above CI as the score CI. Note that this score CI (9) is the set of values of η for which

$$\frac{(\widehat{p}-p_\eta)^2}{p_\eta(1-p_\eta)/m} \leq c^2, \tag{10}$$

where p_η is defined in (5).

Sato (1990) and Graham et al. (2003) developed likelihood-score CI for the ratio of two Poisson means. As noted by Sato (1990), the likelihood-score test can be obtained by using the moment variance estimate under $H_0 : \lambda_1/\lambda_2 = \eta_0$. Sato's test statistic is given by

$$T_{\eta_0} = \frac{\widehat{\lambda}_1 - \eta_0\widehat{\lambda}_2}{\sqrt{\widehat{\operatorname{var}}(\widehat{\lambda}_1 - \eta_0\widehat{\lambda}_2)}} = \frac{\widehat{\lambda}_1 - \eta_0\widehat{\lambda}_2}{\sqrt{\frac{\eta_0(X_1+X_2)}{n_1n_2}}}. \tag{11}$$

The variance estimate in the above statistic is obtained using the results that, under H_0, $E(X_1 + X_2) = \lambda_2(n_2 + n_1\eta_0)$ and $\operatorname{var}(\widehat{\lambda}_1 - \eta_0\widehat{\lambda}_2) = \eta_0\lambda_2(n_2 + n_1\eta_0)/(n_1n_2) = \eta_0 E(X_1 + X_2)/(n_1n_2)$. The likelihood-score CI is the set of values of η_0 for which $T^2_{\eta_0} \leq z^2_{\alpha/2}$.

It is interesting to note that the score CI in (9) and the likelihood-score CI based on (11) are the same. To prove this, it is enough to show that the test statistics in (10) and the one in (11) are the same. Write the statistic in (10) as

$$\left(\frac{X_1}{X_1+X_2} - \frac{n_1\lambda_1}{n_1\lambda_1+n_2\lambda_2}\right) \Big/ \sqrt{\frac{n_1\lambda_1 n_2\lambda_2}{(n_1\lambda_1+n_2\lambda_2)^2(X_1+X_2)}}.$$

After multiplying the numerator and the denominator by $(X_1 + X_2)(n_1\lambda_1 + n_2\lambda_2)$, and minor simplification, we see that the above statistic is equal to the one in (11).

In view of the relations among the CIs, we see that there are only two distinct CIs, namely, the Cox CI in (7) and the Wilson-score (or simply score) CI in (9) for the ratio of Poisson means.

3.2 MLS Confidence Intervals

The MLS CLs for the ratio of Poisson means can be obtained from the one for $\lambda_1 - \eta_0\lambda_2$. Consider testing $H_0 : \lambda_1/\lambda_2 \le \eta_0$ vs. $H_a : \lambda_1/\lambda_2 > \eta_0$, or equivalently, $H_0 : \lambda_1 - \eta_0\lambda_2 \le 0$ vs. $H_a : \lambda_1 - \eta_0\lambda_2 > 0$. The null hypothesis H_0 is rejected if a $1-\alpha$ lower confidence limit for $\lambda_1 - \eta_0\lambda_2$ is greater than zero. The MLS lower CL for $\lambda_1 - \eta_0\lambda_2$ is obtained from (4) as

$$L_{\lambda_1-\eta_0\lambda_2} = \widehat{\lambda}_1 - \eta_0\widehat{\lambda}_2 - \sqrt{(\widehat{\lambda}_1 - l_1)^2 + \eta_0^2(u_2 - \widehat{\lambda}_2)^2}. \tag{12}$$

The null hypothesis is rejected when $L_{\lambda_1-\eta_0\lambda_2} > 0$, so the set of values of η_0 for which $L_{\lambda_1-\eta_0\lambda_2} \le 0$ form a one-sided lower CI for η. Solving the inequality $L_{\lambda_1-\eta_0\lambda_2} \le 0$ for η_0, we can obtain a $1-\alpha$ lower confidence limit for λ_1/λ_2 as

$$L_{\text{mls}} = \frac{\widehat{\lambda}_1\widehat{\lambda}_2 - \sqrt{\widehat{\lambda}_1^2\widehat{\lambda}_2^2 - [\widehat{\lambda}_2^2 - (u_2-\widehat{\lambda}_2)^2][\widehat{\lambda}_1^2 - (l_1-\widehat{\lambda}_1)^2]}}{\widehat{\lambda}_2^2 - (u_2-\widehat{\lambda}_2)^2}. \tag{13}$$

A one-sided upper confidence limit for the ratio can be obtained similarly as

$$U_{\text{mls}} = \frac{\widehat{\lambda}_1\widehat{\lambda}_2 + \sqrt{\widehat{\lambda}_2^2\widehat{\lambda}_1^2 - [\widehat{\lambda}_1^2 - (u_1-\widehat{\lambda}_1)^2][\widehat{\lambda}_2^2 - (l_2-\widehat{\lambda}_2)^2]}}{\widehat{\lambda}_2^2 - (\widehat{\lambda}_2-l_2)^2}. \tag{14}$$

The terms under the radical signs in the above CLs could be negative if X_1 or $X_2 = 0$. This problem can be rectified by defining

$$\widehat{\lambda}_i = \begin{cases} .5/n_i & \text{when } X_i = 0, \\ X_i/n_i & \text{for } X_i \ge 1, \end{cases} \tag{15}$$

$i = 1, 2$. Regarding the choices of individual CLs (l_i, u_i) for λ_i, our preliminary coverage and precision studies indicate that the Jeffrey CLs given by

$$(l_i, u_i) = \left(\frac{1}{2n_i}\chi^2_{2X_i+1;\alpha/2}, \frac{1}{2n_i}\chi^2_{2X_i+1;1-\alpha/2}\right) \tag{16}$$

produce very satisfactory CIs for the ratio λ_1/λ_2.

Remark 1. By interchanging the subscripts (1,2) in (13) and (14) by (2,1), we can obtain CIs for λ_2/λ_1. Furthermore, it should be noted that $\left(U_{\mathrm{mls}}^{-1}, L_{\mathrm{mls}}^{-1}\right)$ is a $1-\alpha$ CI for λ_2/λ_1. To see this, multiply the numerator and the denominator of L_{mls}^{-1} by

$$\widehat{\lambda}_1\widehat{\lambda}_2 + \sqrt{\widehat{\lambda}_1^2\widehat{\lambda}_2^2 - [\widehat{\lambda}_2^2 - (u_2 - \widehat{\lambda}_2)^2][\widehat{\lambda}_1^2 - (l_1 - \widehat{\lambda}_1)^2]}.$$

Similarly, by multiplying the numerator and the denominator of U_{mls}^{-1} by

$$\widehat{\lambda}_1\widehat{\lambda}_2 - \sqrt{\widehat{\lambda}_2^2\widehat{\lambda}_1^2 - [\widehat{\lambda}_1^2 - (u_1 - \widehat{\lambda}_1)^2][\widehat{\lambda}_2^2 - (l_2 - \widehat{\lambda}_2)^2]},$$

it can be seen that U_{mls}^{-1} is the left endpoint of the $1-\alpha$ CI for the ratio λ_2/λ_1.

3.3 Coverage and Precision Studies for the Ratio of Means

Let $\mathbf{x} = (x_1, ..., x_g)$ be an observed value of $X = (X_1, ..., X_g)$, and let $(L(\mathbf{x};\alpha), U(\mathbf{x};\alpha))$ be a $1-\alpha$ CI for $\theta = f(\lambda_1, ..., \lambda_g)$, where f is a real-valued function. Then, for a given α, $(\lambda_1, ..., \lambda_g)$ and $(n_1, ..., n_g)$, the exact coverage probability of the CI $(L(\mathbf{x};\alpha), U(\mathbf{x};\alpha))$ can be calculated using the expression

$$\sum_{x_1=0}^{\infty} \cdots \sum_{x_g=0}^{\infty} \prod_{i=1}^{g} \frac{e^{-n_i\lambda_i}(n_i\lambda_i)^{x_i}}{x_i!} I\left[(L(\mathbf{x};\alpha), U(\mathbf{x};\alpha)) \ni \theta\right], \tag{17}$$

where $I[.]$ is the indicator function. Note that the above expression is conducive to compute the exact coverage probability provided $g \le 2$. For $g \ge 3$, computation of the exact coverage probability is quite time consuming, and in this case, Monte Carlo simulation may be preferred.

We have used the expression (17) to evaluate the coverage probabilities of the CIs for the ratio of Poisson means. The infinite sums in (17) are evaluated by first computing the terms at the modes of the Poisson distributions, and then computing other successive probabilities using forward and backward recurrence relations. The ith series is terminated once the individual probabilities become smaller than 10^{-7} or the number of terms exceeds $\max\{10, n_i\lambda_i + 10\sqrt{n_i\lambda_i}\}$.

The score CI is not defined when $(X_1, X_2) = (0, 0)$, and in this case we take the Cox CI in place of the score CI for evaluating coverage probabilities. For a given value of (n_1, n_2), we calculated the coverage probabilities of CIs for the ratio of Poisson means for 1000 randomly generated values of (λ_1, λ_2) from the indicated uniform distributions in Tables 1–3. The five-number statistics, along with the mean and standard deviation, of the coverage probabilities of 95% one-sided CIs are given in Tables 1 and 2. We first see from the summary results that all CIs are mostly conservative when $n_1 = n_2 = 1$; the coverage probability of the score interval could be as low as .879 when the nominal level is .95 whereas the minimum coverage probability of the MLS CI is .932. We also see that when the expected number of total counts (i.e., values of $n_i\lambda_i$) from each Poisson distribution is at least two, then all CIs maintain coverage probabilities close to the nominal level in most cases. The Cox and the MLS CIs are quite comparable, and the latter has an edge over the former in some cases. In Table 3, we presented summary results of coverage probabilities of two-sided CIs. We once again see that the MLS CIs have better coverage properties than other CIs, and the MLS CIs and the Cox CIs are comparable in most of the cases. Finally, we observe from Tables 1 – 3 that the standard deviation of the coverage probabilities of the MLS CIs is less than or equal to corresponding ones for other CIs. This indicates that the MLS CIs are more consistent in maintaining the coverage probabilities than other methods.

Table 1: Summary statistics of 95% lower confidence limits for λ_1/λ_2

Methods	min	Q_1	med	Q_3	max	mean	s	min	Q_1	med	Q_3	max	mean	s
	$\lambda_1, \lambda_2 \sim$ uniform(.5, 2); $n_1 = n_2 = 1$							$\lambda_1, \lambda_2 \sim$ uniform(.5, 2); $n_1 = n_2 = 4$						
Cox	.896	.948	.964	.982	1	.964	.021	.928	.943	.946	.949	.958	.946	.005
Score	.879	.950	.970	.990	1	.968	.025	.924	.945	.950	.955	.982	.950	.008
MLS	.932	.956	.969	.985	1	.970	.018	.933	.945	.948	.951	.964	.948	.004
	uniform(.5, 2), $n_1 = 5, n_2 = 10$							uniform(.2, 1), $n_1 = 15, n_2 = 15$						
Cox	.934	.947	.948	.950	.956	.948	.003	.935	.947	.948	.950	.956	.948	.003
Score	.928	.942	.945	.948	.958	.945	.005	.934	.947	.950	.953	.965	.950	.006
MLS	.937	.948	.950	.951	.959	.950	.003	.935	.947	.949	.951	.957	.949	.003

Table 2: Summary statistics of 95% upper confidence limits for λ_1/λ_2

Methods	min	Q_1	med	Q_3	max	mean	s	min	Q_1	med	Q_3	max	mean	s
	$\lambda_1, \lambda_2 \sim$ uniform(.5, 2); $n_1 = n_2 = 1$							$\lambda_1, \lambda_2 \sim$ uniform(.5, 2); $n_1 = n_2 = 4$						
Cox	.898	.949	.963	.981	1	.964	.021	.926	.943	.946	.949	.959	.946	.005
Score	.880	.951	.970	.989	1	.968	.025	.921	.945	.950	.954	.979	.950	.008
MLS	.936	.959	.972	.988	1	.973	.017	.926	.945	.948	.951	.966	.948	.005
	uniform(.5, 2), $n_1 = 5, n_2 = 10$							uniform(.2, 1), $n_1 = n_2 = 15$						
Cox	.935	.947	.948	.950	.957	.948	.003	.933	.946	.948	.950	.956	.948	.003
Score	.926	.942	.945	.948	.958	.945	.004	.932	.947	.950	.954	.966	.950	.006
MLS	.935	.947	.948	.949	.957	.948	.003	.934	.947	.949	.950	.956	.948	.003

Table 3: Summary statistics of coverage probabilities of 95% CIs forλ_1/λ_2

Methods	min	Q_1	med	Q_3	max	mean	s	min	Q_1	med	Q_3	max	mean	s
	$\lambda_1, \lambda_2 \sim$ uniform(.5, 2); $n_1 = n_2 = 1$							$\lambda_1, \lambda_2 \sim$uniform(.5, 2); $n_1 = n_2 = 4$						
Cox	.938	.964	.972	.981	.998	.972	.012	.931	.943	.945	.947	.960	.945	.004
Score	.914	.971	.979	.986	.999	.978	.011	.944	.951	.954	.957	.969	.954	.004
MLS	.952	.963	.977	.985	.999	.976	.010	.935	.945	.947	.949	.962	.947	.004
	$\lambda_1, \lambda_2 \sim$uniform(.5, 2), $n_1 = 5, n_2 = 10$							$\lambda_1, \lambda_2 \sim$uniform(.2, 1), $n_1 = n_2 = 15$						
Cox	.934	.945	.947	.948	.952	.946	.003	.940	.948	.949	.949	.955	.947	.002
Score	.934	.951	.955	.959	.972	.956	.005	.937	.948	.950	.953	.960	.952	.003
MLS	.935	.946	.948	.949	.953	.947	.003	.940	.948	.949	.949	.954	.948	.002

Table 4: Expectations of 95% lower confidence limits and of upper confidence limits (in parentheses) for λ_1/λ_2 $n_1 = 4, n_2 = 4$

		λ_1						
λ_2	Methods	1	1.25	1.5	1.75	2	2.5	3.0
1	Cox	.40(45.13)	.52(55.53)	0.65(65.95)	0.79(76.36)	0.92(86.78)	1.19(107.61)	1.46(128.45)
	MLS	.40(42.75)	.52(53.12)	0.65(63.51)	0.78(73.91)	0.91(84.32)	1.18(105.13)	1.45(125.95)
1.25	Cox	.32(18.43)	.43(22.62)	0.53(26.81)	0.64(31.00)	0.75(35.18)	0.98(43.55)	1.20(51.93)
	MLS	.32(17.54)	.43(21.71)	0.53(25.88)	0.64(30.06)	0.75(34.24)	0.97(42.59)	1.20(50.95)
1.5	Cox	.27(8.17)	.36(9.98)	0.45(11.79)	0.54(13.59)	0.64(15.40)	0.83(19.01)	1.03(22.61)
	MLS	.27(7.84)	.36(9.63)	0.45(11.43)	0.54(13.23)	0.64(15.03)	0.83(18.62)	1.02(22.22)
1.75	Cox	.23(4.12)	.31(5.00)	0.39(5.87)	0.47(6.74)	0.56(7.61)	0.72(9.35)	0.89(11.09)
	MLS	.24(3.99)	.31(4.86)	0.39(5.73)	0.47(6.60)	0.56(7.46)	0.72(9.20)	0.89(10.93)
2	Cox	.21(2.44)	.27(2.93)	0.35(3.43)	0.42(3.92)	0.49(4.41)	0.64(5.38)	0.79(6.36)
	MLS	.21(2.39)	.28(2.88)	0.35(3.37)	0.42(3.86)	0.49(4.34)	0.64(5.32)	0.79(6.28)
2.5	Cox	.17(1.31)	.22(1.55)	0.28(1.80)	0.34(2.04)	0.40(2.28)	0.52(2.76)	0.65(3.24)
	MLS	.17(1.30)	.22(1.55)	0.28(1.79)	0.34(2.03)	0.40(2.27)	0.52(2.75)	0.65(3.22)
3	Cox	.14(0.95)	.19(1.12)	0.24(1.29)	0.29(1.46)	0.34(1.63)	0.44(1.96)	0.55(2.30)
	MLS	.14(0.95)	.19(1.12)	0.24(1.29)	0.29(1.46)	0.34(1.63)	0.44(1.96)	0.55(2.29)

To judge the precisions of the CIs for the ratio of means, we evaluated exact expected widths and presented them in Table 4. The score CIs are not included because they are wider for small values of X_1 and X_2, and as a result they tend to have wider expected widths for small values of λ_i's. It is clear from the reported expected widths in Table 4 that the MLS CIs are shorter than the Cox CIs for almost all cases, and the improvement in precision is not appreciable when both λ_1 and λ_2 exceed two.

4 Weighted Average of Poisson Means

Let $X_1, ..., X_g$ be independent Poisson random variables with $X_i \sim \text{Poisson}(n_i\lambda_i)$, $i = 1, ..., g$. In the following, we shall describe two methods of finding CIs for $\mu = \sum_{i=1}^{g} w_i\lambda_i$, where $w_i > 0$ and $\sum_{i=1}^{g} w_i = 1$.

4.1 Confidence Intervals

The MLS CI for $\mu = \sum_{i=1}^{g} w_i\lambda_i$ is given by

$$(L_{WA}, U_{WA}) = \left(\widehat{\mu} - \sqrt{\sum_{i=1}^{g} w_i^2 \left(\widehat{\lambda}_i - l_i\right)^2},\ \ \widehat{\mu} + \sqrt{\sum_{i=1}^{g} w_i^2 \left(\widehat{\lambda}_i - u_i\right)^2}\right), \tag{18}$$

where $\widehat{\mu} = \sum_{i=1}^{g} w_i\widehat{\lambda}_i$. We once again choose (l_i, u_i) to be the Jeffrey CIs given in (16) to compute the above MLS CI.

Recently, Krishnamoorthy and Lee (2010) have proposed an approximate CI for the weighted mean $\mu = \sum_{i=1}^{g} w_i\lambda_i$ based on the fiducial approach, and is given by

$$\left(e\chi^2_{f;\alpha/2}, e\chi^2_{f;1-\alpha/2}\right), \tag{19}$$

where

$$e = \frac{\sum_{i=1}^{g} c_{i*}^2(2m_i+1)}{\sum_{i=1}^{g} c_{i*}(2m_i+1)}, \quad f = \frac{\left(\sum_{i=1}^{g} c_{i*}(2m_i+1)\right)^2}{\sum_{i=1}^{g} c_{i*}^2(2m_i+1)} \text{ and } c_i^* = \frac{c_i}{2n_i}, \ i = 1, ..., g,$$

and $\chi^2_{m;p}$ denotes the p quantile of a chi-square distribution with m degrees of freedom.

To describe the approximate $1-\alpha$ CI proposed by Swift (1995), let $w_i^* = w_i/n_i$, $i = 1, ..., g$, $z_0 = \sum_{i=1}^{g} w_i^{*3}X_i/[6(\sum_{i=1}^{g} w_i^{*2}X_i)^{3/2}]$ and $\widehat{\sigma}^2 = \sum_{i=1}^{g} w_i^{*2}X_i$. In terms of these quantities, the CI is expressed as

$$\left(\widehat{\mu} + \frac{z_0 - z_{1-\alpha/2}}{[1 - z_0(z_0 - z_{1-\alpha/2})]^2}\widehat{\sigma}, \widehat{\mu} + \frac{z_0 + z_{1-\alpha/2}}{[1 - z_0(z_0 + z_{1-\alpha/2})]^2}\widehat{\sigma}\right), \tag{20}$$

where $\widehat{\mu}$ is as defined in (18). When $T = \sum_{i=1}^{g} X_i = 0$, the Swift CI is defined as $\left(0, \ .5\chi^2_{2;1-\alpha/2}\right)$.

Tiwari et al. (2006) have proposed a modification on the CIs by Fay and Feuer (1997), and their approximate CI is expressed as follows. Let $\widehat{\sigma}^2 = \sum_{i=1}^{g} w_i^{*2}X_i$, $f_1 = 2\widehat{\mu}^2/\widehat{\sigma}^2$ and $f_2 = 2\widehat{\mu}^{*2}/\widehat{\sigma}^{*2}$, where $\widehat{\mu}^* = \widehat{\mu} + g^{-1}\sum_{i=1}^{g} w_i/n_i$ and $\widehat{\sigma}^{*2} = \widehat{\sigma}^2 + g^{-1}\sum_{i=1}^{g}(w_i/n_i)^2$. Then the modified CI is given by

$$\left(\frac{\widehat{\sigma}^2}{2\widehat{\mu}}\chi^2_{f_1;\alpha/2}, \ \frac{\widehat{\sigma}^{*2}}{2\widehat{\mu}^*}\chi^2_{f_2;1-\alpha/2}\right). \tag{21}$$

When $\sum_{i=1}^{g} X_i = 0$, we use $\left(0, \ .5\chi^2_{2;1-\alpha/2}\right)$ to estimate μ.

4.2 Coverage Studies for Weighted Sums of Poisson Rates

To assess the accuracy of the MLS CIs and to compare it with the fiducial CI in (19), we estimated their coverage probabilities by Monte Carlo simulation consisting of 10,000 runs. The summary statistics of coverage probabilities at 1,000 random points $(\lambda_1, \lambda_2, \lambda_3)$ generated from uniform$(.5, 2)$ distribution are given in Table 5. The simulation study was carried out for two sets of sample sizes $n_1 = n_2 = n_3 = 1$ and $n_1 = 2, n_2 = 5, n_3 = 4$.

The estimated coverage probabilities clearly indicate that these two CIs are satisfactory and comparable, except that the chisquare-based CI (19) could be too conservative in some cases (see the case $n_1 = n_2 = n_3 = 1$ and $(w_1, w_2, w_3) = (.1, .1, .8)$. Reported expected widths indicate that the MLS CIs are barely shorter than the chisquare-based CIs. The CIs by Swift (1995) could be liberal having coverage probabilities much smaller than the nominal level when the weights are drastically different. The CIs by Tiwari et al. (2006) are, in general, conservative, and they could also be liberal in some cases. Nevertheless, the CIs by Tiwari et al. (2006) are not much wider than other CIs.

We also estimated non-coverage probabilities of the CIs at both tails, and presented them in Table 6. We observe from this table that the MLS CI (18) and the CI (19) undercover on the left-tail and overcover on the right-tail. This indicates that the one-sided upper limits based on the MLS approach and the chi-square approximate CI (19) are conservative and the lower limits must be liberal. So these two CIs are not recommended for applications where one-sided limits are needed. The only CI that seems to control the coverage probabilities in both tails is the one by Swift (1995). However, as noted earlier for the case of $(w_1, w_2, w_3) = (.1, .1, .8)$, the Swift one-sided confidence limits could be too liberal in some cases.

Overall, if the coverage requirement is important, then the CIs by Tiwari et al. are preferred to others. The CIs by Swift (1995) should be used with caution, because when the weights are very different, they could be too liberal.

5 MLS Confidence Intervals for $\Lambda = \prod_{i=1}^{g} \lambda_i^{a_i}$

MLS CIs for a product, geometric mean or for the ratio of geometric means can be obtained as special cases of CIs for $\Lambda = \prod_{i=1}^{g} \lambda_i^{a_i}$ or $\ln \Lambda = \sum_{i=1}^{g} a_i \ln \lambda_i$, where a_i's are known constants. The MLS CLs for $\ln \Lambda$ (follow from (1) and (2)) are given by

$$L_{\ln \Lambda} = \sum_{i=1}^{g} a_i \ln \widehat{\lambda}_i - \sqrt{\sum_{i=1}^{g} a_i^2 (\ln \widehat{\lambda}_i - \ln l_i^*)^2}, \quad \text{with} \quad l_i^* = \begin{cases} l_i & \text{if } a_i > 0, \\ u_i & \text{if } a_i < 0, \end{cases}$$

and

$$U_{\ln \Lambda} = \sum_{i=1}^{g} a_i \ln \widehat{\lambda}_i + \sqrt{\sum_{i=1}^{g} a_i^2 (\ln \widehat{\lambda}_i - \ln u_i^*)^2}, \quad \text{with} \quad u_i^* = \begin{cases} u_i & \text{if } a_i > 0, \\ l_i & \text{if } a_i < 0, \end{cases}$$

where (l_i, u_i) is a $1-\alpha$ CI for λ_i, $i = 1, ..., g$. The above CLs are used with the convention that $\widehat{\lambda}_i = X_i/n_i$ for $X_i \geq 1$ and $\widehat{\lambda}_i = .5/n_i$ at $X_i = 0$, $i = 1, ..., g$. Taking exponentiation, we get the $1-\alpha$ MLS CI (L_Λ, U_Λ) for Λ with the lower limit

$$L_\Lambda = \prod_{i=1}^{g} \widehat{\lambda}_i^{a_i} \exp\left[-\left(\sum_{i=1}^{g} a_i^2 \left(\ln \frac{\widehat{\lambda}_i}{l_i^*} \right)^2 \right)^{1/2} \right] \tag{22}$$

and the upper limit

$$U_\Lambda = \prod_{i=1}^{g} \widehat{\lambda}_i^{a_i} \exp\left[\left(\sum_{i=1}^{g} a_i^2 \left(\ln \frac{\widehat{\lambda}_i}{u_i^*} \right)^2 \right)^{1/2} \right]. \tag{23}$$

A CI for the product, geometric mean or the ratio of geometric means of Poisson parameters can be readily obtained from (22) and (23) by selecting the value of g and the values of a_i's suitably. For instance, for the geometric mean $\prod_{i=1}^{k} \lambda_i^{1/k}$, $a_i = 1/k$ for $i = 1, ..., k$. Regrading the choices of individual CIs (l_i, u_i), our preliminary coverage studies indicated that the CLs (22) and (23) with

Table 5: Summary statistics of coverage probabilities (CP) and expected widths (EW) of 95% CIs for weighted average of Poisson means

			λ_i's $\sim$ uniform(.5, 2), $n_1 = n_2 = n_3 = 1$						uniform(.5, 2), $n_1 = 2, n_2 = 5, n_3 = 4$					
weights	methods		min	Q_1	med	Q_3	max	mean	min	Q_1	med	Q_3	max	mean
.2,.2,.6	CI (18)	CP	.933	.948	.955	.960	.970	.954	.936	.944	.947	.949	.954	.946
		EW	2.53	2.92	3.22	3.55	3.76	3.22	1.11	1.33	1.50	1.71	1.90	1.52
	CI (19)	CP	.934	.955	.958	.961	.966	.957	.938	.947	.948	.951	.956	.949
		EW	2.55	2.94	3.25	3.59	3.80	3.24	1.12	1.35	1.51	1.72	1.91	1.53
	CI (20)	CP	.880	.910	.934	.978	.986	.942	.935	.952	.954	.956	.963	.954
		EW	2.44	2.98	3.35	3.69	3.97	3.31	1.10	1.39	1.56	1.72	1.91	1.55
	CI (21)	CP	.955	.986	.987	.989	.993	.985	.953	.958	.960	.962	.970	.960
		EW	2.58	2.97	3.27	3.58	3.88	3.24	1.12	1.42	1.59	1.75	1.96	1.58
.3,.3,.4	CI (18)	CP	.918	.935	.941	.945	.967	.940	.935	.939	.942	.943	.947	.941
		EW	2.36	2.72	2.90	3.06	3.31	2.88	1.09	1.37	1.49	1.59	1.74	1.48
	CI (19)	CP	.910	.936	.940	.943	.951	.939	.938	.944	.945	.947	.954	.946
		EW	2.38	2.74	2.92	3.09	3.34	2.91	1.10	1.38	1.50	1.60	1.75	1.49
	CI (20)	CP	.961	.977	.979	.980	.986	.978	.941	.950	.952	.954	.960	.952
		EW	2.46	2.73	2.88	3.03	3.37	2.88	1.09	1.35	1.46	1.55	1.75	1.45
	CI (21)	CP	.969	.983	.984	.985	.989	.984	.957	.962	.964	.966	.974	.964
		EW	2.45	2.78	2.96	3.13	3.51	2.95	1.15	1.41	1.53	1.61	1.82	1.51
.1,.1,.8	CI (18)	CP	.952	.966	.970	.972	.978	.969	.924	.934	.941	.951	.969	.943
		EW	2.92	3.51	3.89	4.26	4.65	3.87	1.23	1.44	1.76	2.00	2.25	1.74
	CI (19)	CP	.960	.968	.970	.972	.977	.970	.918	.926	.936	.947	.972	.938
		EW	2.93	3.53	3.92	4.29	4.68	3.90	1.24	1.45	1.77	2.01	2.25	1.75
	CI (20)	CP	.662	.711	.752	.814	.985	.763	.849	.946	.958	.961	.974	.949
		EW	2.52	3.45	4.10	4.67	5.17	4.02	1.28	1.63	1.89	2.12	2.34	1.87
	CI (21)	CP	.917	.987	.989	.990	.994	.985	.921	.945	.950	.953	.963	.948
		EW	2.73	3.60	3.85	4.32	4.66	3.84	1.20	1.55	1.84	2.08	2.32	1.82
.1,.4,.5	CI (18)	CP	.934	.944	.949	.953	.960	.948	.938	.944	.945	.946	.951	.945
		EW	2.50	3.00	3.21	3.41	3.77	3.19	1.03	1.26	1.45	1.55	1.68	1.41
	CI (19)	CP	.939	.948	.952	.955	.959	.951	.941	.947	.948	.950	.954	.948
		EW	2.52	3.03	3.24	3.44	3.81	3.22	1.04	1.27	1.46	1.56	1.69	1.42
	CI (20)	CP	.816	.895	.927	.947	.984	.919	.945	.952	.954	.955	.960	.954
		EW	2.32	3.01	3.29	3.51	3.88	3.25	0.98	1.29	1.43	1.56	1.76	1.42
	CI (21)	CP	.957	.971	.986	.987	.991	.981	.954	.959	.961	.963	.973	.961
		EW	2.62	3.03	3.22	3.46	3.86	3.21	1.00	1.33	1.48	1.60	1.81	1.46

the score CIs for λ_i, given by

$$(l_i, u_i) = \widehat{\lambda}_i + \frac{z^2_{1-\alpha/2}}{2n_i} \mp \sqrt{\left(\widehat{\lambda}_i + \frac{z^2_{1-\alpha/2}}{2n_i}\right)^2 - \widehat{\lambda}_i^2}, \tag{24}$$

are better than the one based on the Jeffrey CIs in (16) for most cases except for $g = 2$ and $a_1 = 1$ and $a_2 = -1$. Specifically, we noticed that, for estimating the geometric mean of Poisson parameters or for estimating the ratio of two geometric means, the CLs in (22) and (23) with the score CIs in (24) for individual λ_i's have better coverage probabilities than the those based on

Table 6: Summary statistics of non-coverage probabilities of 95% CIs for weighted average of Poisson means

L – non-coverage in the left-tail; R – non-coverage in the right-tail

		λ_i's $\sim$ uniform(.5, 2), $n_1 = 2, n_2 = 5, n_3 = 4$					
weights	methods	min	Q_1	med	Q_3	max	mean
		L (R)	L (R)	L (R)	L (R)	L (R)	L (R)
.2,.2,.6	CI (18)	.027(.007)	.034(.014)	.036(.016)	.040(.017)	.049(.022)	.037(.016)
	CI (19)	.030(.007)	.034(.014)	.036(.016)	.038(.017)	.044(.022)	.036(.015)
	CI (20)	.018(.013)	.023(.021)	.024(.023)	.025(.024)	.028(.049)	.024(.023)
	CI (21	.013(.017)	.016(.022)	.017(.023)	.018(.025)	.021(.030)	.017(.023)
.3,.3,.4	CI (18)	.037(.003)	.042(.010)	.044(.012)	.046(.013)	.057(.018)	.044(.012)
	CI (19)	.034(.001)	.039(.009)	.041(.011)	.043(.013)	.052(.017)	.041(.011)
	CI (20)	.019(.018)	.023(.022)	.024(.024)	.025(.026)	.030(.032)	.024(.024)
	CI (21	.013(.005)	.016(.017)	.017(.019)	.018(.020)	.021(.025)	.017(.018)
.1,.4,.5	CI (18)	.029(.007)	.035(.014)	.038(.015)	.040(.017)	.051(.021)	.038(.015)
	CI (19)	.030(.005)	.036(.013)	.037(.015)	.039(.016)	.047(.020)	.038(.014)
	CI (20)	.019(.016)	.023(.021)	.024(.023)	.025(.024)	.030(.029)	.024(.023)
	CI (21	.014(.012)	.017(.020)	.018(.021)	.019(.022)	.024(.028)	.018(.021)

Jeffrey CIs in (16).

Remark 2. Note that an MLS CI for the ratio λ_1/λ_2 can also be obtained from (22) and (23) by setting $g = 2$, $a_1 = 1$ and $a_2 = -1$. The resulting CLs with the Jeffrey individual CIs for λ_i's are in general comparable with the ones in (13) and (14) in terms of coverage probability; however, our preliminary numerical studies indicate that the latter ones seem to be shorter than the former in some cases.

Coverage Studies for the Geometric Means and the Ratio of Geometric Means

We have carried out limited coverage studies for estimating the geometric mean and the ratio of two geometric means. The coverage probabilities were estimated using simulation consisting of 10,000 runs at 1000 points $(\lambda_1, ..., \lambda_4)$ generated randomly from uniform(.5,2) and uniform(2,4) distributions. The summary statistics of these coverage probabilities are reported in Table 7 for estimating geometric mean and in Table 8 for estimating the ratio of geometric means. The summary statistics for estimating the geometric mean indicate that the CIs undercover on the right tail for very small expected counts. The minimum coverage probability of two-sided CIs is close to the nominal level .90, indicating that the CIs are satisfactory for practical purpose. If the expected counts from each Poisson population is not too small, then coverage probabilities of the two-sided CIs are close to the nominal level .90. However, the coverage probabilities of upper CLs indicate that the upper CLs are liberal even for moderate expected counts.

The summary results of CIs for the ratio of geometric means of Poisson rates are presented in Table 8 for some small sample sizes. The performance of the MLS CIs for this problem is better than those for estimating just the geometric mean. In particular, we note that both one-sided and two-sided CIs perform satisfactorily, except that for very small expected counts where the CIs could be overly conservative

Table 7: Summary statistics of coverage probabilities of 95% one-sided CLs and 90% CIs for $\left(\prod_{i=1}^{4}\lambda_i\right)^{1/4}$

L = lower limit; U = upper limit; T = two-sided CI

		min	Q_1	med	Q_3	max	mean	min	Q_1	med	Q_3	max	mean
		λ_i's $\sim$ uniform(.5, 2)						λ_i's $\sim$ uniform(2, 4)					
$(n_1, ..., n_4)$													
(2,1,2,1)	L	.966	.973	.974	.976	.980	.974	.963	.968	.969	.970	.976	.969
	U	.922	.962	.979	.990	.999	.975	.896	.910	.916	.921	.939	.916
	T	.891	.936	.954	.965	.979	.949	.867	.880	.885	.891	.907	.885
(4,3,5,5)	L	.962	.966	.968	.970	.974	.968	.956	.962	.963	.965	.970	.963
	U	.896	.911	.915	.923	.952	.918	.916	.925	.927	.929	.936	.927
	T	.885	.879	.884	.891	.917	.886	.881	.888	.890	.892	.900	.890
(10,6,10,12)	L	.957	.962	.964	.965	.970	.963	.953	.958	.960	.961	.967	.960
	U	.909	.924	.927	.929	.946	.927	.928	.934	.936	.938	.945	.936
	T	.875	.887	.890	.893	.906	.890	.883	.894	.896	.898	.905	.896
(20,10,10,30)	L	.952	.958	.960	.961	.969	.960	.949	.956	.957	.959	.964	.957
	U	.920	.933	.936	.938	.951	.936	.933	.939	.940	.942	.948	.940
	T	.886	.893	.895	.898	.910	.895	.885	.896	.898	.900	.908	.898

Table 8: Summary statistics of coverage probabilities of 95% one-sided CLs for $(\lambda_1\lambda_2)^{1/2}/(\lambda_3\lambda_4)^{1/2}$

L = lower limit; U = upper limit

		min	Q_1	med	Q_3	max	mean	min	Q_1	med	Q_3	max	mean
		λ_i's $\sim$ uniform(.5, 2)						λ_i's $\sim$ uniform(2, 4)					
$(n_1, ..., n_4)$													
(2,1,2,1)	L	.951	.967	.976	.985	.999	.976	.939	.948	.951	.955	.965	.951
	U	.950	.965	.974	.984	.999	.975	.938	.948	.951	.955	.966	.952
(4,3,4,3)	L	.941	.947	.950	.955	.976	.952	.940	.946	.948	.950	.954	.948
	U	.940	.947	.950	.955	.980	.952	.940	.946	.948	.950	.956	.948
(10,3,2,9)	L	.940	.951	.957	.970	.997	.962	.943	.949	.951	.953	.961	.951
	U	.938	.946	.950	.956	.983	.952	.942	.947	.949	.951	.956	.949
(10,5,10,11)	L	.940	.947	.949	.950	.958	.949	.942	.948	.950	.951	.957	.950
	U	.939	.946	.948	.951	.967	.949	.942	.947	.949	.951	.955	.949

6 Examples

Example 1 (CIs for the Ratio of Means) To illustrate the methods in the preceding sections, we shall use the serious adverse experience data analyzed in Liu et al. (2006). The data were collected during the course of a 48-week multi-center clinical trial to compare the tolerability of two drugs losartan and captopril. A total of 722 elderly with heart failure were randomly assigned to double-blind losartan ($n = 352$) or to captopril ($n = 370$). Various adverse events and the patient-years are reported in the study, and for illustration purpose we present here a few events, namely, deaths, heart failure, chest pain and myocardial infarction along with patient-years in Table 9. Note that for each type of event, the numbers of occurrences are not too small and the number

Table 9: 95% CIs for the difference between incidence rates in treatment groups losartan and captopril; X_1, X_2 number of events and t_1, t_2 patient-years

	Losartan		Captopril		CIs			
Methods	X_1	t_1	X_2	t_2	Cox	Score	MLS[1]	MLS[2]
Death	11	309.5	25	295.3	(.202, .835)	(.210, .842)	(.203, .837)	(.202, .833)
Pain, chest	6	309.2	5	295.9	(.353, 3.83)	(.371, 3.53)	(.355, 3.79)	(.354, 3.82)
Heart failure	22	303.7	22	288.6	(.526, 1.72)	(.531, 1.70)	(.520, 1.69)	(.527, 1.72)
Myocardial infarction	5	308.7	12	294.1	(.133, 1.08)	(.146, 1.08)	(.135, 1.08)	(.133, 1.07)

[1]CIs based on (13) and (14); [2]CIs based on (22) and (23)

of patient-years is quite large. As a result all methods produced CIs that are not appreciably different; see Table 9. We also observe that the Cox CIs and the MLS CIs in (22) and (23) are almost identical, and the score CIs are shorter than others.

Example 2 (CIs for a Weighted Average) We shall illustrate the methods of calculating CIs for weighted sums of Poisson means using the incidence rates given in Table III of Dobson et al. (1991). The data were collected from an urban area (reporting unit 1) and a rural area (reporting unit 2) in Federal Republic of Germany. The 1986 incidence rates for non-fatal definite myocardial infarction in women aged 35-64 years stratified by 5-year age group and reporting unit. The incidence rates along with weights c_i for age groups are reproduced here in Table 10.

It is of interest to estimate the age-standardized incidence rates per 10,000 person-years, $\mu = \sum_{i=1}^{6} w_i\lambda_i$ with w_i's as given in Table 10. The sample estimates for reporting units 1 and 2 are 2.75 and 1.41 respectively. The calculated 95% CIs for μ based on (18), (19), (20) and (21) are given in Table 10. The MLS CIs and the one in (19) based on the chi-square approximation are in agreement to some extent, and the Swift CI (20) and the CI (21) are in close agreement. It should be noted that the CIs are in consistent with the coverage studies in Section 4.2. In particular, the CIs (18) and (19) overcover on the right-tail and undercover on the left-tail, and as a result, these CIs are shifted to the right of CIs (20) and (21). Among the four CIs, the Swift CIs for both reporting units are the shortest.

Table 10: 95% Incidence rates for myocardial infarction in women by age and reporting unit

		Reporting unit 1		Reporting unit 2	
Age (years)	w_i	Person-years, n_i	Events, X_i	Person-years, n_i	Events, X_i
35-39	6/31	7,971	0	10,276	0
40-44	6/31	7,084	0	9,365	1
45-49	6/31	9,291	1	11,623	0
50-54	5/31	7,743	2	8,684	4
55-59	4/31	7,798	4	7,926	0
60-64	4/31	8,809	10	8,375	3
Age standardized rate					
per 10,000		2.75		1.41	
CI (18)		(1.83, 4.85)		(0.79, 3.08)	
CI (19)		(2.04, 5.04)		(0.97, 3.26)	
CI (20)		(1.64, 4.36)		(0.64, 2.72)	
CI (21)		(1.59, 4.50)		(0.61, 2.79)	

7 Concluding Remarks

The MLS method has been used extensively for finding confidence limits for a linear combination of variance components, and for finding tolerance intervals in some general linear mixed models (see Krishnamoorthy and Lian, 2011). In most of the problems of finding CIs for a linear combination of parameters, this MLS approach has produced simple closed-form CIs that are comparable or better than the existing CIs; see the articles by Zou et al. (2009a,b) and Zou and Donner (2008). In this article, we have shown that the MLS method produced simple and satisfactory CIs for yet another set of problems involving Poisson rates. In particular, the method produced CIs that are comparable with or slightly better than the existing CIs. For estimating the product or the ratio of product of Poisson means, the MLS CIs are the only intervals that are in closed-form with reasonable accuracy. We also noted that, using the MLS method, one could arrive at two different CIs for the same parametric function of interest, and the better one should be determined by evaluating their properties by numerically or Monte Carlo method.

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