4.4 Problems and Solutions (last update 12 February 2018)

1. At a certain company, 60% of the employees are certified to operate machine A, 30% are certified to operate machine B, and 10% are certified to operate both machine A and machine B. Let A denote the event that a randomly chosen employee is certified to operate machine A, and let B denote the event that a randomly chosen employee is certified to operate machine B.

(a) Find the probability that a randomly chosen employee is certified to operate at least one of machines A and B.
(b) Find the probability that a randomly chosen employee is certified to operate one of machines A and B but not both.
(c) Find the probability that a randomly chosen employee is certified to operate machine A but not machine B.
(d) Find the probability that a randomly chosen employee is certified to operate at most one of machines A and B.

Solution

We know that $P(A) = .6$, $P(B) = .3$, and $P(A \cap B) = .1$.

(a) "at least one" means A or B. Thus the event is $A \cup B$ and
$P(A \cup B) = P(A) + P(B) - P(A \cap B) = .6 + .3 - .1 = .8$.

(b) "one but not both" means (A and not B) or (B and not A). Thus the event is $(A \cap B^c) \cup (A^c \cap B)$ and

$$\begin{aligned} P((A \cap B^c) \cup (A^c \cap B)) &= P(A \cap B^c) + P(A^c \cap B) \qquad \text{note 1} \\ &= P(A) - P(A \cap B) + P(B) - P(A \cap B) \qquad \text{note 2} \\ &= .6 - .1 + .3 - .1 = .7. \end{aligned}$$

note 1: since the events $A \cap B^c$ and $A^c \cap B$ are disjoint.
note 2: since $P(A) = P(A \cap B) + P(A \cap B^c)$.

(c) "A but not B" is the event $(A \cap B^c)$ and $P(A \cap B^c) = P(A) - P(A \cap B) = .6 - .1 = .5$.

(d) "at most one" means that the employee is certified to operate neither machine or exactly one of the machines. Since there are only two machines, this is the complement of the event "the employee is certified to operate both machines". Thus the event is $(A \cap B)^c$ and $P((A \cap B)^c) = 1 - P(A \cap B) = 1 - .1 = .9$.

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2. The water supplies of the cities in a state were tested for two kinds of impurities commonly found in water. It was found that 20% of the water supplies had neither sort of impurity, 40% had an impurity of type A, and 50% had an impurity of type B. If a city is chosen at random, what is the probability its water supply has exactly one type of impurity?

Solution

Let A denote the event that a city water supply has an impurity of type A and let B denote the event that a city water supply has an impurity of type B. The event that a city has neither sort of impurity is $A^c \cap B^c$. We know that $P(A) = .40$, $P(B) = .50$, and $P(A^c \cap B^c) = .20$.

The event of interest, the city water supply has exactly one type of impurity, is $(A \cap B^c) \cup (A^c \cap B)$.

$A^c \cap B^c = (A \cup B)^c$ so that $P(A^c \cap B^c) = P((A \cup B)^c) = 1 - P(A \cup B)$. Thus, $P(A \cup B) = 1 - P(A^c \cap B^c) = 1 - .20 = .80$ and we have $P(A \cap B) = P(A) + P(B) - P(A \cup B) = .40 + .50 - .80 = .10$.
Combining these facts yields the following.

$$\begin{aligned} P((A \cap B^c) \cup (A^c \cap B)) &= P(A \cap B^c) + P(A^c \cap B) \\ &= P(A) - P(A \cap B) + P(B) - P(A \cap B) \\ &= .40 - .10 + .50 - .10 = .70 \end{aligned}$$

Hence the probability a randomly chosen city has a water supply with exactly one type of impurity is 0.7.

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3. John is going to graduate from a university at the end of the semester. After being interviewed at two companies he likes, he assesses that his probability of getting an offer from company A is 0.8, and his probability of getting an offer from company B is 0.6. If he believes that the probability that he will get offers from both companies is 0.5, what is the probability that he will get an offer from at least one of these two companies?

Solution

Let A denote the event that John gets an offer from company A and let B denote the event that John gets an offer from company B.

We know that $P(A) = .8$, $P(B) = .6$, and $P(A \cap B) = .5$. The event of interest, John will get an offer from at least one of these companies, is $A \cup B$.

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = .8 + .6 - .5 = .9$$

Thus, the probability that John will get an offer from at least one of these companies is 0.9.

4. An automobile manufacturer is required to recall all its cars manufactured in a given year for the repair of possible defects in the air–bag system and possible defects in the brake system. Dealers have been notified that 3% of the cars have defective air–bag systems only, and that 6% of the cars have defective brake systems only. If 87% of the cars have neither defect, what percentage of the cars have both defects?

Solution

Let A denote the event that a car has a defective air–bag system and let B denote the event that a car has a defective brake system. We know that $P(A \cap B^c) = .03$, $P(A^c \cap B) = .06$, and $P(A^c \cap B^c) = .87$. It follows that

$$P(A \cup B) = 1 - P((A \cup B)^c) = 1 - P(A^c \cap B^c) = 1 - .87 = .13$$

Since $P(A \cup B) = P(A \cap B^c) + P(A \cap B) + P(A^c \cap B)$, we have

$$P(A \cap B) = P(A \cup B) - P(A \cap B^c) - P(A^c \cap B) = .13 - .03 - .06 = .04$$

This shows that 4% of the cars have both defects.

5. From past experience, a stockbroker believes that under present economic conditions a customer will invest in tax–free bonds with a probability of 0.6, will invest in mutual funds with a probability of 0.3, and will invest in both tax–free bonds and mutual funds with a probability of 0.15. At this time, find the probability that a customer will invest
(a) in tax-free bonds but not mutual funds;
(b) in tax-free bonds or mutual funds (or both);
(c) in neither tax-free bonds nor mutual funds.

Solution

Let A denote the event that a customer will invest in tax–free bonds and let B denote the event that a customer will invest in mutual funds. We know that $P(A) = .6$, $P(B) = .3$, and $P(A \cap B) = .15$. The events of interest are: (a) $A \cap B^c$; (b) $A \cup B$; and, (c) $A^c \cap B^c$.

$$(a) P(A \cap B^c) = P(A) - P(A \cap B) = .6 - .15 = .45$$

$$(b) P(A \cup B) = P(A) + P(B) - P(A \cap B) = .6 + .3 - .15 = .75$$

$$(c) P(A^c \cap B^c) = P((A \cup B)^c) = 1 - P(A \cup B) = 1 - .75 = .25$$

6. At a certain company, 40% of the employees are certified to operate machine A, 50% are certified to operate machine B, 40% are certified to operate machine C, 15% are certified to operate machines A and B, 10% are certified to operate machines A and C, 5% are certified to operate all three machines, and 15% are certified to operate machine C but neither machine A nor machine B. Let A denote the event that a randomly chosen employee is certified to operate machine A, let B denote the event that a randomly chosen employee is certified to operate machine B, and let C denote the event that a randomly chosen employee is certified to operate machine C.

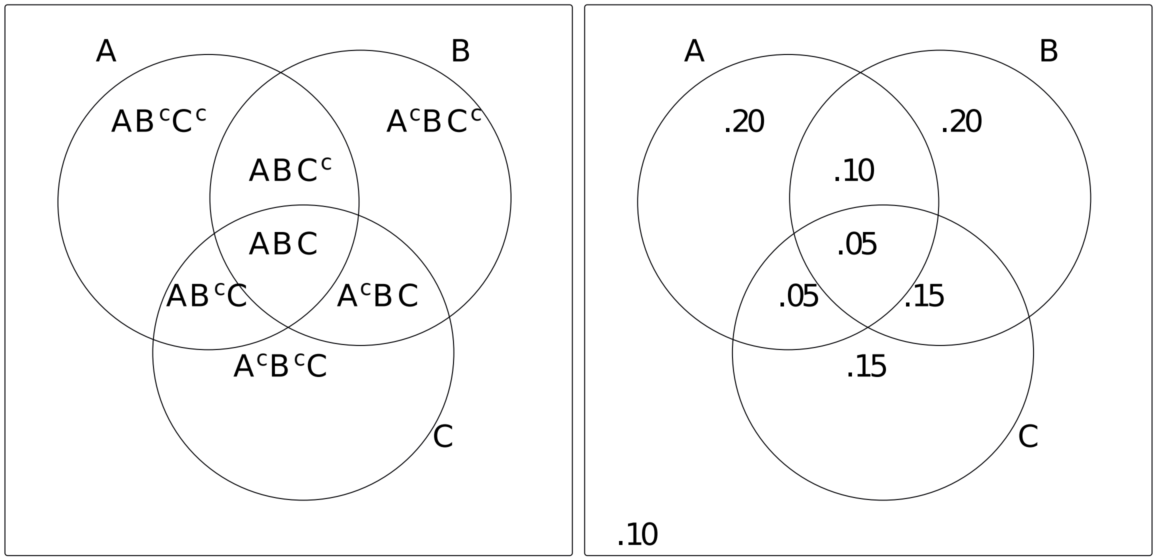
For each of the following, include an expression for the event in terms of set operations on A, B and C, and justify your answer.

(a) Find the probability that a randomly chosen employee is certified to operate at exactly one of the three machines.
(b) Find the probability that a randomly chosen employee is certified to operate machine A and machine C but not machine B.
(c) Find the probability that a randomly chosen employee is certified to operate two of the three machines but not all three.
(d) Find the probability that a randomly chosen employee is certified to operate at least one of the three machines.

Solution

We know that $P(A) = .4$, $P(B) = .5$, $P(C) = .4$, $P(AB) = .15$, $P(AC) = .10$, $P(ABC) = .05$, and $P(A^cB^cC) = .15$.

The Venn diagrams below show a partition of the sample space into eight disjoint events and the associated probabilities for this example. Derivations of these probabilities are provided below the diagrams.



$P(ABC) = .05$

$P(ABC^c) = P(AB) - P(ABC) = .15 - .05 = .10$

$P(AB^cC) = P(AC) - P(ABC) = .10 - .05 = .05$

$P(AB^cC^c) = P(A) - P(ABC^c) - P(AB^cC) - P(ABC) = .40 - .10 - .05 - .05 = .20$

$P(A^cB^cC) = .15$

$P(A^cBC) = P(C) - P(A^cB^cC) - P(AB^cC) - P(ABC) = .40 - .15 - .05 - .05 = .15$

$P(A^cBC^c) = P(B) - P(ABC^c) - P(A^cBC) - P(ABC) = .50 - .10 - .15 - .05 = .20$

$P(A \cup B \cup C) = P(AB^cC^c) + P(A^cBC^c) + P(ABC^c) + P(ABC) + P(AB^cC) + P(A^cBC) + P(A^cB^cC) = .90$ and $P((A \cup B \cup C)^c) = .10$

We will now use these probabilities to find the requested probabilities.

(a) "exactly one" means (only A) or (only B) or (only C). Thus the event is $AB^cC^c \cup A^cBC^c \cup A^cB^cC$ and, since these subevents are disjoint,
$P(AB^cC^c \cup A^cBC^c \cup A^cB^cC) = P(AB^cC^c) + P(A^cBC^c) + P(A^cB^cC) = .20 + .20 + .15 = .55$

(b) "A and C but not B" is the event AB^cC and, as shown above,
$P(AB^cC) = .05$

(c) "two but not all three" or equivalently "exactly two" means (A and B and not C) or (A and not B and C) or (not A and B and C). Thus the event is $ABC^c \cup AB^cC \cup A^cBC$ and, since these events are disjoint,
$P(ABC^c \cup AB^cC \cup A^cBC) = P(ABC^c) + P(AB^cC) + P(A^cBC) = .10 + .05 + .15 = .30$

(d) "at least one" means A or B or C. Thus the event is $A \cup B \cup C$ and, as shown above, $P(A \cup B \cup C) = .90$. We could also compute this as
$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(AB) - P(AC) - P(BC) + P(ABC)$

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7. Among the students in a high school graduating class, 54% studied mathematics during their senior year, 69% studied history during their senior year, and 35% studied both mathematics and history during their senior year. If one of these students is selected at random, find the probability that during their senior year
(a) the student studied mathematics or history (or both);
(b) the student studied neither of these subjects;
(c) the student studied history but not mathematics.

Solution

Let M denote the event that the student studied mathematics during their senior year and let H denote the event that the student studied history during their senior year. We know that $P(M) = .54$, $P(H) = .69$, and $P(M \cap H) = .35$. The events of interest are: (a) $M \cup H$; (b) $M^c \cap H^c = (M \cup H)^c$; and, (c) $M^c \cap H$.

$(a) P(M \cup H) = P(M) + P(H) - P(M \cap H) = .54 + .69 - .35 = .88$

$(b) P(M^c \cap H^c) = 1 - P(M \cup H) = 1 - .88 = .12$

$(c) P(M^c \cap H) = P(H) - P(M \cap H) = .69 - .35 = .34$

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8. Given events A and B, defined for the same sample space, is it possible to have $P(A) = \frac{1}{2}$, $P(A \cap B) = \frac{1}{3}$ and $P(B) = \frac{1}{4}$?

Solution

No, this is not possible. Since the event $A \cap B$ is a subset of the event B, the probability of $A \cap B$ cannot be larger than the probability of B. Since $\frac{1}{3} > \frac{1}{4}$, then probabilities $P(A \cap B) = \frac{1}{3}$ and $P(B) = \frac{1}{4}$ are not compatible.

9. Explain why $P(A \cup B) \leq P(A) + P(B)$.

Solution

There are three mutually exclusive ways for the union of A and B to occur: (1) A occurs but B does not occur, (2) B occurs but A does not occur, and (3) both A and B occur. Thus, the probability of the union is the sum of the probabilities of these three events. In general the events A and B are not disjoint so that the event "both A and B occur" has nonnegative probability, *i.e.*, $P(A \cap B) \geq 0$. Hence when we add the probabilities of A and B the, necessarily nonnegative, probability of their intersection enters the sum twice making the sum $P(A) + P(B)$ at least as large as the probability of the union.

10. Given events A and B, defined for the same sample space, with $P(A) = P(B)$ and $P(A^c \cap B^c) = P(A \cap B) = \frac{1}{6}$. Find

(a) $P(A)$.

(b) $P(A^c \cup B^c)$.

(c) the probability that exactly one of the events A or B occurs.

Solution

We know that $P(A) = P(B)$ and $P(A^c \cap B^c) = P(A \cap B) = \frac{1}{6}$.

(a) Since $A^c \cap B^c = (A \cup B)^c$, we know that $P(A \cup B) = 1 - P(A^c \cap B^c) = 1 - \frac{1}{6} = \frac{5}{6}$. Furthermore, since $P(A \cup B) = P(A) + P(B) - P(A \cap B)$, $P(A) = P(B)$, and $P(A \cap B) = \frac{1}{6}$, we see that $P(A \cup B) = P(A) + P(A) - \frac{1}{6} = \frac{5}{6}$ which implies that $P(A) + P(A) = 1$ giving $P(A) = P(B) = \frac{1}{2}$.

$(b) P(A^c \cup B^c) = P((A \cap B)^c) = 1 - P(A \cap B) = 1 - \frac{1}{6} = \frac{5}{6}$

(c) The event of interest, exactly one of the events A or B occurs, is $(A \cap B^c) \cup (A^c \cap B)$. The two events in this union are disjoint. Thus

$$\begin{aligned} P((A \cap B^c) \cup (A^c \cap B)) &= P(A \cap B^c) + P(A^c \cap B) \\ &= P(A) - P(A \cap B) + P(B) - P(A \cap B) \\ &= \frac{1}{2} - \frac{1}{6} + \frac{1}{2} - \frac{1}{6} = \frac{2}{3} \end{aligned}$$
