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- Outline of proof

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- KU represents the topological K-theory of complex vector bundles...

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- $$[X,\ KU] = \mathrm{Hom}_{HoTop}(X,\ KU).$$

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Examples:

$$\mathrm{rk} : GW_0(\mathbb{C}) \xrightarrow{\cong} \mathbb{Z}$$

$$(i^+, i^-) : GW_0(\mathbb{R}) \xrightarrow{\cong} \mathbb{Z} \oplus \mathbb{Z}$$

$$(\mathrm{rk}, \det) : GW_0(\mathbb{F}_q) \xrightarrow{\cong} \mathbb{Z} \oplus \mathbb{F}_q^\times / F_q^{\times 2} = \begin{cases} \mathbb{Z} & \text{if } q \text{ even} \\ \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} & \text{if } q \text{ odd} \end{cases}$$

$$GW_0(\mathbb{Z}) \xrightarrow{\cong} GW_0(\mathbb{R}) = \mathbb{Z} \oplus \mathbb{Z}$$

$$GW_0(\mathbb{Q}) \xrightarrow{\cong} GW_0(\mathbb{R}) \oplus \bigoplus_p W_0(\mathbb{F}_p)$$

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- There is a set $\mathcal{W}$ of morphisms in $\Delta^{op}Sets$, called weak equivalences, with the property there is an equivalence of categories

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- $\mathcal{K}_0^h =$ the connected component of 0.

Outline of Proof **The map $\mathcal{G}rO \xrightarrow{\hbar} \mathcal{K}^h$.**

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- Commutative diagram in $\mathbb{A}^1$-homotopy category

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BO & \xrightarrow{\gamma} & \mathcal{K}_0^h \\
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\mathcal{G}rO & \xrightarrow{\hbar} & \mathcal{K}^h \\
\downarrow & & \downarrow \\
a_{Nis}(\pi_0^{\mathbb{A}^1}\mathcal{G}rO) & \xrightarrow{\zeta} & a_{Nis}GW_0
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- Theorem: columns $\mathbb{A}^1$-fibrations

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- Theorem: γ is $\mathbb{A}^1$-weak equivalence

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\downarrow & & \downarrow \\
a_{Nis}(\pi_0^{\mathbb{A}^1}\mathcal{G}rO) & \xrightarrow{\zeta} & a_{Nis}GW_0
\end{array}$$

- Theorem: columns $\mathbb{A}^1$-fibrations
- Theorem: γ is $\mathbb{A}^1$-weak equivalence
- Theorem: ζ is an isomorphism.

Outline of Proof $\hbar : \mathcal{G}rO \to \mathcal{K}^h$ is in $\mathcal{W}_{\mathbb{A}^1}$?

- Commutative diagram in $\mathbb{A}^1$-homotopy category

$$\begin{array}{ccc} BO & \xrightarrow{\gamma} & \mathcal{K}_0^h \\ \downarrow & & \downarrow \\ \mathcal{G}rO & \xrightarrow{\hbar} & \mathcal{K}^h \\ \downarrow & & \downarrow \\ a_{Nis}(\pi_0^{\mathbb{A}^1}\mathcal{G}rO) & \xrightarrow{\zeta} & a_{Nis}GW_0 \end{array}$$

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Outline of Proof.

- A diagram of top spaces when evaluated at $X \in Sm/F$

$$\begin{array}{ccc}
(BO)_f & \xrightarrow{\gamma_f} & (\mathcal{K}_0^h)_f \\
\downarrow & & \downarrow \\
(\mathcal{G}rO)_f & \xrightarrow{\hbar_f} & (\mathcal{K}^h)_f \\
\downarrow & & \downarrow \\
(a_{Nis}(\pi_0^{\mathbb{A}^1}\mathcal{G}rO))_f & \xrightarrow{\zeta_f} & (a_{Nis}GW_0)_f
\end{array}$$

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$\longrightarrow$ columns fib, and γ_f weak eq, and ζ_f isom.

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- Theorem: If $\mathcal{G}rO$ is H-space, then

  $\hbar_f$ is a weak equivalence.

Outline of Proof $\hbar : \mathcal{G}rO \to \mathcal{K}^h$ is in $\mathcal{W}_{\mathbb{A}^1}$?

- Commutative diagram in $\mathbb{A}^1$-homotopy category

$$\begin{array}{ccc} BO & \xrightarrow{\gamma} & \mathcal{K}_0^h \\ \downarrow & & \downarrow \\ \mathcal{G}rO & \xrightarrow{\hbar} & \mathcal{K}^h \\ \downarrow & & \downarrow \\ a_{Nis}(\pi_0^{\mathbb{A}^1}\mathcal{G}rO) & \xrightarrow{\zeta} & a_{Nis}GW_0 \end{array}$$

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- **Theorem:** If $\mathcal{G}rO$ is H-space, then

  $\hbar$ is $\mathbb{A}^1$-weak equivalence.