THE LUBIN-TATE SPECTRUM AND ITS HOMOTOPY FIXED POINT SPECTRA

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Abstract. This note is a summary of the results of my Ph.D. thesis (plus slight modifications), completed May 9, 2003, under the supervision of Professor Paul Goerss at Northwestern University.

Let $E_n$ be the Lubin-Tate spectrum with

$$E_n = W(\mathbb{F}_p^n)[[u_1, ..., u_{n-1}]]/[u_1],$$

where the degree of $u$ is $-2$ and the complete power series ring over the Witt vectors is in degree zero. Let $G_n = S_n \rtimes \text{Gal}(\mathbb{F}_p^n/\mathbb{F}_p)$, where $S_n$ is the $n$th Morava stabilizer group (the automorphism group of the Honda formal group law $\Gamma_n$ of height $n$ over $\mathbb{F}_p^n$), and let $G$ be a closed subgroup of $G_n$. Note that $S_n$, $G_n$ and $G$ are all profinite groups.

Morava’s change of rings theorem yields a spectral sequence

$$(1) \quad H^s_c(G_n; \pi_t(E_n \wedge X)) = \Rightarrow \pi_t L_{K(n)}(X),$$

where the $E_2$-term is continuous cohomology and $X$ is a finite spectrum (see [7], [1], [5]). Using the $G_n$-action on $E_n$ by maps of commutative $S$-algebras (work of Goerss and Hopkins ([3], [4]), and Hopkins and Miller [8]), Devinatz and Hopkins [2] constructed spectra $E_n^{hG}$ with strongly convergent spectral sequences

$$(2) \quad H^s_c(G; \pi_t(E_n \wedge X)) = \Rightarrow \pi_t(E_n^{hG} \wedge X).$$

Also, Devinatz and Hopkins showed that $E_n^{hG_n} \wedge X \simeq L_{K(n)}(X)$.

When $K$ is a discrete group and $Y$ is a $K$-spectrum, there is a homotopy fixed point spectrum $Y^{hK} = \text{Map}_K(\mathbb{K}, Y)$, where $\mathbb{K}$ is a free contractible $K$-space. Also, there is a conditionally convergent spectral sequence

$$E_2^{s,t} = H^s(K; \pi_t(Y)) = \Rightarrow \pi_{t-s}(Y^{hK}),$$

where the $E_2$-term is group cohomology [6, §1.1]. Such a spectral sequence is called a descent spectral sequence.

This scenario also occurs in another context. Let $K$ be a profinite group. We say that $Y$ is a discrete $K$-spectrum, if $Y$ is a $K$-spectrum of simplicial sets such that each simplicial set $Y_k$ is a simplicial discrete $K$-set (that is, for each $l \geq 0$, the action map on the $l$-simplices of $Y_k$,
\[ K \times (Y_k)_l \to (Y_k)_l \text{ is a continuous map, where } (Y_k)_l \text{ is given the discrete topology}. \] Using work of Jardine, there is a model category \( Sp_K \) of discrete \( K \)-spectra, and one defines \( Y^{hK} = (Y_f)^K \) to be the homotopy fixed point spectrum of \( Y \), where \( Y \to Y_f \) is a trivial cofibration and \( Y_f \) is fibrant, all in \( Sp_K \). Then, if the cohomological dimension of \( K \) satisfies an appropriate finiteness hypothesis, there is a conditionally convergent descent spectral sequence

\[ H^s_c(K; \pi_t(Y)) \implies \pi_{t-s}(Y^{hK}). \]

Upon comparing (2) with the above descent spectral sequences, \( E_n \) appears to be a continuous \( G_n \)-spectrum with “descent” spectral sequences for “homotopy fixed point” spectra \( E_n^{hG} \wedge X \). Indeed, we explain how [2] implies that \( E_n \) is a continuous \( G_n \)-spectrum - \( E_n \) is an inverse limit of discrete \( G_n \)-spectra. Using this continuous action, we define homotopy fixed point spectra \( (E_n \wedge X)^{hG} \) that are weakly equivalent to \( E_n^{hG} \wedge X \), and, for \( (E_n \wedge X)^{hG} \), we construct a descent spectral sequence that is isomorphic to (2). We remark that this continuous action is not shown to be by \( A_\infty \)- or \( E_\infty \)-maps of ring spectra.

In more detail, the \( K(n)^* \)-local spectrum \( E_n \) has an action by \( G_n \) as a commutative \( S \)-algebra. The \( K(n)^* \)-local commutative \( S \)-algebra \( E_n^{hG} \) has an associated strongly convergent \( K(n)^* \)-local \( E_n \)-Adams spectral sequence

\[ E_2^{s,t} \cong H_c^s(G; E_n^t(Z)) \implies (E_n^{hG})^s(Z), \]

where \( Z \) is any CW-spectrum. Also, [2] proves the remarkable formula

\[ E_n \cong L_{K(n)}(\text{hocolim}_i E_n^{hU_i}), \]

where \( \{U_i\} \) is a cofinal descending chain of open normal subgroups of \( G_n \), and the homotopy colimit, as the notation indicates, is in the category of commutative \( S \)-algebras.

Devinatz and Hopkins prove that the homotopy fixed point spectra \( E_n^{hG} \) have the expected properties and, when one sets \( Z \) equal to the Spanier-Whitehead dual of any finite spectrum \( X \), one obtains a spectral sequence

\[ E_2^{s,t} \cong H_c^s(G; \pi_t(E_n \wedge X)) \implies \pi_{s-t}(E_n^{hG} \wedge X) \]

that has the form of a descent spectral sequence. Thus, their constructions strongly suggest that \( G_n \) acts on \( E_n \) in a continuous sense. However, their highly structured action is not proven to be continuous and their homotopy fixed point spectra are not defined with respect to a continuous action.

Let \( F_n = \text{colim}_i E_n^{hU_i} \). Given \( I = (p^{i_0}, v^{i_1}_1, \ldots, v^{i_{n-1}}_1) \subset BP_\ast \), let \( M_I = M(p^{i_0}, v^{i_1}_1, \ldots, v^{i_{n-1}}_1) \) (when it exists) be the generalized Moore
spectrum with $BP_*(M_I) \cong BP_*/I$. We observe that $F_n \wedge M_I$ is a discrete $G_n$-spectrum, since it is the colimit of $G_n/U_i$-spectra. Then the key fact for getting our work started is

$$E_n \wedge M_I \simeq F_n \wedge M_I,$$

obtained by applying (3), since this implies that $E_n \wedge M_I$ has the stable homotopy type of a discrete $G_n$-spectrum.

Henceforth, $X$ is a finite spectrum only if this is explicitly mentioned.

**Theorem 1.** There is a tower

$$\cdots \rightarrow (F_n \wedge M_I)_f \rightarrow \cdots \rightarrow (F_n \wedge M_I)_f \rightarrow (F_n \wedge M_I)_f$$

of fibrations of fibrant spectra in $Sp_{G_n}$, such that each $(F_n \wedge M_I)_f \simeq F_n \wedge M_I$, and

$$E_n \simeq \lim_j (F_n \wedge M_I)_f$$

is an inverse limit of discrete $G_n$-spectra. Thus, $E_n$ is a continuous $G_n$-spectrum.

**Theorem 2.** Let $G$ be a closed subgroup of $G_n$ and let $X$ have the property that the tower of abelian groups $\{\pi_t(E_n \wedge M_I \wedge X)\}_j$ is Mittag-Leffler for every integer $t$ (e.g. $X$ is finite or $X = E_n \wedge E_n \wedge \cdots \wedge E_n$). Then there is a conditionally convergent descent spectral sequence

$$H_*^{cts}(G; \pi_t(L_K(n)(E_n \wedge X))) \Rightarrow \pi_{-s}(L_K(n)(E_n \wedge X))_hG,$$

where the $E_2$-term is the cohomology of continuous cochains. In particular, if $X$ is a finite spectrum, this descent spectral sequence has the form

$$H_*^c(G; \pi_t(E_n \wedge X)) \Rightarrow \pi_{-s}((E_n \wedge X)_hG),$$

where $H_*^c(G; \pi_t(E_n \wedge X)) \cong \lim_k H_*^c(G; \pi_t(E_n \wedge X)/I^n_k)$, where $I_n = (p, u_1, \ldots, u_{n-1}) \subset E_{ns}$.

We also show that the descent spectral sequence, when $X$ is finite, is isomorphic to the spectral sequence of Devinatz and Hopkins.
Theorem 3. When $X$ is a finite spectrum, descent spectral sequence (4) is isomorphic to the strongly convergent $K(n)_*$-local $E_n$-Adams spectral sequence with abutment $\pi_*(E_n^hG \wedge X)$. In particular, in the stable homotopy category, the morphism $E_n^hG \wedge X \to (E_n \wedge X)^hG$ is an isomorphism.

Finally, we prove that the $K(n)_*$-localization of any finite complex is a homotopy fixed point spectrum. In particular, $L_{K(n)}(S^0)$ is the $G_n$-homotopy fixed points of $E_n$, in a continuous sense.

Theorem 4. Let $X$ be a finite complex. Then

$$L_{K(n)}(X) \cong (E_n \wedge X)^{hG_n},$$

in the stable homotopy category. In particular, if $X$ is also of type $n$, then

$$L_nX \cong (F_n \wedge X)^{hG_n}.$$

Acknowledgements. This work is indebted to the beautiful work of Ethan Devinatz and Mike Hopkins [2], to [3] and [4] by Paul Goerss and Hopkins, and to the work of Hopkins and Haynes Miller [8].

Many thanks to my thesis advisor Paul Goerss for a plethora of helpful conversations and many suggestions that improved the organization and writing of my thesis. His fingerprints are present throughout. I am grateful to Mark Mahowald and Stewart Priddy for their careful examination of my thesis and their comments about various drafts of it.

I thank Ethan Devinatz for helpful conversations and e-mails that gave me a better understanding of his paper [2] with Hopkins. I thank Charles Rezk and Rick Jardine for answering various questions, and I had many useful conversations with Christian Haesemeyer during the course of my research. Also, I thank Halvard Fausk, Jeff Smith and Thomas Wenger for helpful discussions.

References


