E_n as a continuous G_n-spectrum and its homotopy fixed point spectra

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§1. Notation

- Fix $n \geq 1$, a prime p.

- $S_n = \mathrm{Aut}_{\Gamma_n}(\mathbb{F}_{p^n})$ is the *nth Morava stabilizer group*, where Γ_n is the unique $p-$typical height n formal group law with $[p]_{\Gamma_n}(x) = x^{p^n}$, over $\mathbb{F}_{p^n}$.

- $G_n = S_n \rtimes \mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)$.

- G_n is a profinite group and there exists a collection of open normal subgroups

  $G_n = U_0 \gneq U_1 \gneq \ldots \gneq U_i \gneq \ldots$, with $\bigcap_i U_i = \{e\}$,

  where

  $$G_n \approx \varprojlim_i G_n/U_i$$

  has the profinite topology.

- $E_{n*} = W(\mathbb{F}_{p^n})[[u_1, ..., u_{n-1}]][u^{\pm 1}]$,

$$|u_i| = 0, |u| = -2.$$

- E_{n*} is Landweber exact; the corresponding homology theory is represented by E_n, the Lubin-Tate spectrum.

- $I = (p^{i_0}, ..., u_{n-1}^{i_{n-1}}) \subset E_{n,0}$. There is a collection of these ideals I such that

$$E_{n,0} \cong \varprojlim{}_I \, E_{n,0}/I.$$

  - $E_{n,0}$ is a profinite topological space.
  - $E_{n*} = \bigoplus_{k \in \mathbb{Z}} E_{n,0} \cdot u^{-k}$ is also a topological space.

- M_I is a generalized Moore spectrum where

$$BP_*(M_I) = BP_*/I.$$

- L_n and $L_{K(n)}$ denote Bousfield localization with respect to $E(n)_*$ and $K(n)_*$, respectively.

- A completion result for any spectrum E:

$$L_{K(n)}(E) \simeq \operatorname{holim}_I\, (L_n E \wedge M_I).$$

- X always denotes a finite spectrum.

§2. Context and Motivation

- G_n acts continuously on E_{n*} by ring automorphisms.

- This continuous action appears in the **Morava Change of Rings Theorem**, which via the Adams-Novikov spectral sequence yields a spectral sequence

$$H_c^{*,*}(G_n; \pi_*(E_n \wedge X)) \Rightarrow \pi_*(L_{K(n)}(X)).$$

- By Brown representability and the graded profiniteness of E_{n*}, G_n acts on the spectrum $E_n \wedge X$ *in the stable homotopy category*. This does not imply that E_n is a G_n–spectrum.

The appearance of continuous cohomology in the spectral sequence

$$H_c^{*,*}(G_n; \pi_*(E_n \wedge X)) \Rightarrow \pi_*(L_{K(n)}(X))$$

is a hint that something more is going on with the action and the spectral sequence.

If Y is a $G-$spectrum, where G is a finite or discrete group, then there is a *descent spectral sequence*

$$H^s(G; \pi_t(Y)) \Rightarrow \pi_{t-s}(Y^{hG}),$$

$$\text{where } Y^{hG} := \mathrm{Map}_G(EG_+, Y).$$

By the work of Jardine and Thomason, this scenario can be generalized when G is a profinite group ...

- We say Y is a *discrete G-spectrum* if:
  - Y is a G-spectrum of simplicial sets;
  - Each pointed simplicial set Y_k is a simplicial discrete pointed $G-$set.

- There is a homotopy fixed point spectrum $Y^{h'G}$ and a descent spectral sequence:

$$H_c^s(G; \pi_t(Y)) \Rightarrow \pi_{t-s}(Y^{h'G}).$$

- Comparison with

$$H_c^{*,*}(G_n; \pi_*(E_n \wedge X)) \Rightarrow \pi_*(L_{K(n)}(X))$$

  leads to some natural questions:
  - Are E_n and $E_n \wedge X$ continuous G_n-spectra?
  - Is $L_{K(n)}X$ the homotopy fixed point spectrum $(E_n \wedge X)^{h'G_n}$? Is $L_{K(n)}S^0 \simeq E_n^{h'G_n}$?
  - E_n is not a discrete G_n-spectrum: if it is, then E_{n*} is a discrete G_n-module.

The **Hopkins-Miller Theorem** implies that G_n acts on E_n through A_∞-maps, so that E_n is actually a G_n-spectrum.

§3. More Motivation, from the Work of Devinatz, Goerss, and Hopkins

- G_n acts on E_n through maps of commutative S-algebras.

- For any open subgroup U in G_n, there is a spectrum E_n^{hU}. In particular, there is the G_n/U_i-spectrum $E_n^{hU_i}$.

- $E_n^{hG_n} \simeq L_{K(n)}S^0$.

- Henceforth, G is a closed subgroup of G_n. If G is not finite, then G is a non-discrete profinite group. Any open subgroup is closed.

- $E_n^{hG} := L_{K(n)}(\operatorname{colim}_i E_n^{hU_iG})$.

- There is an Adams spectral sequence

$$H_c^s(G; \pi_t(E_n \wedge X)) \Rightarrow \pi_{t-s}(E_n^{hG} \wedge X),$$

so that $E_n^{hG} \wedge X$ behaves like the G-homotopy fixed point spectrum of $E_n \wedge X$.

- If $X = S^0$,

$$H_c^s(G; \pi_t(E_n)) \Rightarrow \pi_{t-s}(E_n^{hG}),$$

and when $G = G_n$,

$$E_n^{hG_n} \wedge X \simeq L_{K(n)} S^0 \wedge X \simeq L_{K(n)} X \text{ yields}$$

$$H_c^s(G_n; \pi_t(E_n \wedge X)) \Rightarrow \pi_{t-s}(L_{K(n)} X).$$

- Therefore, the work of Devinatz and Hopkins provides more evidence that:
  - E_n and $E_n \wedge X$ are continuous G_n-spectra.
  - There are homotopy fixed point spectra $(E_n \wedge X)^{h'G}$ defined with respect to the continuous G-action with associated descent spectral sequences.

§4. E_n is a continuous G_n-spectrum

- Henceforth, “spectrum” means Bousfield-Friedlander spectrum of simplicial sets.

- Note that

$$E_n \simeq \text{holim}_I\, E_n \wedge M_I, \text{ and}$$

  $\pi_*(E_n \wedge M_I) \cong E_{n*}/I$ is a discrete G_n-module. Is there a spectrum E_n/I that is weakly equivalent to $E_n \wedge M_I$ and that is a discrete G_n-spectrum?

- $F_u := \text{colim}_i\, E_n^{hU_i}$ is a discrete G_n-spectrum. Thus, $F_u \wedge M_I$ is a discrete G_n-spectrum.

Theorem 1 (Hovey/Strickland). *If E is any spectrum and F is a finite spectrum of type n, then*

$$(L_{K(n)}E) \wedge F \simeq L_n(E) \wedge F.$$

Theorem 2. $F_u \wedge M_I \xrightarrow{\simeq} E_n \wedge M_I$.

Proof. By the work of Devinatz and Hopkins,

$$E_n \simeq E_n^{h\{e\}} := L_{K(n)}(F_u).$$

Thus,

$$\begin{aligned} E_n \wedge M_I &\simeq (L_{K(n)}F_u) \wedge M_I \\ &\simeq L_n(F_u) \wedge M_I \\ &\simeq F_u \wedge M_I. \end{aligned}$$

□

Corollary 3. *Since* $E_n \simeq \operatorname{holim}_I F_u \wedge M_I$ *is a homotopy inverse limit of discrete* G_n*-spectra,* E_n *is a continuous* G_n*-spectrum. Similarly,*

$$E_n \wedge X \simeq \operatorname{holim}_I (F_u \wedge M_I \wedge X)$$

is a continuous G_n*−spectrum.*

§5. Homotopy fixed point spectra for $E_n \wedge X$

- $(E_n \wedge X)^{h'G} :=$ holim$\,_I\,(F_u \wedge M_I \wedge X)^{h'G}$.

- By construction, $F_u^{h'G}$ is $E(n)_*$-local, so
$$E_n^{h'G} \simeq L_{K(n)}(F_u^{h'G})$$
is $K(n)_*$-local.

Theorem 4. *Let G be a closed subgroup of G_n. Then there is a descent spectral sequence*
$$H_c^s(G; \pi_t(E_n \wedge X)) \Rightarrow \pi_{t-s}((E_n \wedge X)^{h'G}).$$

Key Ingredients of Proof. Let Y be a discrete G-spectrum. Let

Γ: discrete $G-$spectra $\longrightarrow$ discrete $G-$spectra

be the functor defined by
$$Y \mapsto \Gamma(Y) = \mathrm{Map}_c(G, Y).$$
This gives a cosimplicial spectrum $\Gamma^{\cdot}(Y)$.

- $Y^{h'G}$:= $\operatorname{holim}_{\Delta}(\Gamma^{\cdot}(Y))^G$. This is a special case of Thomason's hypercohomology spectrum. Since G has finite virtual cohomological dimension, this definition agrees with Jardine's globally fibrant replacement $G(\mathrm{Hom}_G(-,Y))(*)$.

- Let $F^{\cdot}$ be a cosimplicial fibrant spectrum. There is the Bousfield-Kan spectral sequence

$$\pi^s\pi_t(F^{\cdot}) \Rightarrow \pi_{t-s}(\operatorname{holim}_{\Delta} F^{\cdot}).$$

- Set

  $$F^{\cdot} := \operatorname{holim}_I \Big(\Gamma^{\cdot}\Big(\operatorname{colim}_i Q(E_n^{hU_i} \wedge M_I \wedge X)\Big)\Big)^G$$

  and apply the Bousfield-Kan spectral sequence.

$\square$

§6. Comparison between $(E_n \wedge X)^{h'G}$ and $E_n^{hG} \wedge X$

"Theorem" 5.

$$(\mathrm{colim}_i\, E_n^{hU_iG}) \wedge M_I \xrightarrow{\simeq} F_u^{h'G} \wedge M_I.$$

Corollary 6. *There is a weak equivalence*

$$(E_n \wedge X)^{h'G} \simeq E_n^{hG} \wedge X.$$

In particular,

$$E_n^{h'G} \simeq E_n^{hG}.$$

Corollary 7. $(E_n \wedge X)^{h'G_n} \simeq L_{K(n)}(X)$. *Thus, the $K(n)_*$-localization of a finite spectrum is a homotopy fixed point spectrum.*