THE $G_n$-ACTION ON $E_n$ IN THE STABLE CATEGORY

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Abstract. It is a well-known fact that, by Brown representability, the extended Morava stabilizer group $G_n$ acts on the Lubin-Tate spectrum $E_n$, in the stable category. Though it is not hard to prove this, as far as the author knows, the proof is not written down in the literature. Thus, he wrote out the details for himself, and is making it available, in case it can be helpful to others.

1. Introduction

Let $S_n$ be the $n$th Morava stabilizer group, and let $G_n = S_n \rtimes \text{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)$. Let $E_n$ be the Lubin-Tate spectrum with

$$E_n = W(\mathbb{F}_{p^n})[[u_1, \ldots, u_{n-1}][u^\pm 1]],$$

where the degree of $u$ is $-2$ and the complete power series ring over the Witt vectors is in degree zero.

As a part of his change of rings theorem [8], Jack Morava pointed out that, by the deformation theory of formal group laws, the profinite group $G_n$ acts continuously on the graded profinite coefficient ring $E_{n*}$. It is a well-known fact that, by Brown representability, one can conclude that the action on the level of topological algebra is actually induced by an action of $G_n$ on $E_n$ in the stable homotopy category; that is, there is a well-defined action through homotopy classes of maps.

In the early 1990’s, Mike Hopkins and Haynes Miller showed that $G_n$ actually acts on the spectrum $E_n$ on the point-set level, in the category of $A_\infty$-ring spectra [9]. They showed that $E_n$ is an $A_\infty$-ring spectrum and $G_n$ acts on the spectrum through $A_\infty$-ring spectrum maps. This point-set level action induces the homotopy action of $G_n$ on $E_n$.

Subsequently, Paul Goerss and Hopkins showed that $E_n$ is a commutative $S$-algebra, and the $G_n$-action is by maps of commutative $S$-algebras ([5], [6], [4]). Then, using the beautiful and deep paper [4], by Ethan Devinatz and Hopkins, the author showed in his thesis [2] that $G_n$ actually acts continuously on $E_n$, on the point-set level: there is a model of $E_n$ that is an inverse limit of $G_n$-spectra of simplicial sets, such that each set constituting these spectra is a discrete $G_n$-set.

As far as the author knows, the proof that the $G_n$-action on $\pi_*(E_n)$ implies a homotopy action of $G_n$ on $E_n$ is not written down anywhere in the literature. Thus, I wrote down the details for myself, and am making them available in case it can be useful to others who are beginning to study the beautiful story of the relationship between $G_n$ and $E_n$. I thank Paul Goerss for explaining to me how Brown representability yields the homotopy action.
2. $G_n$ acts on $E_n$ in $\text{Ho}(\text{Sp})$, by ring spectrum maps

The precise statement we will prove is that the $G_n$-action on $E_n^*$ implies that there is a unique action of $G_n$ on $E_n$ by ring spectrum maps in the stable homotopy category ([3, pg. 767], [4, pg. 9]).

The stable homotopy category of spectra is a monogenic Brown category, satisfying a particular form of Brown representability (see [1, pp. 342-344] and [7, pp. 6, 54-61]). For us, this means the following. A natural transformation $E_*(-) \to F_*(-)$ of homology theories on the category of finite spectra is induced by a map $E \to F$, in the stable category, of spectra. The map $E \to F$ is unique if the set $\mathcal{P}(E, F)$ of phantom maps is zero. Also, $\mathcal{P}(E, F) = 0$ if and only if $[E, F]$ is Hausdorff.

Now consider the fact that the $G_n$-action on $E_n^*$ makes $E_n^*(X)$ a natural $G_n$-module. This means that, given any map $X \to X'$ of finite spectra, there is a commutative diagram

$$
\begin{array}{ccc}
E_n^*(X) & \xrightarrow{g} & E_n^*(X) \\
\downarrow & & \downarrow \\
E_n^*(X') & \xrightarrow{g} & E_n^*(X'),
\end{array}
$$

where $g$ is induced by the action of $g \in G_n$. Thus, $g: E_n^*(-) \to E_n^*(-)$ is a natural transformation of homology theories induced by a map $g: E_n \to E_n$ of spectra. Letting $E_n = \text{colim}_\alpha E_n$ be a colimit of finite subspectra over a directed set $\{\alpha\}$ implies that

$$[E_n, E_n] \cong \lim_{\alpha} \pi_0(E_n \wedge DE_\alpha) \cong \lim_{\alpha, t} \pi_0(E_n \wedge M_t \wedge DE_\alpha)$$

is a profinite and hence, Hausdorff, topological space. Thus, there are no phantom maps $E_n \to E_n$ and the map $g$ is unique. This shows that there is a unique action of $G_n$ on $E_n$ in the stable category that induces the action of $G_n$ on $E_n^*$.

Now we explain why this action is by maps of ring spectra. Since $E_n$ is a ring spectrum, there are the unit and multiplication maps $\eta: S^0 \to E_n$ and $\mu: E_n \wedge E_n \to E_n$, respectively, in the stable category. Now consider the map $g: E_n \to E_n$. The action of $G_n$ on $E_n^*$ is by ring homomorphisms so that $g: E_n^* \to E_n^*$ sends $\eta$ to itself. Since this homomorphism is induced by $g: E_n \to E_n$, the following diagram commutes:

$$
\begin{array}{ccc}
S^0 & \xrightarrow{\eta} & E_n \\
\downarrow & & \downarrow \\
& \eta \downarrow & E_n.
\end{array}
$$

The pairing $E_n^*(X) \otimes_{E_n^*} E_n^*(Y) \to E_n^*(X \wedge Y)$ is $G_n$-equivariant, where the tensor product is given the diagonal action. Thus, given the maps $S^0 \to E_n \wedge X$ and $S^0 \to E_n \wedge Y$ in the stable category, the following diagram is commutative:

$$(2.1)$$

$$
\begin{array}{cccccccc}
S^0 & \xrightarrow{\eta} & E_n \wedge X \wedge E_n \wedge Y & \xrightarrow{\eta \wedge 1} & E_n \wedge E_n \wedge X \wedge Y & \xrightarrow{\mu \wedge 1} & E_n \wedge X \wedge Y \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
E_n \wedge X \wedge E_n \wedge Y & \xrightarrow{g \wedge 1 \wedge 1} & E_n \wedge X \wedge E_n \wedge Y & \xrightarrow{g \wedge 1} & E_n \wedge E_n \wedge X \wedge Y & \xrightarrow{\mu \wedge 1} & E_n \wedge X \wedge Y.
\end{array}
$$
As a map between spectra, $g\mu: E_n \wedge E_n \to E_n$ gives a natural transformation of homology theories $(E_n \wedge E_n)_*(-) \to E_{n*}(-)$. Thus, any map $W \to Z$ of finite spectra gives a commutative diagram

\[
\begin{array}{c}
(E_n \wedge E_n)_*(W) \longrightarrow E_{n*}(W) \\
\downarrow \quad \downarrow \\
(E_n \wedge E_n)_*(Z) \longrightarrow E_{n*}(Z).
\end{array}
\]

(2.2)

Setting $X = S^0$, and $Y = W$ and then $Z$, in 2.1 implies that 2.2 is also induced by $\mu(g \wedge g)$. Since the natural transformations $(E_n \wedge E_n)_*(-) \to E_{n*}(-)$ coming from $g\mu$ and $\mu(g \wedge g)$ give the same collection of commutative squares, they are the same natural transformation. Since the same argument as above shows that $[E_n \wedge E_n, E_n]$ is a profinite and hence, Hausdorff, topological space, $P(E_n \wedge E_n, E_n) = 0$ and the natural transformation is represented by a unique map. Thus, $g\mu = \mu(g \wedge g)$, and $g: E_n \to E_n$ is a map of ring spectra.

References


