For the Ausoni-Rognes conjecture at \( n = 1, \ p > 3 \): a strongly convergent descent spectral sequence

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Basic objects to be utilized; Description of Ausoni-Rognes Conj.

Our progress on this conjecture

Tools & theorems that played a role in the proof of Theorem 1

A “draft theorem” |

What about for higher $n$?

- $n \geq 1$
- $p$, a prime
- $E_n$ is the Lubin-Tate spectrum, with
  $\pi_*(E_n) = W(\mathbb{F}_{p^n})[u_1, ..., u_{n-1}][u^\pm 1]$. Here:
  - $W(\mathbb{F}_{p^n})$ is the ring of Witt vectors of the field $\mathbb{F}_{p^n}$
  - the complete power series ring is in degree zero
  - $|u| = 2$
    (this is the “choice” Ausoni makes in his Inventiones paper)
- $G_n = S_n \rtimes \text{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)$, the extended Morava stabilizer group
- $E_n$, a commutative $S$–algebra
- $G_n$ acts on $E_n$ by maps of commutative $S$–algebras.
- (The preceding two points are an application of the Goerss-Hopkins-Miller Theorem.)
(A, a commutative S–algebra) ↦ (K(A), the algebraic K–theory spectrum of A, a commutative S–algebra)

⇒ K(E_n), a commutative S–algebra

By the functoriality of K(−), $\mathbb{G}_n$ acts on $K(E_n)$ by maps of commutative S–algebras.

$L_{K(n)}(S^0)$, the Bousfield localization of the sphere spectrum with respect to $K(n)$, the $n$th Morava K–theory spectrum.

$\mathbb{G}_n$, a profinite group (more: a compact $p$–adic analytic group; finite v.c.d.)

The $K(n)$–local unit map of $K(n)$–local commut. S–algebras

$L_{K(n)}(S^0) \rightarrow E_n$

is a consistent profaithful $K(n)$–local profinite $\mathbb{G}_n$–Galois extension (due to Rognes, Behrens-D.).
$V_n$, a finite $p$–local complex of type $n + 1$

$v: \Sigma^d V_n \to V_n$, a $v_{n+1}$–self-map ($d$, some positive integer)

Thus, $v$ induces a sequence

$$V_n \to \Sigma^{-d} V_n \to \Sigma^{-2d} V_n \to \cdots$$

of maps of spectra.

We set

$$v_{n+1}^{-1} V_n = \colim_{j \geq 0} \Sigma^{-jd} V_n,$$

the colimit of the above sequence, the mapping telescope associated to the $v_{n+1}$–self-map $v$. 
Conjecture (Ausoni, Rognes)

The $\mathbb{G}_n$–Galois extension $L_{K(n)}(S^0) \to E_n$ induces a map

$$K(L_{K(n)}(S^0)) \wedge v_{n+1}^{-1} V_n \to (K(E_n))^{h\mathbb{G}_n} \wedge v_{n+1}^{-1} V_n$$

that is a weak equivalence, and associated with the target of this weak equivalence is a homotopy fixed point spectral sequence that has the form

$$E_2^{s,t} \Rightarrow (V_n)_{t-s}((K(E_n))^{h\mathbb{G}_n})[v_{n+1}^{-1}],$$

with

$$E_2^{s,t} = H_c^s(\mathbb{G}_n; (V_n)_t(K(E_n))[v_{n+1}^{-1}]).$$
Remark

We supplement the statement of the conjecture with the following comments:

- the $E_2$-term

$$E_2^{s,t} = H_*^c(\mathbb{G}_n; (V_n)_t(K(E_n))[v_n^{-1}])$$

of its spectral sequence is given by continuous cohomology;

- its object $(K(E_n))^{h\mathbb{G}_n}$ is a continuous homotopy fixed point spectrum; and

- the conjecture is actually just a piece of the family of conjectures made by Ausoni and Rognes – we only stated the part that we have been focusing on.
For every integer $t$, there is an isomorphism

$$(V_n)_t(K(E_n))[v_{n+1}^{-1}] \cong \pi_t(K(E_n) \wedge v_{n+1}^{-1} V_n).$$

When the conjectured spectral sequence

$$E^{s,t}_2 \Rightarrow (V_n)_{t-s}((K(E_n))^{hG_n})[v_{n+1}^{-1}],$$

exists, since it is for homotopy fixed points, there should also be an equivalence

$$(K(E_n))^{hG_n} \wedge v_{n+1}^{-1} V_n \cong (K(E_n) \wedge v_{n+1}^{-1} V_n)^{hG_n}.$$

Obtaining this equivalence and a homotopy fixed point spectral sequence

$$E^{s,t}_2 = H^s_c(G_n; \pi_t(K(E_n) \wedge v_{n+1}^{-1} V_n)) \Rightarrow \pi_{t-s}((K(E_n) \wedge v_{n+1}^{-1} V_n)^{hG_n})$$

immediately implies the existence of the conjectured spectral sequence.
To make progress on the conjecture, one obstacle that must be overcome is that there are no known constructions of the (continuous) homotopy fixed point spectra

\[(K(E_n))^{hG_n}, (K(E_n) \wedge v_{n+1}^{-1} V_n)^{hG_n}\]

for any \(n\) and \(p\).

Also, there are no known constructions of the two homotopy fixed point spectral sequences that we have referred to.
In this talk, we are reporting on progress on this conjecture for $n = 1$, $p \geq 5$, with

$$V_1 = V(1), \text{ the type 2 Smith-Toda complex } S^0/(p, \nu_1).$$

Thus, we have

$$E_1 = KU_p, \ p\text{-completed complex } K\text{-theory},$$

$$\mathbb{G}_1 = \mathbb{Z}_p^\times \cong \mathbb{Z}_p \times \mathbb{Z}/(p - 1).$$

**N.B.**

Henceforth, we will use the term “descent spectral sequence” in place of “homotopy fixed point spectral sequence.”
Theorem 1 (D.)

Let $p \geq 5$. Given any closed subgroup $K$ of $\mathbb{Z}_p^\times$, there is a strongly convergent descent spectral sequence

$$E_2^{s,t} \rightarrow \pi_{t-s}((K(KU_p) \wedge v_2^{-1} V(1))^{hK}),$$

where

$$E_2^{s,t} = H_c^s(K; \pi_t(K(KU_p) \wedge V(1))[v_2^{-1}]),$$

with $E_2^{s,t} = 0$, for all $s \geq 2$ and any $t \in \mathbb{Z}$. Also, there is an equivalence of spectra

$$(K(KU_p) \wedge v_2^{-1} V(1))^{hK} \simeq \text{colim}_{j \geq 0} (K(KU_p) \wedge \Sigma^{-j} V(1))^{hK}.$$
Remarks about the theorem

- Each subgroup $K$ is a profinite group.
- Both occurrences of $(-)^{hK}$ in the theorem are for the application of the (continuous) $K$–homotopy fixed points functor for the category of discrete $K$-spectra.
- The term “spectrum” means symmetric spectrum: the theorem, its proof, and the underlying theory are worked out in the setting of symmetric spectra of simplicial sets.
- Letting $K = \mathbb{Z}_p^\times$ gives some progress on the Ausoni-Rognes conjecture: for $n = 1$, $p \geq 5$, and $V_1 = V(1)$, we have obtained the desired descent spectral sequence
  \[
  H^s_c(G_n; \pi_t(K(E_n) \wedge \nu_{n+1}^{-1} V_n)) \Longrightarrow \pi_{t-s}( (K(E_n) \wedge \nu_{n+1}^{-1} V_n)^{hG_n} ).
  \]
- The construction of $K(KU_p)^{h\mathbb{Z}_p^\times}$ remains open.
Theorem 2 (D.)

For $p \geq 5$, there is a canonical map of symmetric spectra

$$\eta: K(L_{K(1)}(S^0)) \wedge v_2^{-1} V(1) \rightarrow (K(KU_p) \wedge v_2^{-1} V(1))^{h\mathbb{Z}_p^\times}.$$

Remark

It is easy to see that if $\eta$ is a weak equivalence and if

$$(K(KU_p) \wedge v_2^{-1} V(1))^{h\mathbb{Z}_p^\times} \cong K(KU_p)^{h\mathbb{Z}_p^\times} \wedge v_2^{-1} V(1),$$

then there would be an equivalence

$$K(L_{K(1)}(S^0)) \wedge v_2^{-1} V(1) \cong K(KU_p)^{h\mathbb{Z}_p^\times} \wedge v_2^{-1} V(1),$$

which would be close to proving part of the Ausoni-Rognes conjecture in the $n = 1, p \geq 5$ case.
For the next result ...

Let $G$ be a profinite group, with $\mathcal{N} = \{N_\alpha\}_{\alpha \in \Lambda}$ an inverse system of open normal subgroups of $G$ that satisfies

(a) the maps in the diagram $\mathcal{N}$ (indexed by the directed poset $\Lambda$) are given by inclusions (that is, $\alpha_1 \leq \alpha_2$ in $\Lambda$ if and only if $N_{\alpha_2}$ is a subgroup of $N_{\alpha_1}$), and
(b) the intersection $\bigcap_{\alpha \in \Lambda} N_\alpha$ is the trivial group $\{e\}$.

Also, let $X$ be a $G$–spectrum such that the $G$–module $\pi_t(X)$ is a discrete $G$–module, for every $t \in \mathbb{Z}$. 
Theorem (D.)

Let $G$ and $X$ be as on the previous slide. Suppose that the map

$$\lambda^s_{\pi_t(X)} : H^s_c(N_{\alpha}; \pi_t(X)) \to H^s(N_{\alpha}; \pi_t(X))$$

is an isomorphism for all $s \geq 0$, every integer $t$, and each $\alpha \in \Lambda$. If

- there exists a natural number $r$, such that for all integers $t$ and every $\alpha \in \Lambda$, $H^s_c(N_{\alpha}; \pi_t(X)) = 0$, for all $s > r$; or
- there exists some fixed integer $l$, such that $\pi_t(X) = 0$, for all $t > l$,

then there is a zigzag of $G$–equivariant maps

$$X \xrightarrow{\sim} \text{holim}_{\Delta} \text{Sets}(G^{\bullet+1}, X_f) \xleftarrow{\sim} X^\text{dis}_N$$

that are weak equivalences in $\text{Sp}^\Sigma$, with $X^\text{dis}_N \in \Sigma \text{Sp}_G$. 
For Theorem 1, we didn’t need the full power of the preceding theorem; we only needed a simpler version ...
Definition

A spectrum $X$ is an $f$–spectrum if $\pi_\ast(X)$ is degreewise finite.

Theorem (D.)

Let $p$ be any prime and let $H$ be any finite discrete group. If $X$ is a $(\mathbb{Z}_p \times H)$–spectrum and an $f$–spectrum, then there is a zigzag

$$X \xrightarrow{\sim} X' \xleftarrow{\sim} X_{\mathcal{N}}^\text{dis}$$

of $(\mathbb{Z}_p \times H)$–spectra and $(\mathbb{Z}_p \times H)$–equivariant maps that are weak equivalences of symmetric spectra, and $X_{\mathcal{N}}^\text{dis}$ is a discrete $(\mathbb{Z}_p \times H)$–spectrum.

$$X = X_{\mathcal{N}}^\text{dis} \implies X^{h(\mathbb{Z}_p \times H)} := (X_{\mathcal{N}}^\text{dis})^{h(\mathbb{Z}_p \times H)}$$
The construction of $X^\text{dis}_\mathcal{N}$ is elementary!:

$$X^\text{dis}_\mathcal{N}$$

is equal to

$$\text{colim} \, \text{holim} \left( \bigcup_{m \geq 0} \left( \text{Sets}(\mathbb{Z}_p \times H, \cdots, \text{Sets}(\mathbb{Z}_p \times H, X_f) \cdots) \right) \times \left( p^m \mathbb{Z}_p \right) \times \{e\}, \right)$$

where each $(p^m \mathbb{Z}_p) \times \{e\}$ is an (open normal) subgroup of $\mathbb{Z}_p \times H$ and $p^m \mathbb{Z}_p$ has its usual meaning.
Now we build on these tools for the situation of filtered colimits.
Let $G$ be any profinite group and $X$ any $G$–spectrum.

**Definition**

If $G$, $X$, and $\mathcal{N}$ (an inverse system of open normal subgroups of $G$) satisfy the hypotheses of either of the last two theorems, then we say that the triple $(G, X, \mathcal{N})$ is *suitably finite*.

**Definition**

Let $G$ be a profinite group with $\mathcal{N}$ a fixed inverse system of open normal subgroups of $G$, and let $\{X_\mu\}_\mu$ be a filtered diagram of $G$–spectra such that for each $\mu$, $(G, X_\mu, \mathcal{N})$ is a suitably finite triple and $X_\mu$ is a fibrant spectrum. We refer to $(G, \{X_\mu\}_\mu, \mathcal{N})$ as a *suitably filtered triple*.
Let \((G, \{X_\mu\}_\mu, \mathcal{N})\) be a suitably filtered triple. There is a zigzag of \(G\)-equivariant maps

\[
\colim_{\mu} X_\mu \xrightarrow{\simeq} \colim_{\mu} \operatorname{holim} \operatorname{Sets}(G^{\bullet+1}, (X_\mu)_f) \xleftarrow{\simeq} \colim_{\mu} (X_\mu)^{\text{dis}}_{\mathcal{N}}
\]

that are weak equivalences in \(\Sigma \text{Sp}\). The composition

\[
\colim_{\mu} \pi_t(X_\mu) \xrightarrow{\simeq} \pi_t(\colim_{\mu} (X_\mu)^{\text{dis}}_{\mathcal{N}}) \xrightarrow{\simeq} \colim_{\mu} \pi_t((X_\mu)^{\text{dis}}_{\mathcal{N}})
\]

consists of two isomorphisms in the category of discrete \(G\)-modules (in particular, each of the above abelian groups is a discrete \(G\)-module).
Definition

Given a suitably filtered triple \((G, \{X_{\mu}\}_{\mu}, \mathcal{N})\), we have seen that the \(G\)–spectrum \(\text{colim}_\mu X_{\mu}\) can be identified with the discrete \(G\)–spectrum \(\text{colim}_\mu (X_{\mu})^\text{dis}_\mathcal{N}\). Thus, it is natural to define

\[
(\text{colim}_\mu X_{\mu})^{hG} = (\text{colim}_\mu (X_{\mu})^\text{dis}_\mathcal{N})^{hG}.
\]

We can extend this definition to an arbitrary closed subgroup \(K\) in \(G\): since the \(K\)–spectrum \(\text{colim}_\mu X_{\mu}\) can be regarded as the discrete \(K\)–spectrum \(\text{colim}_\mu (X_{\mu})^\text{dis}_\mathcal{N}\), we define

\[
(\text{colim}_\mu X_{\mu})^{hK} = (\text{colim}_\mu (X_{\mu})^\text{dis}_\mathcal{N})^{hK}.
\]
We say that a profinite group $G$ has \textit{finite virtual cohomological dimension} ("finite v.c.d.") if $G$ contains an open subgroup that has finite c.d.

\textbf{Theorem (D.)}

Let $G$ be a profinite group with finite v.c.d. If $(G,\{X_{\mu}\}_{\mu},\mathcal{N})$ is a suitably filtered triple and $K$ is a closed subgroup of $G$, then there is a conditionally convergent descent spectral sequence $E_{r,*,*}(K)$ that has the form

$$E_2^{s,t}(K) = H_c^s(K; \pi_t(\colim_{\mu} X_{\mu})) \Rightarrow \pi_{t-s}(\colim_{\mu} X_{\mu})^{hK}.$$
An almost complete sketch of the proof of this theorem ...

Let \( U \) be an open subgroup of \( G \) that has finite c.d. Then \( U \cap K \) is an open subgroup of \( K \), and since \( U \) has finite c.d. and \( U \cap K \) is closed in \( U \), there exists some \( r \) such that for any discrete \((U \cap K)\)-module \( M \),

\[
H^s_c(U \cap K; M) \cong H^s_c(U; \text{Coind}_{U \cap K}^U(M)) = 0, \quad \text{whenever } s > r,
\]

by Shapiro’s Lemma. This shows that \( K \) has finite v.c.d.

Then, as a special case of a result due to [Behrens-D., D.], we obtain the conditionally convergent spectral sequence

\[
E^{s, t}_{2} = H^s_c(K; \pi_t(\text{colim}_\mu (X_\mu^\text{dis}_\mathcal{N}))) \Rightarrow \pi_{t-s} \left( (\text{colim}_\mu (X_\mu^\text{dis}_\mathcal{N}))^{hK} \right),
\]

and this is the desired spectral sequence.
A little more useful detail ...

Since $K$ has finite v.c.d.,

$$(\operatorname{colim}_\mu (X_\mu)_\mathcal{N})^{hK} \simeq \operatorname{holim}_\Delta \Gamma_K \operatorname{colim}_\mu (X_\mu)_\mathcal{N},$$

and for each $m \geq 0$, the $m$-cosimplices of the cosimplicial spectrum $\Gamma_K \operatorname{colim}_\mu (X_\mu)_\mathcal{N}$ satisfy the isomorphism

$$(\Gamma_K \operatorname{colim}_\mu (X_\mu)_\mathcal{N})^m \cong \operatorname{colim}_{\vee \triangleleft_o K^m/\vee} \operatorname{colim}_\mu (X_\mu)_\mathcal{N},$$

where $K^m$ is the $m$-fold Cartesian product of $K$ ($K^0$ is the trivial group $\{e\}$, equipped with the discrete topology).
Our spectral sequence is the homotopy spectral sequence for the spectrum

$$\text{holim}_\Delta \Gamma_K \text{colim}_\mu (X_\mu)^{\text{dis}}_{N_f}.$$ 

Based on [Behrens-D., D.], one might expect us to instead form the homotopy spectral sequence for

$$\text{holim}_\Delta \Gamma_K (\text{colim}_\mu (X_\mu)^{\text{dis}}_{N_f})_{fK}.$$ 

But since each $(X_\mu)^{\text{dis}}_{N_f}$ is a fibrant spectrum, $\text{colim}_\mu (X_\mu)^{\text{dis}}_{N_f}$ is already a fibrant spectrum, so that we do not need to apply $(-)_{fK}$ to it (so that we are taking the homotopy limit of a cosimplicial fibrant spectrum).

This completes our sketch-proof.
Theorem (D.)

Let $G$ be a profinite group with finite v.c.d., let $(G, \{X_\mu\}_\mu, \mathcal{N})$ be a suitably filtered triple such that $\{\mu\}_\mu$ is a directed poset, and let $K$ be a closed subgroup of $G$. If there exists a nonnegative integer $r$ such that for all $t \in \mathbb{Z}$ and each $\mu$, $H^s_c(K; \pi_t(X_\mu)) = 0$ whenever $s > r$, then descent spectral sequence $E_r^{*,*}(K)$ is strongly convergent and there is an equivalence of spectra

$$(\colim_\mu X_\mu)^{hK} \simeq \colim_\mu (X_\mu)^{hK}.$$
We give the complete proof of this result ... 

For all $t \in \mathbb{Z}$, when $s > r$, we have

$$E_2^{s,t}(K) = H_c^s(K; \pi_t(\text{colim } X_\mu)) \cong \text{colim } H_c^s(K; \pi_t(X_\mu)) = 0,$$

so that the spectral sequence is strongly convergent, by [Thomason’s Lemma 5.48].
If $V$ is an open normal subgroup of $K^m$, where $m \geq 0$, then $K^m/V$ is finite, and hence, an earlier isomorphism implies that

$$
\left( \Gamma^\bullet_K \colim_{\mu}(X_\mu)_{\mathcal{N}}^{\text{dis}} \right)^m \cong \colim_{\mu} \colim_{V \triangleleft_o K^m} \prod K^m/V (X_\mu)_{\mathcal{N}}^{\text{dis}}
$$

$$
\cong \colim_{\mu} (\Gamma^\bullet_K (X_\mu)_{\mathcal{N}}^{\text{dis}})^m,
$$

so that there is an isomorphism

$$
\Gamma^\bullet_K \colim_{\mu}(X_\mu)_{\mathcal{N}}^{\text{dis}} \cong \colim_{\mu} \Gamma^\bullet_K (X_\mu)_{\mathcal{N}}^{\text{dis}}
$$

of cosimplicial spectra.
Therefore, we have

\[(\text{colim}_\mu (X_\mu)^{\text{dis}})_{hK} \simeq \text{holim} \Gamma_K \text{colim}_\mu (X_\mu)^{\text{dis}} \simeq \text{holim} \text{colim}_\mu \Gamma_K (X_\mu)^{\text{dis}},\]

which gives

\[(\text{colim}_\mu X_\mu)^{hK} \simeq \text{holim} \text{colim}_\mu \Gamma_K (X_\mu)^{\text{dis}} \leftarrow \text{colim} \text{holim}_\mu \Gamma_K (X_\mu)^{\text{dis}} \simeq \text{colim}_\mu ((X_\mu)^{\text{dis}})^{hK} \simeq \text{colim}_\mu (X_\mu)^{hK},\]

and the canonical colim/holim exchange map above is a weak equivalence if there exists a nonnegative integer \(r\) such that for every \(t\) and all \(\mu\),

\[H^s[\pi_t(\Gamma_K^* (X_\mu)^{\text{dis}})] = 0, \text{ when } s > r.\]
The proof is completed by noting that there are isomorphisms

\[ H^s \left[ \pi_t \left( \Gamma_K^* (X_\mu)_{\text{dis}} \right) \right] \cong H_c^s(K; \pi_t((X_\mu)_{\text{dis}})) \cong H_c^s(K; \pi_t(X_\mu)), \]

for all \( s \geq 0 \).
Ausoni showed that $K(ku_p) \wedge V(1)$ is an $f$–spectrum.

$$
\implies
$$

$K(KU_p) \wedge V(1)$ is an $f$–spectrum.

Then our tools give

$$
K(KU_p) \wedge v_2^{-1} V(1) = \operatorname{colim}_{j \geq 0} \left( (K(KU_p) \wedge \Sigma^{-jd} V(1))_{N} \right)^{\operatorname{dis}} \in \Sigma \operatorname{Sp}_{\mathbb{Z}_p}^\times
$$

and

$$
(K(KU_p) \wedge v_2^{-1} V(1))^{hK} = \left( \operatorname{colim}_{j \geq 0} \left( (K(KU_p) \wedge \Sigma^{-jd} V(1))_{N} \right)^{\operatorname{dis}} \right)^{hK}.
$$
We have not constructed \((K(KU_p))^{h\mathbb{Z}_p^\times}\) for any \(p\). Nevertheless, we have a “draft result” related to this conjectural object ...

### A Recollection

If \(G\) is any profinite group and \(X\) is a (naive) \(G\)–spectrum, then \(G\) can be regarded as a discrete group and one can always form the “discrete homotopy fixed point spectrum”

\[ X^{\tilde{h}G} = \text{Map}_G(EG_+, X). \]

### “Draft theorem”

When \(p \geq 5\), there is an equivalence of spectra

\[ (K(KU_p) \wedge v_2^{-1} V(1))^{h\mathbb{Z}_p^\times} \simeq (K(KU_p))^{\tilde{h}\mathbb{Z}_p^\times} \wedge v_2^{-1} V(1). \]
Does this work shed any light on the Ausoni-Rognes Conjecture for higher $n$?

- For any $n$ and $p$: there exists $K \triangleleft_c \mathbb{G}_n$, with $\mathbb{G}_n/K \cong \mathbb{Z}_p$.
- I believe it is reasonable to think that there exists an equivalence

$$
(K(E_n) \wedge v_{n+1}^{-1} V_n)^{h\mathbb{G}_n} \cong (K(E_n) \wedge v_{n+1}^{-1} V_n)^{hK})^{h\mathbb{Z}_p}.
$$

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For the Ausoni-Rognes conjecture at $n = 1, p > 3$
Also, if

- $K(E_n) \wedge V_n$ can be shown to be an $f$-spectrum, and
- $(K(E_n) \wedge V_n)^{hK}$ can be constructed as an $f$-spectrum,

then I believe that the tools and techniques of this work will yield a construction of

$$\left((K(E_n) \wedge \nu_n^{-1} V_n)^{hK}\right)^{h\mathbb{Z}_p}.$$