Nonlinear Quantization of Phase Patterns in a Continuum as a Hybrid System

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Abstract. The centers of localized wave solutions (*solitons*) propagating over the surface of an elastic shell appear to behave as particles while the action at a distance between the centers of the solitons is a typical manifestation of the internal stresses in the continuum. The solitons are the solutions of an nonlinear eigen value problem for the flexural deformations of the middle surface of the shell. Since the spectrum is discrete a natural quantization of the model arises. We present here numerical results for the shapes of the solitons when the shell is subject to hydrostatic pressure.

INTRODUCTION

The apparent duality between the point particles and field(s) underlies the foundation of modern physics. Newton advanced the concept of point particles (corpuscles) moving freely in a void geometrical space which gave rise of a fruitful quantitative description of the motion of bodies known nowadays as *Newtonian Mechanics*. On a new level, the continuum description resurrected in the Hamiltonian-Schrödinger wave mechanics. From the far side de Broglie quantified the wave properties of the particles.

The new concept we propose here is to call particles the localized waves of the field which propagate *over* the field as phase patterns. Such individualized waves are well known from the soliton theory. The solitons preserve their shapes and energies upon collisions and can be called "quasi-particles".

We chose the elastic shell as a model for a unified field because of its closeness to the original Schroedinger's equation of wave mechanics. There are other candidates for the field possessing solutions of type of "quasi-particles", e.g., the *sine*-Gordon equation considered in [6] which is believed to model the meson field. Nowadays there are many "fields" and more are being introduced *ad hoc* for explaining one or another of the hundreds of elementary particles already known.

In order to distinguish the material unified field from the mechanical continuous media (bodies, liquids, gases, etc.), we refer to it as to *metacontinuum* in the sense that it is beyond – *meta* the observable phenomena and is their progenitor.

CURVATURE WAVES IN A METACONTINUUM AS QUASI-PARTICLES

A $N^{\mathcal{D}}$ hypersurface is a mathematical abstraction for the material construct known as thin shell (or a membrane). We consider an elastic shell of a $(N+1)^{\mathcal{D}}$ material whose shear Lamé coefficient is much smaller than the dilational one.

In the technological applications of shell theory an assumption is tacitly made, namely that the length scale L of the flexural deformations is large. Thus the so-called "shallow shells" are considered whose deflections are of unit order, the strains (gradients) are $O(L^{-1})$, and the curvatures are of second-order in smallness. Here we treat the case $L \gg h$ when the deflections are small, the strains (gradients) are of unit-order, and curvatures are large. The standard shell theory is not sufficient for describing such an object which is geometrically strongly nonlinear. A cubic dispersive equation was derived in [3, 4] for the flexural deformations of very thin but stiff elastic shell, i.e., the case when a shell is essentially not a membrane. In case of no traction, for the amplitude ζ of normal deflection in direction of $N+1$-st dimension we get

$$\mu_0 \frac{\partial^2 \zeta}{\partial t^2} = \mu_0 \mathcal{F} + D\left[-\Delta\Delta\zeta + (\Delta\zeta)^3\right] + \sigma_0 \Delta\zeta \,. \tag{1}$$

We render the last equation dimensionless by introducing the scales

$$\zeta = L\zeta', \quad \boldsymbol{x} = L\boldsymbol{x}', \quad t = \frac{L}{c_f} t', \quad c_f = \sqrt{\frac{\sigma_0}{\mu_0}} \equiv c\sqrt{|k_0|},, \tag{2}$$

where c_f has dimension of velocity. Note that the scale for ζ and the length scale of the localized wave coincide (the length L). This fits the original assumptions: small deflections of order L, unit strains and large curvatures of order L^{-1}. Thus

$$\frac{\partial^2 \zeta'}{\partial t^2} = \beta\left[-\Delta\Delta\zeta' + \left(\Delta\zeta'\right)^3\right] - \Delta\zeta', \tag{3}$$

where $\beta = D|\sigma_0|^{-1}L^{-2}$ is the dispersion parameter and $\mathcal{F}$ can be canceled. Eq. (3) is our "master equation of wave mechanics".

Some remarks are due concerning the "master wave equation" (3). Its linear part has the form originally proposed by Schrödinger (Eq. (4) from [7]) who also mentioned the analogy with the plate equation and injected a remark in the cited paper.

FLEXURAL LOCALIZED SOLUTIONS – THE FLEXONS

In the stationary case the localized waves possess spherical symmetry and for them the following dimensionless boundary value problem in infinite domain is posed

$$b - b^3 + \frac{1}{r^{N-1}} \frac{d}{dr} r^{N-1} \frac{d}{dr} b = 0, \quad b \to 0 \quad \text{for} \quad r \equiv |\mathbf{x}| \to \pm\infty \,, \tag{4}$$

where $b = \Delta\zeta$ is the curvature of the transverse deformation. The linearized version of Eq. (4) possesses along with the trivial solution a localized non-trivial one: the *sinc* function $\zeta = ar^{-1}\sin r$, where a is an arbitrary constant. The *sinc*-shape solution has been recently interpreted as a "single event" of the Schrödinger equation [1] rather than as an equation for the probability density of a particle. In this instance the present work lends support to [1] for interpreting (3) as a field equation.

We solve (4) by means of the Method of Variational Imbedding (MVI) proposed in [2] (see, also [5]). Because of the slow algebraic decay of the tails of the solution the interval has to be truncated at very large r. At the same time the grid has to be dense enough to allow resolving the oscillations. The results presented here employ grids with up to 40000 points and spacing 0.01. Fig. 1 presents the numerically obtained solutions.

The model under consideration is a conservative one, but to prove that our flexural localized waves live up to the definition of *soliton* it is necessary either to find analytically the two-soliton solution of (3) or to demonstrate the interactions numerically. Numerical solution of a time dependent 2D problem would require enormous computational resources. Until their solitonic properties are strictly established we will call the localized waves discovered here *flexons* which carries also a hint of their origin (flexural deformations or deflections).

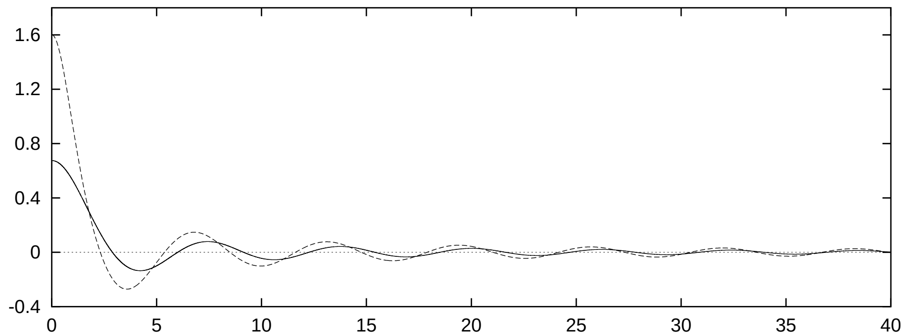

Figure 1: Flexon solutions with two different amplitudes.

The amplitude of a *flexon* can be interpreted as its *mass* when considered as s quasi-particle. Naturally, a *flexon* with negative amplitude will be an *anti-particle*. One sees in Fig. 1 that the amplitudes (and hence – the masses) of flexons can be quite different.

QUASI-NEWTONIAN DYNAMICS OF CENTERS OF LOCALIZED WAVES

Consider Eq. (3) in the domain D with boundary conditions for the wave-amplitude ζ

$$\zeta = \psi_0(\boldsymbol{x}), \quad \Delta\zeta = \psi_2(\boldsymbol{x}), \quad \text{for} \quad \boldsymbol{x} \in \partial D. \tag{5}$$

The trivial boundary conditions correspond to an isolated system of wave-particles. A Hamiltonian representation is readily derived upon multiplying (3) by $\Delta\zeta_t$ and integrating over the infinite domain.

$$H \equiv \frac{1}{2}\int_D \left[(\nabla\zeta_t)^2 - (\Delta\zeta)^2 + \frac{1}{2}\beta(\Delta\zeta)^4 + \beta(\nabla\Delta\zeta)^2\right] \mathrm{d}^N\boldsymbol{x}\,. \tag{6}$$

For boundary conditions that do not depend on time, the total energy H is conserved $dH/dt = 0$.

The wave mass and wave momentum are defined as

$$M \equiv \int_{-\infty}^{\infty} \Delta\zeta\, \mathrm{d}^N\boldsymbol{x}\,, \quad \boldsymbol{P} \equiv \int_{-\infty}^{\infty} \zeta_t \nabla\Delta\zeta\, \mathrm{d}^N\boldsymbol{x}. \tag{7}$$

For this model the meaning of M is the "mass of curvature" of the Riemann proposal. The *mass* is conserved if $\partial\zeta_{tt}/dn = 0$ at ∂D which is a natural requirement for an isolated system. For the wave momentum $\boldsymbol{P}$ we have

$$\frac{d\boldsymbol{P}}{dt} = \oint_{\partial D} \boldsymbol{n}\left[-\frac{1}{2}(\nabla\zeta_t)^2 - \frac{1}{2}(\Delta\zeta)^2 + \frac{\beta}{4}(\Delta\zeta)^4 + \frac{\beta}{2}(\nabla\Delta\zeta)^2\right]\mathrm{d}\boldsymbol{s} - \oint_{\partial D} \beta(\nabla\Delta\zeta)\frac{\partial\Delta\zeta}{\partial n}\mathrm{d}\boldsymbol{s} = \mathcal{F}_{\text{psu}}, \tag{8}$$

where $\mathcal{F}_{\text{psu}}$ is called *pseudoforce* and $\boldsymbol{n}$ is the outer normal to the region D. The wave momentum is conserved if the *pseudoforce* is equal to zero.

The main significance of the Hamiltonian formulation is that it provides the means to build up the dynamical model for the *discrete* phase objects (*solitons*). If the shape of localized wave is known, the Hamiltonian dynamics for the discrete system of centers of "particles" can be derived from (3) with a good approximation provided that they do not interact so strongly as to change appreciably their shapes. For two localized waves the wave amplitude can be decomposed as follows:

$$\zeta = F_1(\boldsymbol{x} - \boldsymbol{X}_1(t)) + F_2(\boldsymbol{x} - \boldsymbol{X}_2(t)) + F_{12}(\boldsymbol{x} - \boldsymbol{X}_1(t), \boldsymbol{x} - \boldsymbol{X}_2(t)). \tag{9}$$

Here F_i are the shape functions of the quasi-particles and $\boldsymbol{X}_i(t)$ are the trajectories of their centers. When the particles are far from each other F_{12} is negligible. Now the time derivative of each shape function can be expressed as follows

$$\frac{\partial \zeta}{\partial t} = -\nabla F_1 \cdot \frac{d\boldsymbol{X}_1}{dt} - \nabla F_2 \cdot \frac{d\boldsymbol{X}_2}{dt}, \tag{10}$$

and the term $(\nabla \zeta_t)^2$ yields in the discrete-system Hamiltonian a term that contains quadratic forms of the velocities of centers

$$\mathcal{A}_{12} \cdot \dot{\boldsymbol{X}}_1 \dot{\boldsymbol{X}}_2, \tag{11}$$

where the matrix $\mathcal{A}_{12}$ is positive definite and almost diagonal for well separated quasi-particles. Then (11) can be interpreted as definitive relation for the kinetic energy of the center of particle. Then the Euler-Lagrange equations for discrete system of centers of quasi-particles will contain the acceleration $\ddot{\boldsymbol{X}}$ which is Newton's law for inertia.

QUASI, PSEUDO, META: CONCLUDING REMARKS

From the point of view of soliton paradigm, the notion of discrete versus continuous (wave-particle dualism) is revisited. As a featuring example a very thin layer of Hookean elastic medium of very large dilational modulus is considered: a special kind of N-dimensional shell. A dispersive and nonlinear equation of Boussinesq type ("master" equation of the wave mechanics) is derived for the flexural deformations of the shell. Its linear part is the linearized Schrödinger equation.

We obtain the localized solution of the "master" equation (called *flexons*) numerically by means of Method of Variational Imbedding [5].

The Hamiltonian formulation of the wave-mechanics "master" equation (the *metadynamics*) define the Hamiltonian (Newtonian) dynamics of the centers of the phase objects (*quasi-particles*). Quasi-particles are not moving *through* the metacontinuum, rather they are *phase patterns* propagating *over* it without disturbing its contiguity.

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References

[1] A. O. Barut. Quantum theory of single events: Localized de broglie wavelets, schroedinger waves,and classical trajectories. *Foundations of Physics*, 20:1233–1240, 1990.

[2] C. I. Christov. A method for identification of homoclinic trajectories. In *Proc. 14-th Spring Conf, Sunny Beach*, pages 571–577, Sofia, Bulgaria, 1985. Union of Bulg. Mathematicians.

[3] C. I. Christov. A first-order in thickness model for flexural deformations of geometrically nonlinear membranes. *Ann. Univ. Sofia*, 89(libre 1 - Mecanique):129–140, 1995.

[4] C. I. Christov. Discrete out of continuous: Dynamics of phase patterns in continua. In K. Markov, editor, *Continuum Models and Discrete Systems – Proceedings of CMDS8*, pages 370–394, Singapore, 1996. World Scientific.

[5] C. I. Christov and M. G. Velarde. On localized solutions of an equation governing Bénard-Marangoni convection. *Appl. Math. Modelling*, 17:311–320, 1993.

[6] J. K. Perring and T. H. R. Skyrme. A model unified field equation. *Nuclear Physics*, 31:550–555, 1962.

[7] E. Schrödinger. *Annalen der Physik*, 79:743, 1926.