EMBEDDING GROUPS OF CLASS TWO AND PRIME EXPONENT IN CAPABLE AND NON-CAPABLE GROUPS

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Abstract. We show that if G is any p-group of class at most two and exponent p, then there exist groups G_1 and G_2 of class two and exponent p that contain G, neither of which can be expressed as a central product, and with G_1 capable and G_2 not capable. We provide upper bounds for $\mathrm{rank}({G_i}^{\mathrm{ab}})$ in terms of $\mathrm{rank}(G^{\mathrm{ab}})$ in each case.

1. Introduction.

A group is called *capable* if it is a central factor group. Capability plays an important role in P. Hall's scheme of classifying p-groups up to isoclinism [5], and has interesting connections to other branches of group theory. The finitely generated capable abelian groups were classified by Baer [1]. It is not too difficult to determine wether a given finitely generated group of class two is capable or not by finding its *epicentre* $Z^*(G)$, the smallest subgroup of G such that $G/Z^*(G)$ is capable. Computing the epicentre in this setting reduces to relatively straightfoward computations with finitely generated abelian groups; see for example [3, Thms. 4 and 7]. However, currently available techniques seem insufficient to give a classification of capable finitely generated groups of class two along the lines of Baer's result for the abelian case.

A full classification for the p-groups of class two and prime exponent seems a modest and possibly attainable goal; some purely numerical necessary conditions [6, Thm. 1] and sufficient ones [8, Thm. 5.26] are known, and a number of results allow us to reduce the problem to a restricted subclass. If we let G be a p-group of class at most two and odd prime exponent, it is not hard to show (for example, using [2, Prop. 6.2]) that if G is a nontrivial direct product, then G is capable if and only if each direct factor is either capable or nontrivial cyclic. We can write G as $G = K \times C_p^n$ where K is a group that satisfies $Z(K) = K'$ and C_p is cyclic of order p, and so G is capable if and only if K is nontrivial capable, or K is trivial and $n > 1$. If G can be decomposed as a nontrivial central product, $G = CD$ with $[D, C] = \{e\}$ and $\{e\} \neq [C, C] \cap [D, D]$ (where e is the identity of the group), then G is not capable [6, Prop. 1]. Ellis proved that if $\{x_1, \ldots, x_n\}$ is a transversal for $G/Z(G)$, and the nontrivial commutators $[x_j, x_i]$, $1 \leq i < j \leq n$ form a basis for $[G, G]$, then G is capable [3, Prop. 9]. We are then reduced to considering groups in a restricted class; we give it a name for future reference:

Definition 1.1. We will denote by $\mathcal{R}_p$ the class of all p-groups G of odd prime exponent p that cannot be decomposed into a nontrivial central product, with $Z(G) = [G, G]$, and such that if $\{x_1, \ldots, x_n\}$ is a transversal for $G/Z(G)$, then

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there is a nontrivial relation among the nontrivial commutators of the form $[x_j, x_i]$, $1 \leq i < j \leq n$.

Unfortunately, the situation appears to be far from straightforward once we reach this point. In particular, as the two main results in this paper show, to determine whether such a G is capable we need a "holistic" examination of G: there are no forbidden-subgroup criteria for the capability or non-capablity of G. Explicitly, we show that if G is any group of class at most two and odd prime exponent p, then G is contained in groups G_1 and G_2, both in $\mathcal{R}_p$, and with G_1 capable and G_2 not capable.

In Section 2 we prove some properties of the 2-nilpotent product with amalgamation, which is our main construction tool. Section 3 contains our main results.

2. The 2-nilpotent product with amalgamation.

Groups will be written multiplicatively, and we will use e to denote the identity element of G; C_p denotes the cyclic group of order p.

Definition 2.1. Let A and B be nilpotent groups of class at most 2. The *2-nilpotent product of A and B* is defined to be the group $F/[[F,F],F]$, where $F = A * B$ is the free product of A and B. We denote the 2-nilpotent product of A and B by $A \amalg^{\mathfrak{N}_2} B$.

The 2-nilpotent product was introduced by Golovin [4], with a more general definition that applies to any two groups A and B. If A and B are nilpotent of class at most two, then their 2-nilpotent product is their coproduct (in the sense of Category Theory) within the variety of all groups of class at most two, hence our choice of notation. The elements of $A \amalg^{\mathfrak{N}_2} B$ can be written uniquely as $\alpha\beta\gamma$, with $\alpha \in A$, $\beta \in B$, and $\gamma \in [B, A]$; multiplication in $A \amalg^{\mathfrak{N}_2} B$ is then given by:

$$(\alpha_1\beta_1\gamma_1)(\alpha_2\beta_2\gamma_2) = (\alpha_1\alpha_2)(\beta_1\beta_2)(\gamma_1\gamma_2[\beta_1, \alpha_2]).$$

A theorem of T. MacHenry [7] shows that $[B, A] \cong B^{\mathrm{ab}} \otimes A^{\mathrm{ab}}$ via $[b, a] \mapsto \overline{b} \otimes \overline{a}$ (where $\overline{x}$ denotes the image of x under the canonical maps $A \to A^{\mathrm{ab}}$ and $B \to B^{\mathrm{ab}}$, and $A^{\mathrm{ab}} \otimes B^{\mathrm{ab}}$ is the usual tensor product of abelian groups). Note that $A \amalg^{\mathfrak{N}_2} B$ contains isomorphic copies of A and B and is generated by these copies, so if A and B are both of odd exponent n, then so is $A \amalg^{\mathfrak{N}_2} B$.

Definition 2.2. Let A and B be nilpotent groups of class at most 2, let $H \leq [A, A]$, $K \leq [B, B]$, and let $\varphi\colon H \to K$ be an isomorphism. The *amalgamated coproduct of A and B along φ* is defined to be the group

$$A \amalg^{\mathfrak{N}_2}_{\varphi} B = \frac{A \amalg^{\mathfrak{N}_2} B}{\{h\varphi(h)^{-1} \mid h \in H\}}.$$

It is again easy to verify that $A \amalg^{\mathfrak{N}_2}_{\varphi} B$ contains isomorphic copies of A and B whose intersection is exactly the image of H identified with the image of K as indicated by φ.

Recall that a group G is said to be a *central product* of subgroups C and D if and only if $G = CD$ and $[C, D] = \{e\}$. The direct product is thus a special case of the central product. If in addition we have that $C \cap D = Z(G)$ then we say that G is the *full central product* of C and D. The central product is said to be trivial if $C \subseteq D$ or $D \subseteq C$. We prove that if A and B are both nontrivial, then $A \amalg^{\mathfrak{N}_2}_{\varphi} B$ cannot be decomposed as a nontrivial central product.

Theorem 2.3. *Let A and B be nontrivial groups of class at most 2 and odd prime exponent p, let $H \leq [A,A]$, $K \leq [B,B]$, and let $\varphi\colon H \to K$ be an isomorphism. If $G = A \amalg_{\varphi}^{\mathfrak{N}_2} B$, then $Z(G) = [G,G]$ and G cannot be expressed as a nontrivial central product.*

Proof. Identify A and B with their images in G. Let $\{a_1,\ldots,a_m\}$ be a transversal for $A/[A,A]$, and $\{b_1,\ldots,b_n\}$ a transversal for $B/[B,B]$. Since A is of exponent p, every element of A can be written uniquely in the form $a_1^{r_1}\cdots a_m^{r_m}a'$ with $0 \leq r_i < p$ and $a' \in [A,A]$, and similarly for every element of B. From the construction of the amalgamated coproduct it follows that $\{a_1,\ldots,a_m,b_1,\ldots,b_n\}$ is a transversal for $G/[G,G]$, and that $[G,G] \cong [A,A][B,B] \times [B,A]$. In particular, the commutators $[b_j,a_i]$, $1 \leq i \leq m$, $1 \leq j \leq n$ form a linearly independent subset of $[G,G]$ (viewing the latter as an $\mathbb{F}_p$-vector space).

Let $g \in G$; then we may write $g = a_1^{r_1}\cdots a_m^{r_m}b_1^{s_1}\cdots b_n^{s_n}g'$, with $0 \leq r_i, s_j < p$ and $g' \in [G,G]$. We first assert that $g \in Z(G)$ if and only if $r_i = s_j = 0$ for all i and j. Indeed, we have:

$$[g,a_i] = \prod_{j=1}^{m}[a_j,a_i]^{r_j} \cdot \prod_{j=1}^{m}[b_j,a_i]^{s_j}.$$

For this to be trivial, we must have $s_j = 0$ for each j. Symmetrically, computing $[g,b_j]$ we obtain that $r_i = 0$ for each i. Thus, $Z(G) = [G,G]$, as claimed.

Now suppose that G is decomposed as a central product, $G = CD$. We can express $a_1 = c_1d_1$ for some $c_1 \in C$, $d_1 \in D$. Let

$$\begin{aligned} c_1 &= a_1^{r_1}\cdots a_m^{r_m}b_1^{s_1}\cdots b_m^{s_m}c',\\ d_1 &= a_1^{\rho_1}\cdots a_m^{\rho_m}b_1^{\sigma_1}\cdots b_m^{\sigma_m}d', \end{aligned}$$

with $c',d' \in [G,G]$; since $c_1d_1 = a_1$, we have $r_1 + \rho_1 \equiv 1 \pmod{p}$, and $r_i + \rho_i \equiv s_j + \sigma_j \equiv 0 \pmod{p}$ for $2 \leq i \leq m$, $1 \leq j \leq n$. Since $[d_1,c_1] = e$, and

$$[d_1,c_1] = a''b''\prod_{i=1}^{m}\prod_{j=1}^{n}[b_j,a_i]^{\sigma_j r_i - s_j\rho_i}, \qquad a'' \in [A,A],\ d'' \in [B,B],$$

we have $0 \equiv \sigma_j r_1 - s_j\rho_1 \equiv \sigma_j r_1 + \sigma_j(1-r_1) \equiv \sigma_j \equiv -s_j \pmod{p}$. That is, $c_1, d_1 \in A[G,G]$. Since $a_1 \notin Z(G)$, either $c_1 \notin Z(G)$ or $d_1 \notin Z(G)$. If both hold, then we have $C \subseteq C_G(d_1) \subseteq A[G,G]$ and $D \subseteq C_G(c_1) \subseteq A[G,G]$, so we conclude that $G = CD \subseteq A[G,G]$, which is impossible. Thus, exactly one of c_1 and d_1 is noncentral. Without loss of generality say $c_1 \notin Z(G)$ and $d_1 \in Z(G)$, so $a_1 = c_1d_1 \in CZ(G)$.

Since none of $b_1,\ldots,b_n$ commute with a_1, and $a_1 \in CZ(G)$, we must have $b_1,\ldots,b_n \in CZ(G)$ as well; that is, $B \subseteq C[G,G]$. And since $b_1 \in C[G,G]$ and none of $a_1,\ldots,a_m$ commute with b_1, we must also have $a_1,\ldots,a_m \in C[G,G]$. Thus, $G = C[G,G]$, and so $[G,G] = [C,C]$ and $D \subseteq C$. Hence the central product decomposition $G = CD$ is trivial, as claimed. □

We finish this section by describing the epicentre of an amalgamated coproduct.

Theorem 2.4. *Let A and B be nontrivial groups of class at most 2 and odd prime exponent p, let $H \leq [A,A]$, $K \leq [B,B]$, and let $\varphi\colon H \to K$ be an isomorphism. If $G = A \amalg_{\varphi}^{\mathfrak{N}_2} B$, then $Z^*(G) = \{h \in H \mid h \in Z^*(A) \text{ and } \varphi(h) \in Z^*(B)\}$; that is, if we identify A and B with their images in G, then $Z^*(G) = Z^*(A) \cap Z^*(B)$.*

Proof. Let $a_1,\ldots,a_m$ be a transversal for $A/[A,A]$, and $b_1,\ldots,b_n$ a transversal for $B/[B,B]$. Following Ellis, for $x,y,z\in\{a_i,b_j\,|\,1\leq i\leq m, 1\leq j\leq n\}$ let

$$J(x,y,z) = [x,y]\otimes\overline{z} + [y,z]\otimes\overline{x} + [z,x]\otimes\overline{y},$$

and let S be the subgroup of $[G,G]\otimes G^{\rm ab}$ generated by all such $J(x,y,z)$. By [3, Theorem 7], an element $g\in Z(G)=[G,G]$ lies in $Z^*(G)$ if and only if $g\otimes w\in S$ for all $w\in\{a_i,b_j\}$. Since $[G,G]\cong [A,A][B,B]\times(B^{\rm ab}\otimes A^{\rm ab})$, $G^{\rm ab}\cong A^{\rm ab}\times B^{\rm ab}$, and all factors are elementary abelian p-groups, we have that $[G,G]\otimes G^{\rm ab}$ is isomorphic to

$$\Bigl(C\otimes (A^{\rm ab}\times B^{\rm ab})\Bigr)\oplus\left(\bigoplus_{i,k=1}^{m}\bigoplus_{j=1}^{n}\langle [b_j,a_i]\otimes\overline{a_k}\rangle\right)\oplus\left(\bigoplus_{j,k=1}^{n}\bigoplus_{i=1}^{m}\langle [b_j,a_i]\otimes\overline{b_k}\rangle\right),$$

where C is the central product of $[A,A]$ and $[B,B]$ obtained by identifying H with K along φ. An easy computation shows that none of the elements $J(x,y,z)$ has a nontrivial $[b_j,a_i]\otimes\overline{b_j}$ component. Thus, if $g\in[G,G]$ has a nontrivial $[B,A]$ component, say $[b_j,a_i]$, then it follows that $g\otimes\overline{b_j}$ does not lie in S, so g is not in $Z^*(G)$. Thus, $Z^*(G)\subseteq [A,A][B,B]$.

Consider the elements $J(x,y,z)$ in which at least one of x, y, or z is equal to b_1. Unless the other two are in B, the generators include nontrivial $[B,A]\otimes(B^{\rm ab}\times A^{\rm ab})$ components that do not occur in any other generator, and occur in pairs. It is straightforward then that if $g\in[G,G]$ lies in $Z^*(G)$, since $g\otimes\overline{b_1}$ must lie in S we have that g can be expressed in terms of commutators of $b_1,\ldots,b_m$; that is, $g\in[B,B]$. By a symmetric argument considering a_1 instead, we obtain that if $g\in Z^*(G)$ then $g\in[A,A]$. Thus, $Z^*(G)$ is contained in $[A,A]\cap[B,B]$; recall that this intersection is equal to the identified subgroups $H=K$.

If $g\in H$ lies in $Z^*(G)$, then $g\otimes\overline{a_i}\in S$ for each i, and this readily yields that for all $a\in A$, $g\otimes\overline{a}$ lies in the subgroup of $[A,A]\otimes A^{\rm ab}$ generated by all $J(a_i,a_j,a_k)$; thus $g\in Z^*(A)$; symmetrically, since $g\otimes\overline{b_j}\in S$ we obtain that g (considered now as an element of K) lies in $Z^*(B)$, so $Z^*(G)\subseteq Z^*(A)\cap Z^*(B)$. Conversely, if $h\in H\cap Z^*(A)$ is such that $\varphi(h)\in Z^*(B)$, then $h\otimes\overline{a_i}\in\langle J(a_r,a_s,a_t)\rangle$ for all i and $\varphi(h)\otimes\overline{b_j}\in\langle J(b_r,b_s,b_t)\rangle$ for all j, hence $h=\varphi(h)\in Z^*(G)$, giving the desired equality. □

3. Main results.

We now give the promised results.

Theorem 3.1. *Let G be any nontrivial group of class at most 2 and odd prime exponent p. Then there exists a capable group $G_1\in\mathcal{R}_p$ that contains G. If G is nonabelian and capable, then we may choose G_1 so that* $\mathrm{rank}(G_1^{\rm ab})\leq\mathrm{rank}(G^{\rm ab})+2$. *Otherwise, we may choose G_1 such that* $\mathrm{rank}(G_1^{\rm ab})\leq\mathrm{rank}(G^{\rm ab})+3$.

Proof. We construct G_1 in two steps. If G is nonabelian and capable, set $G_0=G$; otherwise, let $G_0 = G\amalg^{\mathfrak{N}_2} C_p$. To obtain G_1, let $H=C_p\amalg^{\mathfrak{N}_2} C_p$, and let φ be an isomorphism between $[H,H]$ and a nontrivial cyclic subgroup of $[G_0,G_0]$. Finally, let $G_1 = G_0\amalg_{\varphi}^{\mathfrak{N}_2} H$. Since G_0 is capable, G_1 is capable; by Theorem 2.3 G_1 is not a nontrivial central product. The identification of the generator of $[H,H]$ with a nontrivial element of $[G_0,G_0]$ guarantees the existence of a nontrivial relation among nontrivial commutators of any transversal, hence $G_1\in\mathcal{R}_p$, as desired. The rank inequality is immediate. □

Theorem 3.2. *Let G be any nontrivial group of class at most 2 and exponent p. Then there exists a non-capable group $G_2 \in \mathcal{R}_p$ that contains G. If G is nonabelian, then we may choose G_2 such that* $\mathrm{rank}(G_2^{\mathrm{ab}}) \leq \mathrm{rank}(G^{\mathrm{ab}}) + 6$*. If G is abelian, then we may choose G_2 with* $\mathrm{rank}(G_2^{\mathrm{ab}}) \leq \mathrm{rank}(G^{\mathrm{ab}}) + 7$.

Proof. If G is nonabelian, let $H_1 = C_p \amalg^{\mathfrak{N}_2} C_p$, let $g \in [G,G]$ be nontrivial, and let H be the central product of G and H_1 identifying g with a generator of $[H_1, H_1]$. Since this is a nontrivial central product with $[G,G] \cap [H_1,H_1] \neq \{e\}$, it is not capable and $g \in Z^*(H)$ by [6, Prop. 1]. Now let $G_2 = H \amalg_{\varphi}^{\mathfrak{N}_2} E$, where E is an extraspecial group of order p^5 and exponent p, and φ identifies g with a generator of $[E,E]$ (which is in $Z^*(E)$); since this is an amalgamated coproduct that identifies elements of the epicentres, Theorems 2.3 and 2.4 yield that G_2 is not capable and lies in $\mathcal{R}_p$ (the theorem of Ellis mentioned in the introduction guarantees the existence of a nontrivial relation among nontrivial commutators of any transversal).

If G is abelian, then let H be the central product of $G \amalg^{\mathfrak{N}_2} C_p$ with $C_p \amalg^{\mathfrak{N}_2} C_p$ identifying a generator of $C_p \amalg^{\mathfrak{N}_2} C_p$ with a nontrivial commutator in $G \amalg^{\mathfrak{N}_2} C_p$; this is a non-capable group. We now let $G_2 = H \amalg_{\varphi}^{\mathfrak{N}_2} E$ where E is again the extraspecial group of order p^5 and exponent p, and φ identifies elements of the epicentres. Again, G_2 is not capable and lies in $\mathcal{R}_p$. The rank inequalities are immediate. □

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