Math 566 - Homework 7
Due Wednesday April 3, 2024

1. Let R be a ring with unity and let $I \lhd R$ be an ideal of R. Prove that $I[x]$ is an ideal of $R[x]$ and $R[x]/I[x] \cong (R/I)[x]$.

2. Let R be the ring of 2×2 matrices with coefficients in $\mathbb{Z}$
   (i) Show that for all $A \in R$, $(x+A)(x-A) = x^2 - A^2$ holds in $R[x]$.
   (ii) Show that there are matrices A and C in R such that
   $$(C+A)(C-A) \neq C^2 - A^2.$$

3. Let R be a commutative ring. Given $p \in R[x]$,
   $$p = a_0 + a_1 x + \cdots + a_n x^n,$$
   we have a function $\mathbf{p}\colon R \to R$ given by
   $$\mathbf{p}(r) = a_0 + a_1 r + \cdots + a_n r^n.$$
   The assignment $p \mapsto \mathbf{p}$ defines a ring homomorphism $\varphi\colon R[x] \to R^R$, the ring of all functions from R to itself with pointwise addition and product. (You may take this for granted).

   Show that if R is finite and nonzero, then φ is not one-to-one.

4. Let D be an integral domain. Show that the morphism $\varphi\colon D[x] \to D^D$ from the previous problem is one-to-one if and only if D is infinite.

5. Show that if F is a field, then (x) is a maximal ideal of $F[x]$, but it is not the only maximal ideal of F.

6. Let D be an integral domain, and let $c \in D$ be an irreducible element. Prove that the ideal (x, c) of $D[x]$ is not principal.

7. Let R be a commutative ring with unity, and let
   $$f(x) = a_0 + a_1 x + \cdots + a_n x^n \in R[x].$$
   Define the *formal derivative* of $f(x)$ by
   $$f'(x) = a_1 + 2a_2 x + \cdots + n a_n x^{n-1}.$$
   with $f'(x) = 0$ if $f = 0$.
   (i) Prove that $(f+g)' = f' + g'$, $(af)' = af'$, and $(fg)' = f'g + fg'$ for all $f, g \in R[x]$ and all $a \in R$.
   (ii) Prove that if R is an integral domain, $\deg(f) > 0$, and $\mathrm{char}(R) = 0$, then $f' \neq 0$.
   (iii) Prove that if R is an integral domain, $\deg(f) > 0$, and $\mathrm{char}(R) = p$, then $f' = 0$ if and only if f is a polynomial in x^p. That is,
   $$f = a_0 + a_p x^p + a_{2p} x^{2p} + \cdots + a_{mp} x^{mp}, \qquad a_i \in R.$$