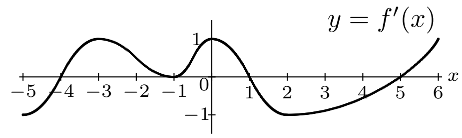
MATH 270-005 — SPRING 2024

CHAPTER 4 TEST

Solutions

Prof Arturo Magidin

1. The following sketch represents the graph of **the derivative** $f'(x)$ of a function $f(x)$ that is defined and continuous at every point on $[-5, 6]$. Use this graph to answer the following questions about the function $f(x)$. Note: the questions are **not** about the function whose graph you are seeeing.



   (a) On what intervals is $f(x)$ decreasing? *(2 points)*

   Answer. $f(x)$ is decreasing on the intervals in which the derivative is negative, with endpoints included. So for this function, it will be on $[-5, -4]$ and on $[1, 5]$.

   (b) On what intervals is $f(x)$ concave down? *(2 points)*

   Answer. $f(x)$ is concave down on intervals where the derivative is decreasing, not including endpoints. For this function, that happens on $(-3, -1)$, and on $(0, 2)$.

   (c) At which points does $f(x)$ have points of inflection? *(2 points)*

   Answer. $f(x)$ has points of inflection at points where $f'(x)$ changes directions; so the points where $f'(x)$ has local extremes. That happens at $x = -3$, $x = -1$, $x = 0$, and $x = 2$.

   (d) What are the critical points of $f(x)$? *(4 points)*

   Answer. the critical points of $f(x)$ are the points where $f'(x)$ is either undefined or 0.

   From the graph, $f'(x)$ is always defined, and it is equal to 0 at $x = -4$, $x = -1$, $x = 1$, and $x = 5$.

   (e) For each critical point, determine whether $f(x)$ has a local maximum, a local minimum, or neither, at that point. Justify your answer using either the first or second derivative test. *(4 points)*

   Answer. Using the First Derivative Test, $f(x)$ has a local minimum at the critical point c if $f'(x)$ changes from negative to positive at c. This happens at $x = -4$ and at $x = 5$.

   Also from the First Derivative Test, $f(x)$ has a local maximum at the critical point c if $f'(x)$ changes from positive to negative at c; this happens at $x = 1$.

   Finally, if $f'(x)$ is positive on both sides of the critical point c, or negative on both sides of c, then f does not have a local extreme at c. This happens for $x = -1$.

2. At the annual basketball game between the Springfild University Atoms and the Shelbyville Technical Institute Tireburners, the Atoms' cheerleaders discover that the amount of time they spend singing their fight song, "Springfield Enbiggens the Score," has a direct impact on the number of points their team scores.

   The cheerleaders discover that if they sing for x minutes, then the Atoms will score

$$A(x) = \frac{1}{2}x^2 - 4x + 63 \text{ points.}$$

   The cheerleaders must sing for at least two minutes, and can sing for a maximum of 10 minutes.

   How long should they sing if they want to maximize the number of points scored by the Atoms?

Your answer needs to be justified through the use of Calculus. It is not enough to graph the function on your calculator and use the graph. Show all your work. *(10 points)*

Answer. We want to find the maximum of $A(x)$ on the inverval $[2, 10]$. This is a continuous function on a finite closed interval, so we can find $A'(x)$, determine the critical points, and then compare the value of A at the critical points and the endpoints. Largest value is the maximum.

We have that $A'(x) = x - 4$. So the only critical point is $x = 4$. Now we evaluate $A(x)$ at $x = 2$, $x = 4$, and $x = 10$ to figure out the maximum:

$$\begin{aligned} A(2) &= 2 - 8 + 63 = 57; \\ A(4) &= 8 - 16 + 63 = 55; \\ A(10) &= 50 - 40 + 63 = 73. \end{aligned}$$

So the maximum is 73, achieved at $x = 10$. Thus, in order maximize the number of points scored by the Atoms, their cheerleaders should sing for the full 10 minutes.

3. Use the first and second derivatives of the function

$$f(x) = x^4 - 6x^2$$

to determine all intervals in which $f(x)$ is both **increasing** and **concave up**; that is, the intervals in which both things happen together. NOTE: You **cannot** use a graphing calculator as justification for this question, you must use the first and second derivative. *(8 points)*

Answer. We find the first and second derivatives, and find the places where they are 0 (which is where they can switch signs; note $f'(x)$ and $f''(x)$ will be polynomials, so they are continuous everywhere):

$$\begin{aligned} f(x) &= x^4 - 6x^2 \\ f'(x) &= 4x^3 - 12x = 4x(x^2 - 3) \\ &= 4x\left(x + \sqrt{3}\right)\left(x - \sqrt{3}\right) \\ f''(x) &= 12x^2 - 12 = 12(x^2 - 1) \\ &= 12(x + 1)(x - 1). \end{aligned}$$

So the first derivative can change signs at $-\sqrt{3}$, 0, and $\sqrt{3}$; the second derivative can change signs at $x = -1$ and at $x = 1$. We do a sign table:

	$(-\infty, -\sqrt{3})$	$(-\sqrt{3}, -1)$	$(-1, 0)$	$(0, 1)$	$(1, \sqrt{3})$	$(\sqrt{3}, \infty)$
x	$-$	$-$	$-$	$+$	$+$	$+$
$x + \sqrt{3}$	$-$	$+$	$+$	$+$	$+$	$+$
$x - \sqrt{3}$	$-$	$-$	$-$	$-$	$-$	$+$
$f'(x)$	$-$	$+$	$+$	$-$	$-$	$+$
$x + 1$	$-$	$-$	$+$	$+$	$+$	$+$
$x - 1$	$-$	$-$	$-$	$-$	$+$	$+$
$f''(x)$	$+$	$+$	$-$	$-$	$+$	$+$

The function is both increasing and concave up when we have both $f'(x) > 0$ and $f''(x) > 0$. From the bale, we see that this happens on $(-\sqrt{3}, -1)$ and on $(\sqrt{3}, \infty)$.

4. Compute the following limits. You may use L'Hopital's Rule when appropriate. If you use L'Hopital's Rule, indicate this by writing LH on the equal sign at the step where you apply the rule. *(3 points each, 9 points total)*

(a) $\lim_{x\to 0} \frac{e^x - 1}{3x^2 + 5x}$
Answer. If we plug in $x = 0$, we get a $\frac{0}{0}$ indeterminate, so we can try to Use L'Hopital's Rule:

$$\lim_{x\to 0} \frac{e^x - 1}{3x^2 + 5x} \stackrel{\text{LH}}{=} \lim_{x\to 0} \frac{(e^x - 1)'}{(3x^2 + 5x)'} = \lim_{x\to 0^+} \frac{e^x}{6x + 5} = \frac{1}{5}.$$

So the limit equals $\frac{1}{5}$.

(b) $\lim_{x\to 3} \frac{x^2 - 7x + 12}{3x - 3}$
Answer. If we try plugging in, the denominator is **not** equal to 0, so we can just do that:

$$\lim_{x\to 3} \frac{x^2 - 7x + 12}{3x - 3} = \frac{9 - 21 + 12}{6 - 3} = \frac{0}{3} = 0.$$

(c) $\lim_{x\to 0^+} x \ln x$
Answer. This is a $0 \times \infty$ indeterminate, so we first need to rewrite it as a fraction before using L'Hopital's Rule. We have:

$$\lim_{x\to 0^+} \frac{\ln x}{x} = \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x}} \stackrel{\text{LH}}{=} \lim_{x\to 0^+} \frac{(\ln x)'}{\left(\frac{1}{x}\right)'} = \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}}$$
$$= \lim_{x\to 0^+} -\frac{x^2}{x} = \lim_{x\to 0^+} -\frac{x}{1} = 0.$$

5. We want to use a local linear approximation to estimate the value of

$$\ln(1.05)$$

(i) Write the function $f(x)$ that you will use to approximate $\ln(1.05)$. *(1 point)*
Answer. Although there are many possibilities, the simplest one is $f(x) = \ln(x)$.

(ii) Write the value of a (the "easy point") you will use for the approximation. *(1 point)*
Answer. Given $f(x)$, it makes sense to let $a = 1$.

(iii) Write the exact values of $f(a)$ and $f'(a)$. *(2 points)*
Answer. We have $f(x) = \ln(x)$, and so $f'(x) = \frac{1}{x}$. Therefore, $f(1) = \ln(1) = 0$, and $f'(1) = \frac{1}{1} = 1$.

(iv) Using the function $f(x)$ from (i) and the values of a, $f(a)$, and $f'(a)$ from (ii) and (iii), give value you get as an approximation of $\ln(1.05)$. *(3 points)*
Answer. The formula is:

$$f(x) \approx f(a) + f'(a)(x - a) \text{ for } x \text{ near } a;$$

so we have:

$$\ln(x) \approx 0 + 1(x - 1) = (x - 1)\text{for } 1 \text{ near } a.$$

Therefore,

$$\ln(1.05) \approx 0 + 1(1.05 - 1) = 0.05.$$

(v) Use $f''(a)$ to determine if your approximation in (iv) is an overestimate or an underestimate. *(2 points)*
Answer. Since $f''(x) = -\frac{1}{x^2}$, we have that $f''(1) < 0$. So $y = f(x)$ is concave down, hence the approximation is an OVERESTIMATE.