MATH 270-005 — SPRING 2024

CHAPTER 3 TEST

SOLUTIONS

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1. Find $f^{(4)}$, the **fourth derivative** of $f(x)$, where

$$f(x) = x^4 - 2e^x + x^{-1} + \sin(x) + \pi^6.$$

(6 points)

Answer. Note that π^6 is a constant. We have:

$$\begin{aligned}
f(x) &= x^4 - 2e^x + x^{-1} + \sin(x) + \pi^6\\
f'(x) &= 4x^3 - 2e^x - x^{-2} + \cos(x)\\
f''(x) &= 12x^2 - 2e^x + 2x^{-3} - \sin(x)\\
f'''(x) &= 24x - 2e^x - 6x^{-4} - \cos(x)\\
f^{(4)}(x) &= 24 - 2e^x + 24x^{-5} + \sin(x).
\end{aligned}$$

2. This page contains two functions; the next page contains three. For each of them, find the derivative $f'(x)$. Your final answer should contain no derivative left indicates, no complex fractions (fractions of fractions), and no easy simplifications left (no expressions like $2+3$ or $3x-6x$), but you do not need to write it in simplest possible

(3 points each, 15 points total)

(i) $f(x) = \arctan(x)$

Answer. This is one of our basic functions. We have $f'(x) = \dfrac{1}{1+x^2}$.

(ii) $f(x) = e^{\cos x}$

Answer. This is a Chain Rule. We have:

$$f'(x) = e^{\cos x}(\cos x)' = e^{\cos x}(-\sin x) = -\sin(x)e^{\cos x}.$$

(iii) $f(x) = x\ln(x)$

Answer. This is a Product Rule:

$$f'(x) = (x)'\ln(x) + x\,(\ln x)' = \ln(x) + x\left(\frac{1}{x}\right) = \ln(x) + 1.$$

(iv) $f(x) = \dfrac{x}{x^2-1}$

Answer. This is a Quotient Rule:

$$f'(x) = \frac{(x^2-1)(x)' - x(x^2-1)'}{(x^2-1)^2} = \frac{(x^2-1) - x(2x)}{(x^2-1)^2} = \frac{x^2-1-2x^2}{(x^2-1)^2} = \frac{-1-x^2}{(x^2-1)^2} = -\frac{1+x^2}{(x^2-1)^2}.$$

(v) $f(x) = \dfrac{e^x + e^{-x}}{2}$

Answer. Note that we can write $f(x) = \frac{1}{2}(e^x + e^{-x})$. So

$$f'(x) = \left(\frac{e^x+e^{-x}}{2}\right)' = \frac{1}{2}(e^x + e^{-x})' = \frac{1}{2}(e^x + e^{-x}(-x)') = \frac{1}{2}(e^x - e^{-x}) = \frac{e^x - e^{-x}}{2}.$$

3. Use implicit differentiation to find the equation of the tangent line to the curve defined implicitly by

$$x + y^2 + xy = 3$$

at the point $(1, 1)$. Write the equation of the tangent line in the form $y = mx + b$. *(10 points)*

Do not attempt to solve for y first; you must use implicit differentiation.

Answer. Note that $(1, 1)$ satisfies the equation, as both sides evaluate to 3. Now we have

$$\begin{aligned} \frac{d}{dx}(x + y^2 + xy) &= \frac{d}{dx}(3) \\ 1 + 2yy' + (x)'y + x(y)' &= 0 \\ 1 + 2yy' + y + xy' &= 0 \\ 2yy' + xy' &= -1 - y \\ (2y + x)y' &= -(1 + y) \\ y' &= -\frac{1 + y}{2y + x}. \end{aligned}$$

So at $(1, 1)$, we have $y' = -\frac{1+1}{2+1} = -\frac{2}{3}$.

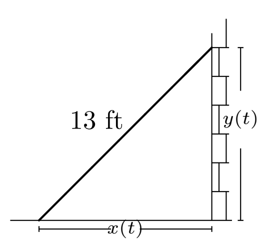
(Alternatively, you can plug in $x = y = 1$ in the third line to get $1 + 2y' + 1 + y' = 0$, so $3y' = -2$, and $y' = -\frac{2}{3}$).

Now we have the tangent equation:

$$\begin{aligned} y - 1 &= -\frac{2}{3}(x - 1) \\ y &= -\frac{2}{3}x + \frac{2}{3} + 1 \\ y &= -\frac{2}{3}x + \frac{5}{3}. \end{aligned}$$

4. A 13 foot ladder is leaning against a vertical wall. The foot of the ladder starts slipping away from the wall at a rate of $\frac{1}{2}$ ft/sec. The top of the ladder remains on the wall, but starts sliding down. How fast is the top of the ladder sliding down the wall the instant that the foot of the ladder is 5 ft from the wall? *(10 points)*

Answer. Let $x(t)$ be the distance from the foot of the ladder to the wall at time t, in feet, and $y(t)$ the distance from the top of the ladder to the ground at time t, also in feet. We measure t in seconds:



We are told that $\frac{dx}{dt} = \frac{1}{2}$ ft/sec (positive because it is moving away from the wall). We want to know the value of $\frac{dy}{dt}$ when $x = 5$.

An equation that connects $x(t)$ and $y(t)$ is

$$13^2 = x^2 + y^2.$$

Differentiating implicitly, we get

$$0 = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}.$$

We want to plug in the values of x and $\frac{dx}{dt}$ and solve for $\frac{dy}{dt}$; but we also need the value of y when $x = 5$.

If $13^2 = x^2 + y^2$, then at $x = 5$ we have $169 = 25 + y^2$, so $y^2 = 144$. Therefore, $y = 12$. Plugging in and solving for $\frac{dy}{dt}$, we have

$$\begin{aligned}
0 &= 2x\frac{dx}{dt} + 2y\frac{dy}{dt} \\
0 &= 2(5)\left(\frac{1}{2}\right) + 2(12)\frac{dy}{dt} \\
0 &= 5 + 24\frac{dy}{dt} \\
-5 &= 24\frac{dy}{dt} \\
-\frac{5}{24} &= \frac{dy}{dt}.
\end{aligned}$$

Negative because the ladder is sliding down.

So when the foot of the ladder is 5 feet from the wall, the top of the ladder is sliding down the wall at a rate of $\frac{5}{24}$ feet per second.

5. The position of an object moving horizontally along a line after t seconds is given by the function

$$s(t) = t^2 - 4t, \qquad 0 \le t \le 5,$$

with $s(t)$ measured in inches and t measured in seconds.

(i) In what units will the velocity $s'(t)$ be measured? *(1 point)*

Answer. It is measured in units of s over units of t, so in inches/second.

(ii) In what units will the acceleration $s''(t)$ be measured? *(1 point)*

Answer. It is measured in units of s' over units of t, which becomes inches/second^2.

(iii) Determine when the object is stationary. *(4 points)*

Answer. When the velocity, $v(t) = s'(t)$, is zero. We have $v'(t) = 2t - 4$, so the object is stationary when $2t - 4 = 0$, that is when $t = 2$ seconds.

(iv) Determine the time interval(s) during which the object is speeding up. *(3 points)*

Answer. The object is speeding up when the velocity and acceleration have the same sign (both positive or both negative). We have:

$$\begin{aligned}
s(t) &= t^2 - 4t \\
v(t) &= s'(t) = 2t - 4 = 2(t - 2) \\
a(t) &= s''(t) = v'(t) = 2.
\end{aligned}$$

So the acceleration is always positive. The velocity is positive when $2 < t \le 5$, and negative when $0 \le t < 2$. So the object is speeding up on the interval $(2, 5]$. (Note that the problem statement specifies that t must be in $[0, 5]$).