Math 270–005: Calculus I
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Homework 7
SOLUTIONS

§4.1

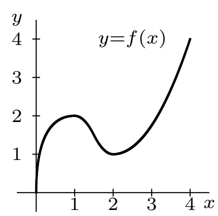
11. Please see the book for the graph. The funciton has an absolute maximum at b, and an absolute minimum at c_2.

13. Please see the book for the graph. This function has an absolute minimum at $x = a$, but has no absolute maximum.

15. Please see the book for the graph. The absolute maximum of $f(x)$ occurs at $x = b$, and the absolute minimum occurs at $x = a$. Both are also local extremes.

    There are also local maxima that are not absolute maxima at $x = p$ and $x = r$; and local minima that are not absolute minima at $x = q$ and at $x = s$.

19. We want a continuous function $f(x)$ defined on $[0, 4]$ which has $f'(x) = 0$ at $x = 1$ and at $x = 2$, an absolute maximum at $x = 4$, an absolute minimum at $x = 0$, and a local minimum at $x = 2$. There are many possible graphs; here is one:



23. To find the critical points of $f(x) = 3x^2 - 4x + 2$, we first take the derivative:

$$f'(x) = 6x - 4.$$

    This is always defined, and is equal to 0 at $x = \frac{2}{3}$. This is the only critical point, which is a stationary point.

35. First we find the derivative of $f(x) = \frac{1}{x} + \ln x$. Note that the function is only defined when $x > 0$.

$$f'(x) = -\frac{1}{x^2} + \frac{1}{x} = \frac{1}{x} - \frac{1}{x^2}.$$

    This is not defined at $x = 0$, but $x = 0$ is not in the domain of f, so it doesn't count as a critical point.

    The stationary points are the points where $f'(x) = 0$. This will occur if $\frac{1}{x} = \frac{1}{x^2}$, which requires $x^2 = x$, or $x^2 - x = x(x - 1) = 0$. This means $x = 0$ (which we already noted is not in the domain), or $x = 1$. Indeed, $x = 1$ is a stationary point.

    Thus, the only critical point is $x = 1$.

45. We want to find the absolute extremes of $f(x) = x^3 - 3x^2$ on $[-1, 3]$. Note the function is continuous on a finite closed interval, so we can find the absolute extremes by finding all critical points in $[-1, 3]$, and evaluating the function at them and the endpoints.

We have $f'(x) = 3x^2 - 6x = 3x(x-2)$. The critical points are then $x = 0$ and $x = -2$. We have:

$$\begin{aligned} f(-1) &= -1 - 3 = -4, \\ f(3) &= 27 - 27 = 0, \\ f(0) &= 0, \\ f(2) &= 8 - 12 = -4. \end{aligned}$$

So the absolute maximum is 0, achieved at $x = 0$ and at $x = 3$; and the absolute minimum is -4, achieved at $x = -1$ and at $x = 2$.

55. Here we have $f(x) = x^2 + \arccos(x)$ on $[-1, 1]$. Note the function is continuous on that interval. So proceeding as above, we have

$$f'(x) = 2x - \frac{1}{\sqrt{1-x^2}} = \frac{2x\sqrt{1-x^2} - 1}{\sqrt{1-x^2}}.$$

This is undefined at $x = -1$ and $x = 1$, which are the endpoints.

The stationary points are where $2x\sqrt{1-x^2} - 1 = 0$, or when $2x\sqrt{1-x^2} = 1$. Squaring both sides, we get $4x^2(1-x^2) = 1$, or $4x^2 - 4x^4 - 1 = 0$. This is a quadratic equation on x^2,

$$4(x^2)^2 - 4(x^2) + 1 = 0$$

And we can rewrite as

$$0 = 4(x^2)^2 - 4x^2 + 1 = (2x^2)^2 - 2(2x^2) + 1 = (2x^2 - 1)^2.$$

So the critical points occur when $2x^2 = 1$, or when $x^2 = \frac{1}{2}$, which occurs at $x = \frac{\sqrt{2}}{2}$ and $x = -\frac{\sqrt{2}}{2}$. However, note that $x = -\frac{\sqrt{2}}{2}$ is not a critical point, since $f'(-\frac{\sqrt{2}}{2}) < 0$; this spurious solution arises because we squared both sides of the equation. So the only critical points are $x = 1$, $x = -1$ (which are also the endpoints), and $x = \frac{\sqrt{2}}{2}$.

Evaluating, we have:

$$\begin{aligned} f(-1) &= 1 + \arccos(-1) = 1 + \pi. \\ f(1) &= 1 + \arccos(1) = 1. \\ f\left(\frac{\sqrt{2}}{2}\right) &= \frac{1}{2}\arccos\left(\frac{\sqrt{2}}{2}\right) = \frac{1}{2} + \frac{\pi}{4} = \frac{2+\pi}{4}. \end{aligned}$$

Of these, the smallest value is 1, achieved at $x = 1$; and the largest value is $1 + \pi$, achieved at $x = -1$.

§4.2

5. Please see the book for the graph of $f(x) = \frac{x^2}{4} + 1$ on $[-2, 4]$.

   (a) From the graph, it would seem that the tangent is parallel to the green line around $c = 1$.

   (b) To verify this, we compute. First, $f'(x) = \frac{x}{2}$. The slope of the secant line is

   $$\frac{f(4) - f(-2)}{4 - (-2)} = \frac{5-2}{6} = \frac{1}{2}.$$

   And we see that $f'(c) = \frac{c}{2}$ takes the value $\frac{1}{2}$ at $c = 1$, as we guessed.

11. We have the function $f(x) = x(x-1)^2$ on the interval $[0,1]$.

    The function is a polynomial, so it is continuous everywhere; in particular, on $[0,1]$. It is also differentiable everywhere, so it is differentiable on $(0,1)$. Finally, wee verify that $f(0) = f(1)$ (both are equal to 0. So the hypotheses of Rolle's Theorem are true for $f(x)$ on $[0,1]$, and the theorem applies: there exists at least one point c in $(0,1)$ where $f'(c) = 0$.

    To find all such points, we compute $f'(x)$:

    $$f'(x) = (x)'(x-1)^2 + x((x-1)^2)' = (x-1)^2 + 2x(x-1) = (x-1)(x-1+2x) = (x-1)(3x-1).$$

    The only solution to $f'(x) = 0$ in $(0,1)$ is $x = \frac{1}{3}$. So the one value where $f'(c) = 0$ in $(0,1)$ is $c = \frac{1}{3}$.

15. We now consider $f(x) = 1 - x^{2/3}$ on $[-1,1]$. The function is continuous everywhere, so it is continuous on $[-1,1]$. The derviative is $f'(x) = -\frac{2}{3}x^{-1/3} = -\frac{2}{3\sqrt[3]{x}}$. We note that this is undefined at $x = 0$. This means that $f(x)$ is **not** differentiable on $(-1,1)$, and so the hypotheses of Rolle's Theorem are not satisfied. The theorem does not apply.

    (And although $f(-1) = f(1)$, in this case there are no points where $f'(c) = 0$.)

21. (a) The function $f(x) = 7 - x^2$ is continuous everywhere, so it is continuous on $[-1,2]$. It is differentiable everywhere, so it is differentiable on $(-1,2)$. The hypotheses of the Mean Value Theorem are satisfied, so it applies to $f(x)$ on $[-1,2]$.

    (b) We have

    $$\frac{f(2) - f(-1)}{2 - (-1)} = \frac{3 - 6}{3} = -1.$$

    Now, $f'(x) = -2x$. The only point where $f'(c) = -1$ is then $c = \frac{1}{2}$.