Math 270–005: Calculus I
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Homework 6
SOLUTIONS

§3.9

21. If $y = \ln(x^4+1)$, then using the Chain Rule we have:

$$y' = \frac{1}{x^4+1}(x^4+1)' = \frac{4x^3}{x^4+1}.$$

25. If $y = (x^2+1)\ln x$, then using the Product Rule we have

$$y' = (x^2+1)'\ln x + (x^2+1)(\ln x)' = 2x\ln x + \frac{x^2+1}{x}.$$

29. If $y = \ln(\ln x)$, then by the Chain Rule we have:

$$y' = \frac{1}{\ln x}(\ln x)' = \frac{1}{\ln x}\left(\frac{1}{x}\right) = \frac{1}{x\ln x}.$$

37. If $y = 8^x$, then $y' = \ln(8)8^x$.

43. And if $y = x^3 3^x$, then using the Product Rule we have:

$$y' = (x^3)'3^x + x^3(3^x)' = 3x^2 3^x + x^3(\ln(3)3^x) = 3^x x^2(3 + x\ln(3)).$$

§3.10

13. If $f(x) = \arcsin(2x)$, then by the Chain Rule we have:

$$f'(x) = \frac{1}{\sqrt{1-(2x)^2}}(2x)' = \frac{2}{\sqrt{1-4x^2}}.$$

17. If $f(x) = \arcsin(e^{-2x})$, then we apply the Chain Rule twice:

$$\begin{aligned} f'(x) &= \frac{1}{\sqrt{1-(e^{-2x})^2}}(e^{-2x})' \\ &= \frac{1}{\sqrt{1-e^{-4x}}}e^{-2x}(-2x)' \\ &= -\frac{2e^{-2x}}{\sqrt{1-e^{-4x}}}. \end{aligned}$$

21. If $f(y) = \arctan(2y^2-4)$, then by the Chain Rule we have:

$$f'(y) = \frac{1}{1+(2y^2-4)^2}(2y^2-4)' = \frac{4y}{1+(2y^2-4)^2} = \frac{4y}{4y^4-16y^2+17}.$$

25. If $f(x) = x^2 + 2x^3\text{arccot}(x) - \ln(1+x^2)$, we have:

$$\begin{aligned}
f'(x) &= 2x + 2\Big((x^3)'\text{arccot}\,x + x^3(\text{arccot}\,x)'\Big) - \frac{1}{1+x^2}(1+x^2)' \\
&= 2x + 2\left(3x^2\text{arccot}\,x - x^3\left(\frac{1}{1+x^2}\right)\right) - \frac{2x}{1+x^2} \\
&= 2x + 6x^2\text{arccot}\,x - \frac{2x^3}{1+x^2} - \frac{2x}{1+x^2} \\
&= 2x + 6x^2\text{arccot}\,x - \frac{2x^3+2x}{1+x^2} \\
&= 2x + 6x^2\text{arccot}\,x - \frac{2x(x^2+1)}{1+x^2} \\
&= 2x + 6x^2\text{arccot}\,x - 2x \\
&= 6x^2\text{arccot}\,x.
\end{aligned}$$

§3.11

21. Let $V(t)$ be the volume of the snowball at time t, measured in cm^3. Let $S(t)$ be the surface area of the snowball at time t, measured in cm^2, and let $r(t)$ be the length of the radius at time t, measured in cm. We will measure time in minutes.

    We are told that the rate of change of the volume is proportional to the size of the surface; that is, we are told that $\frac{dV}{dt} = kS$ for some constant k.

    The volume is given by $V(t) = \frac{4}{3}\pi r^3$. Differentiating implicitly and remembering that $S(t) = 4\pi r^2$ we get:

$$\begin{aligned}
\frac{dV}{dt} &= 4\pi r^2\frac{dr}{dt} \\
kS(t) &= (4\pi r^2)\frac{dr}{dt} \\
kS(t) &= S(t)\frac{dr}{dt} \\
k &= \frac{dr}{dt}.
\end{aligned}$$

    That is, the rate of change of the length of the radius is constant, as claimed.

36. See the picture in the book. Let $h(t)$ be the height of the water inside the tank at time t, measured in feet; let $r(t)$ be the radius of the surface of the water at time t, measured in feet. We will measure time in seconds.

    Using similar triangles, we have that

$$\frac{h(t)}{r(t)} = \frac{12}{6} = 2,$$

    so $r(t) = \frac{1}{2}h(t)$.

    We want to know how fast the depth is changing when the depth is 3 feet. So we want to know the value of $\frac{dh}{dt}$ when $h = 3$.

    We are told that the volume is changing at a rate of -2 ft^3/sec (negative because the water is draining). So we know that $\frac{dV}{dt} = -2$ cubic feet per second.

We want an equation that relates volume and height. We have:

$$V = \frac{1}{3}\pi r^2 h$$
$$V = \frac{1}{3}\pi \left(\frac{h}{2}\right)^2 h$$
$$V = \frac{\pi}{12}h^3.$$

Differentiating both sides implicitly with respect to t, we get:

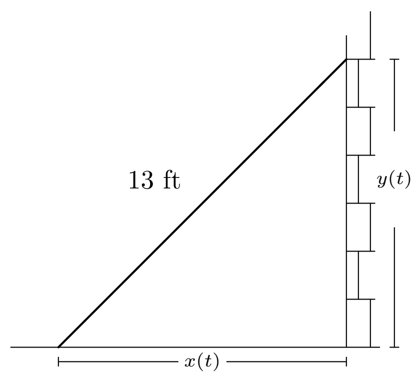
$$\frac{dV}{dt} = \frac{\pi}{4}h^2\frac{dh}{dt}.$$

Now, plugging in the known values and solving, we get:

$$-2 = \frac{\pi}{4}(3)^2\frac{dh}{dt}$$
$$-\frac{8}{9\pi} = \frac{dh}{dt}.$$

So when the water is 3 feet deep, the depth is decreasing at a rate of $\frac{8}{9\pi}$ feet per second.

41. Let $x(t)$ be the distance from the foot of the ladder to the wall at time t, in feet, and $y(t)$ the distance from the top of the ladder to the ground at time t, also in feet. We measure t in seconds:



We are told that $\frac{dx}{dt} = 0.5$ ft/sec (positive because it is moving away from the wall). We want to know the value of $\frac{dy}{dt}$ when $x = 5$.

An equation that connects $x(t)$ and $y(t)$ is

$$13^2 = x^2 + y^2.$$

Differentiating implicitly, we get

$$0 = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}.$$

We want to plug in the values of x and $\frac{dx}{dt}$ and solve for $\frac{dy}{dt}$; but we also need the value of y when $x = 5$.

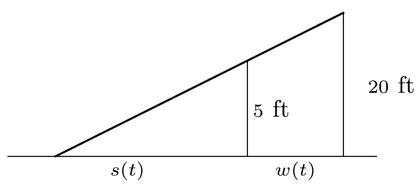
If $13^2 = x^2 + y^2$, then at $x = 5$ we have $169 = 25 + y^2$, so $y^2 = 144$. Therefore, $y = 12$. Plugging in, we have

$$\begin{aligned} 0 &= 2x\frac{dx}{dt} + 2y\frac{dy}{dt} \\ 0 &= 2(5)(0.5) + 2(12)\frac{dy}{dt} \\ -5 &= 24\frac{dy}{dt} \\ -\frac{5}{24} &= \frac{dy}{dt}. \end{aligned}$$

Negative because the ladder is sliding down.

So when the foot of the ladder is 5 feet from the wall, the top of the ladder is sliding down the wall at a rate of $\frac{5}{24}$ feet per second.

43. Let $w(t)$ be the distance from the woman to the streetlamp, and $s(t)$ the length of the shadow she casts, both measured in feet with t in seconds:



By similar triangles, we have that $\frac{s+w}{20} = \frac{s}{5}$, or equivalently, $s + w = 4s$, or $3s = w$.

Now, first we want to know the rate of change of the length of the shadow, that is $\frac{dw}{dt}$, when $s = 15$. But the value of s is irrelevant: from $3s = w$ we get $3s'(t) = w'(t)$. Now, we are told that $\frac{dw}{dt} = -8$ ft/sec (because she is approaching the lamp), so this means that $s'(t) = -\frac{8}{3}$ feet per second.

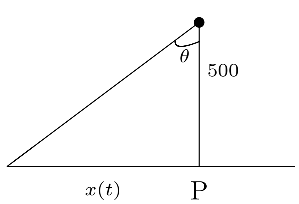
So the length of the shadow is shrinking at a rate of $\frac{8}{3}$ feet per second.

Now, the tip of the shadow is $s + w$ away from the lamp. If we want to know the rate of change of $s + w$, then we want

$$\frac{d}{dt}(s + w) = s'(t) + w'(t) = -8 - \frac{8}{3} = -\frac{32}{3},$$

so the tip of the shadow is moving at a speed of $\frac{32}{3}$ feet per second (towards the lamp).

50. Let θ be the angle made by the beam, and let $x(t)$ be the distance of the spotlight to the point P, mesured in meters: I'm imagining the beam rotating anticlockwise, so the beam is moving left to right.



We are told the beam rotates four times each minute, so the rate of change of the angle is $\frac{d\theta}{dt} = 8\pi$ radians/minute. We want to know the value of $\frac{dx}{dt}$ when $x(t) = 200$.

We need an equation that connects θ to $x(t)$. The simplest is to use the tangent, which will give $\tan\theta = \frac{x}{500}$, or $500\tan\theta = x(t)$.

Differentiating implicitly, we obtain:

$$500\sec^2(\theta)\frac{d\theta}{dt} = \frac{dx}{dt}.$$

We need to know the value of $\sec^2(\theta)$ when $x(t) = 200$.

We can calculate it directly, or recall that

$$\sec^2\theta = 1 + \tan^2\theta.$$

Since $\tan(\theta) = \frac{200}{500} = \frac{2}{5}$, we obtain that

$$\sec^2\theta = 1 + \frac{4}{25} = \frac{29}{25}.$$

Plugging in the values we get

$$\begin{aligned} 500\sec^2(\theta)\frac{d\theta}{dt} &= \frac{dx}{dt} \\ 500\left(\frac{29}{25}\right)(8\pi) &= \frac{dx}{dt} \\ 4640\pi &= \frac{dx}{dt}. \end{aligned}$$

So the beam is moving at a speed of 4640π meters/minute at that instant.

How does the speed of the beam along the shore vary when the distance between P and Q? From the equation

$$\frac{dx}{dt} = 500\pi\sec^2(\theta)\frac{d\theta}{dt},$$

we see that when $\sec(\theta)$ gets smaller, the beam slows down. The secant achieves its smaller value at 0, and grows as we move further away from zero in either direction. So the further we are from the point P, the faster the beam seems to move along the shore.