Math 270–005: Calculus I
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Homework 1
SOLUTIONS

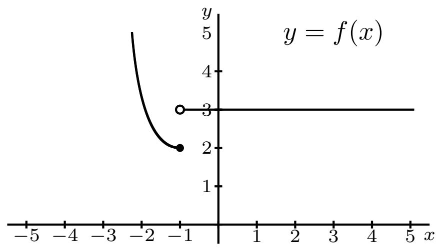
§2.2

5. See the book for the graph.
   (a) From the graph, $f(1) = -1$.
   (b) On the other hand, the graph suggests that $\lim_{x\to 1} f(x) = 1$, since the values of $f(x)$ approach 1 as x approaches 1.
   (c) The graph indicates that $f(0) = 2$.
   (d) Here, we also have $\lim_{x\to 0} f(x) = 2$, as the values of $f(x)$ are also approaching 2 as x approaches 0.

15. Please see the book for the graph.
   (a) From the graph, $f(1) = 0$.
   (b) The limit of $f(x)$ as x approaches 1 from the left is 1, $\lim_{x\to 1^-} f(x) = 1$, as seen in the graph.
   (c) The limit from the right is 0, though: $\lim_{x\to 1^+} f(x) = 0$.
   (d) Because the one-sided limits do not agree, we conclude that $\lim_{x\to 1} f(x)$ does not exist.

17. Please see the book for the graph.
   (a) $f(1) = 3$, the dot on the graph above $x = 1$.
   (b) $\lim_{x\to 1^-} f(x) = 2$.
   (c) $\lim_{x\to 1^+} f(x) = 2$.
   (d) $\lim_{x\to 1} f(x) = 2$.
   (e) $f(3) = 2$.
   (f) $\lim_{x\to 3^-} f(x) = 4$.
   (g) $\lim_{x\to 3^+} f(x) = 1$.
   (h) $\lim_{x\to 3} f(x)$ does not exist.
   (i) $f(2) = 3$.
   (j) $\lim_{x\to 2^-} f(x) = 3$.
   (k) $\lim_{x\to 2^+} f(x) = 3$.
   (l) $\lim_{x\to 2} f(x) = 3$.

19. The graph of
$$f(x) = \begin{cases} x^2+1 & \text{if } x \leq -1,\\ 3 & \text{if } x > -1,\end{cases}$$
looks like the graph of $f(x) = x^2 + 1$ (the standard parabola $y = x^2$ raised by one unit) up to $x = -1$, and then is just a horizontal line at height $x = 3$.



At $a = -1$, we see that $f(a) = f(-1) = (-1)^2 + 1 = 2$. On the other hand, $\lim_{x \to -1^-} f(x) = 2$, and $\lim_{x \to -1^+} f(x) = 3$. It is also evident from the graph that $\lim_{x \to -1} f(x)$ does not exist.

§2.3

7. We are assuming that $\lim_{x \to 1} f(x) = 8$, so

$$\lim_{x \to 1} (4f(x)) = 4 \lim_{x \to 1} f(x) = 4(8) = 32.$$

11. Here, in addition to $\lim_{x \to 1} f(x) = 8$, we also assume $\lim_{x \to 1} g(x) = 3$ and $\lim_{x \to 1} h(x) = 2$. We note that

$$\lim_{x \to 1} (g(x) - h(x)) = \lim_{x \to 1} g(x) - \lim_{x \to 1} h(x) = 3 - 2 = 1 \neq 0,$$

so we have

$$\lim_{x \to 1} \frac{f(x)}{g(x) - h(x)} = \frac{\lim_{x \to 1} f(x)}{\lim_{x \to 1} (g(x) - h(x))} = \frac{8}{1} = 8.$$

15. If

$$g(x) = \begin{cases} 2x + 1 & \text{if } x \neq 0, \\ 5 & \text{if } x = 0, \end{cases}$$

then from the definition we see that $g(0) = 5$. And because $g(x)$ takes the exact same values as $2x + 1$ everywhere near 0, except *at* 0, we have

$$\lim_{x \to 0} g(x) = \lim_{x \to 0} (2x + 1) = 2(0) + 1 = 1.$$

19. To be explicit with the limit laws, we have

$$\lim_{x \to 4} (3x - 7) = 3 \lim_{x \to 4} x - \lim_{x \to 4} 7 = 3(4) - 7 = 5.$$

25. Since the limit of the denominator is not 0, we can proceed directly:

$$\begin{aligned} \lim_{x \to 1} \frac{5x^2 + 6x + 1}{8x - 4} &= \frac{\lim_{x \to 1} (5x^2 + 6x + 1)}{\lim_{x \to 1} (8x - 4)} \\ &= \frac{5(1)^2 + 6(1) + 1}{8(1) - 4} = \frac{5 + 6 + 1}{8 - 4} \\ &= \frac{12}{4} = 3. \end{aligned}$$

27. Since the limit of the denominator is not 0, we can proceed directly:

$$\lim_{p\to 2} \frac{3p}{\sqrt{4p+1}-1} = \frac{\lim_{p\to 2} 3p}{\lim_{x\to 2} \sqrt{4p+1}-1} = \frac{3(2)}{\sqrt{4(2)+1}-1}$$
$$= \frac{6}{\sqrt{9}-1} = \frac{6}{2} = 3.$$

35. The denominator has a limit of 0; so we factor, and cancel:

$$\lim_{x\to 4} \frac{x^2-16}{4-x} = \lim_{x\to 4} \frac{(x-4)(x+4)}{-(x-4)} \stackrel{*}{=} \lim_{x\to 4} \frac{x+4}{-1}$$
$$= -8.$$

where $*$ holds because the function $\frac{(x-4)(x+4)}{4-x}$ and the function $\frac{x+4}{-1}$ are equal at every point near $x = 4$, *except for* at $x = 4$; so they have the same limit as x approaches 4.

49. This looks daunting, but if we do the algebra it becomes similar to problems we have already solved:

$$\lim_{x\to 4} \frac{\frac{1}{x}-\frac{1}{4}}{x-4} = \lim_{x\to 4} \frac{\frac{4-x}{4x}}{x-4} = \lim_{x\to 4} \frac{4-x}{(4x)(x-4)}$$
$$= \lim_{x\to 4} \frac{-(x-4)}{4x(x-4)} = \lim_{x\to 4} \frac{-1}{4x}$$
$$= -\frac{1}{16}.$$

73. The function is

$$f(x) = \begin{cases} x^2+1 & \text{if } x < -1, \\ \sqrt{x+1} & \text{if } x \geq -1. \end{cases}$$

(a) Consider what happens as x approaches -1 from the left. Since x will be smaller than -1, the function f just takes the value $x^2 + 1$. So we have

$$\lim_{x\to -1^-} f(x) = \lim_{x\to -1^-} (x^2+1) = (-1)^2 + 1 = 2.$$

(b) On the other hand, if we approach -1 from the right, the function f will take the exact same values as the function $\sqrt{x+1}$. Using the limit laws, we have

$$\lim_{x\to -1^+} f(x) = \lim_{x\to -1^+} \sqrt{x+1} = \sqrt{-1+1} = \sqrt{0} = 0.$$

(c) Since the one-sided limits are not equal to each other, $\lim_{x\to -1} f(x)$ does not exist.

87. We have the function

$$f(x) = \begin{cases} \dfrac{x^2-5x+6}{x-3} & \text{if } x \neq 3, \\ a & \text{if } x = 3. \end{cases}$$

We want to know what value of a will result in $\lim_{x\to 3} f(x) = f(3)$.

Note that $f(3) = a$; so what we really need to find out is the value of $\lim_{x\to 3} f(x)$. Since $f(x)$ takes the exact same values as $\frac{x^2 - 5x + 6}{x + 3}$ for any x near (but not equal to) x, we have

$$\lim_{x\to 3} f(x) = \lim_{x\to 3} \frac{x^2 - 5x + 6}{x - 3}.$$

Now, this rational function has both a numerator and denominator that evaluate to 0 when we plug in 3, so we factor out $x - 3$ from the numerator and cancel:

$$\begin{aligned} \lim_{x\to 3} f(x) &= \lim_{x\to 3} \frac{x^2 - 5x + 6}{x - 3} \\ &= \lim_{x\to 3} \frac{(x-3)(x-2)}{x - 3} \\ &= \lim_{x\to 3} \frac{x - 2}{1} = 3 - 2 = 1. \end{aligned}$$

So in order for $f(3)$ to equal 1, we need $a = 1$.