MATH 270-005 – SPRING 2024
FINAL EXAM
SOLUTIONS
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1. Evaluate each of the following limits. You may **not** use sampling or approximations, but you can use limit laws, algebraic manipulations, and L'Hôpital's Rule (if applicable). If you use L'Hôpital's Rule, mark explicitly the step in which you use it. If the limit does not exist and equals ∞ or $-\infty$, your answer must indicate that it equals ∞ or that it equals $-\infty$, whichever is appropriate. If the limit does not exist and does not equal ∞ or $-\infty$, you must explicitly write that the limit DOES NOT EXIST. *(2 points each, 8 points total)*

   (a) $\displaystyle\lim_{x\to 1}\frac{5x^3-2x+1}{x^2-x+1}$
   Answer. Since the denominator has limit 1 at $x=1$, we can just evaluate:
$$\lim_{x\to 1}\frac{5x^3-2x+1}{x^2-x+1} = \frac{5(1)^3-2(1)+1}{(1)^2-1+1} = \frac{5-2+1}{1-1+1} = 4.$$

   (b) $\displaystyle\lim_{x\to\infty}\frac{x^2-1}{3x^2+1}$
   Answer. This is an $\frac{\infty}{\infty}$ indeterminate: we can use L'Hopital's Rule, either once followed by cancelling, or twice:
$$\lim_{x\to\infty}\frac{x^2-1}{3x^2+1} \stackrel{\text{LH}}{=} \lim_{x\to\infty}\frac{(x^2-1)'}{(3x^2+1)'} = \lim_{x\to\infty}\frac{2x}{6x} = \lim_{x\to\infty}\frac{2}{6} = \frac{1}{3}.$$
   Alternatively, we can divide the numerator and the denominator by the highest power of x that occurs in the denominator, to get:
$$\lim_{x\to\infty}\frac{x^2-1}{3x^2+1} = \lim_{x\to\infty}\frac{\frac{1}{x^2}(x^2-1)}{\frac{1}{x^2}(3x^2+1)} = \lim_{x\to\infty}\frac{1-\frac{1}{x^2}}{3+\frac{1}{x^2}} = \frac{1-0}{3+1} = \frac{1}{3}.$$

   (c) $\displaystyle\lim_{x\to 0^+} x\ln(x)$
   Answer. This is a $0\times\infty$ indeterminate, so first we rewrite as a fraction to be able to apply L'Hopital's Rule, then we apply the rule, simplify, and compute the limit:
$$\lim_{x\to 0^+} x\ln(x) = \lim_{x\to 0^+}\frac{\ln(x)}{\frac{1}{x}} \stackrel{\text{LH}}{=} \lim_{x\to 0^+}\frac{(\ln(x))'}{(\frac{1}{x})'} = \lim_{x\to 0^+}\frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x\to 0^+}\frac{-x^2}{x} = \lim_{x\to 0^+}(-x) = 0.$$

   (d) $\displaystyle\lim_{x\to\frac{\pi}{4}}\big(\sin(x)+\cos(x)\big)$
   Answer. These two functions are continuous, so we just need to plug in and evaluate:
$$\lim_{x\to\frac{\pi}{4}}\big(\sin(x)+\cos(x)\big) = \sin\left(\frac{\pi}{4}\right)+\cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2} = \sqrt{2}.$$

2. On this page there are three functions; and two on the next page. For each function, find the derivative $f'(x)$. Your final answer should contain no indicated derivatives, no complex fractions, and no easy arithmetic or algebraic simplifications to be done. But you are not required to obtain the simplest possible expression. *(2 points each, 10 points total)*

   (a) $f(x)=\arcsin(x)$
   Answer. This is one of our basic functions: $f'(x)=\frac{1}{\sqrt{1-x^2}}$.

(b) $f(x) = x^2 \ln(x)$
Answer. A Product Rule:

$$f'(x) = (x^2)' \ln(x) + x^2(\ln(x))' = 2x \ln(x) + x^2 \left(\frac{1}{x}\right) = 2x \ln(x) + x = x(2\ln(x) + 1).$$

(c) $f(x) = e^{\sqrt{x}} + \pi^e$
Answer. This is a Chain Rule, plus the derivative of a constant:

$$f'(x) = e^{\sqrt{x}}(\sqrt{x})' + 0 = \frac{1}{2\sqrt{x}} e^{\sqrt{x}} = \frac{e^{\sqrt{x}}}{2\sqrt{x}}.$$

(d) $f(x) = \dfrac{\tan(x)}{2x+1}$
Answer. This is Quotient Rule:

$$f'(x) = \frac{(2x+1)(\tan(x))' - (\tan(x))(2x+1)'}{(2x+1)^2} = \frac{(2x+1)\sec^2(x) - 2\tan(x)}{(2x+1)^2}.$$

(e) $f(x) = \displaystyle\int_1^x \frac{e^t}{\sin t}\, dt.$
HINT: Do not try to compute the integral.
Answer. By the Fundamental Theorem of Calculus, Part 2, we have:

$$f'(x) = \frac{d}{dx} \int_1^x \frac{e^t}{\sin t}\, dt = \frac{e^x}{\sin x}.$$

3. In this page and the next there are five integrals. Solve each integral by any valid method, **except plugging it into your calculator or using a table of integrals**. For indefinite integrals, give the most general antiderivative. For definite integrals, give the exact value. If you use u-substitution, you must state clearly and explicitly what substitution you are using, as well as the value of du. *(2 points each, 10 points total)*

(a) $\displaystyle\int \left(x + \sqrt{x}\right)\, dx$
Answer. We have: $\int (x + \sqrt{x})\, dx = \int \left(x + x^{1/2}\right)\, dx = \frac{1}{2}x^2 + \frac{2}{3}x^{3/2} + C.$

(b) $\displaystyle\int_0^1 \frac{1}{1+x^2}\, dx$
Answer. This is the integral of the derivative of $\arctan(x)$, so

$$\int_0^1 \frac{1}{1+x^2}\, dx = \arctan(x) \Big|_0^1 = \arctan(1) - \arctan(0) = \frac{\pi}{4}.$$

(c) $\displaystyle\int_0^{\pi/6} \sin(3x)\, dx$
Answer. We can do a substitution. Set $u = 3x$, so $du = 3\, dx$. Then $\frac{1}{3} du = dx$. For the limits of integration, when $x = 0$ we have $u = 0$; when $x = \frac{\pi}{6}$, we have $u = \frac{\pi}{2}$. So:

$$\begin{aligned}\int_0^{\pi/6} \sin(3x)\, dx &= \int_0^{\pi/2} \frac{1}{3} \sin(u)\, du = -\frac{1}{3} \cos(u) \Big|_0^{\pi/2} \\ &= -\frac{1}{3} \cos\left(\frac{\pi}{2}\right) + \frac{1}{3} \cos(0) = -\frac{1}{3}(0) + \frac{1}{3}(1) = \frac{1}{3}.\end{aligned}$$

(d) $\int 2xe^{x^2}\,dx$

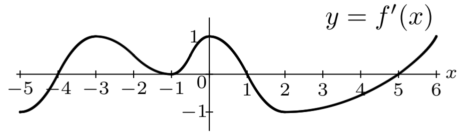
Answer. Another substitution: set $u = x^2$. Then $du = 2x\,dx$, so we have

$$\int 2xe^{x^2}\,dx = \int e^u\,du = e^u + C = e^{x^2} + C.$$

(e) $\int \sec^2(x)\,dx$

Answer. The derivative of $\tan(x)$ is $\sec^2(x)$, so $\int \sec^2(x)\,dx = \tan(x) + C$.

4. The following sketch represents the graph of **the derivative** $f'(x)$ of a function $f(x)$ that is defined and continuous at every point on $[-5, 6]$. Use this graph to answer the following questions about the function $f(x)$. NOTE: the questions are **not** about the function whose graph you are seeeing.



For parts (a), (b), (c), and (d), you do not need to provide explanations, just the answer. For part (e), you must justify your answers.

(a) On what intervals is $f(x)$ increasing? *(2 points)*

Answer. This happens when $f'(x)$ is positive, including endpoints.. So this happens at $[-4, 1]$ and $[5, 6]$. (You can also write it as $[-4, -1]$, $[-1, 1]$, and $[5, 6]$.)

(b) On what intervals is $f(x)$ concave up? *(2 points)*

Answer. This happens when $f'(x)$ is increasing, without including the endpoints. Here, it is on $(-5, -3)$, $(-1, 0)$, and $(2, 6)$.

(c) Where are the points of inflection of $f(x)$? *(2 points)*

Answer. This happens where the derivative changes directions (local extremes); so at $x = -3$, $x = -1$, $x = 0$, and $x = 2$.

(d) What are the critical points of $f(x)$? *(2 points)*

Answer. Where the derivative is 0 or undefined. Here, $x = -4$, $x = -1$, $x = 1$, and $x = 5$.

(e) For each critical point, determine whether $f(x)$ has a local maximum, a local minimum, or neither at that point; justify your answer. *(4 points)*

Answer. Using the First Derivative Test: if the derivative changes from negative to positive at the critical point, we have a relative minimum. This happens at $x = -4$ and at $x = 5$.

If the derivative changes from positive to negative at the critical point, we have a relative maximum. This happens at $x = 1$.

If the derivative has the same sign on both sides of the critical point, it is not a local extreme. This happens at $x = -1$.

5. The following equation defines a function implicitly:

$$x^3 + y^3 = 3xy.$$

Use implicit differentiation to find the equation of the tangent line to the graph of this function at the point $(\frac{3}{2}, \frac{3}{2})$. Express the equation of the tangent in the form $y = mx + b$.

Do not attempt to solve for y first. You must use implicit differentiation. *(10 points)*

Answer. Note that the point lies on the curve: $(\frac{3}{2})^3 + (\frac{3}{2})^2 = \frac{27}{8} + \frac{27}{8} = \frac{27}{4}$; and $3(\frac{3}{2})(\frac{3}{2}) = \frac{27}{4}$. So the point satisfies the equation.

Taking derivatives we have:

$$\begin{aligned}
x^3 + y^3 &= 3xy \\
\frac{d}{dx}(x^3 + y^3) &= \frac{d}{dx}(3xy) \\
3x^2 + 3y^2y' &= 3(y + xy') \\
3x^2 + 3y^2y' &= 3y + 3xy'.
\end{aligned}$$

At this point we can plug in the values before solving for y'; we get

$$\begin{aligned}
3x^2 + 3y^2y' &= 3y + 3xy' \\
3\left(\frac{3}{2}\right)^2 + 3\left(\frac{3}{2}\right)^2 y' &= 3\left(\frac{3}{2}\right) + 3\left(\frac{3}{2}\right)y' \\
\frac{27}{4} + \frac{27}{4}y' &= \frac{9}{2} + \frac{9}{2}y' \\
\frac{27}{4}y' - \frac{9}{2}y' &= \frac{9}{2} - \frac{27}{4} \\
\left(\frac{27}{4} - \frac{18}{4}\right)y' &= \frac{18}{4} - \frac{27}{4} \\
\frac{9}{4}y' &= -\frac{9}{4} \\
y' &= \frac{-\frac{9}{4}}{\frac{9}{4}} \\
y' &= -1.
\end{aligned}$$

So the equation of the tangent line is the equation of the line with slope $m = -1$ through the point $(\frac{3}{2}, \frac{3}{2})$, which is

$$\begin{aligned}
y - \frac{3}{2} &= -\left(x - \frac{3}{2}\right) \\
y &= -x + \frac{3}{2} + \frac{3}{2} \\
y &= -x + 3.
\end{aligned}$$

6. Spaceman Spiff is trying to escape from the Ravenous Bugblatter Beast of Traal. He is running away and will reach his spaceship exactly two hours after he began running. As he weaves, zigs, and zags, concentrating on his escape, too late does Spaceman Spiff realize that the gravity on Traal is much lower than he is accostumed to. If his velocity rises above 8 miles per hour, he will be going so fast that he will launch himself into low orbit and begin floating away. If this happens, he will be unable to avoid becoming easy prey (and tasty dinner) for the Ravenous Beast. In addition, the Ravenous Beast is so close behind him that if Spaceman Spiff's velocity drops below 1 mile per hour, the Beast will catch up and end his illustrious career. But if

Spaceman Spiff's velocity stays between 1 and 8 miles per hour (inclusively), he will be able to reach his spaceship safely and escape. His velocity is the function $v(t)$:

$$\textbf{velocity:} \quad v(t) = t^3 + \frac{3}{2}t^2 - 6t + 5 \quad \text{miles per hour.}$$

He begins to run at $t = 0$; he will reach his spaceship at $t = 2$.

Determine whether Spaceman Spiff will escape and live to fight another day, or whether the Ravenous Beast will have Tasty Barbecued Spare Spiff Ribs for dinner tonight.[1] Your answer must be justified mathematically. Looking at a graph in your calculator is not sufficient justification. *(10 points)*

Answer. The problem boils down to this: we have a function $v(t)$ defined on the interval $[0, 2]$. Will the function ever take values greater than 8 or less than 1?

We can answer the question by figuring out the maximum and minimum of $v(t)$ on $[0, 2]$. If the maximum is less than or equal to 8, and the minimum is greater than or equal to 1, then Spiff will escape. If the maximum is greater than 8, or the minimum is less than 1, then Spiff will be eaten.

And $v(t)$ is continuous on a finite closed interval. So we just need to find the critical points, and evaluate $v(t)$ at the critical points and at the endpoints.

Since $v(t) = t^3 + \frac{3}{2}t^2 - 6t + 5$, we have

$$v'(t) = 3t^2 + 3t - 6 = 3(t^2 + t - 2) = 3(t + 2)(t - 1).$$

The only critical point in $[0, 2]$ is $t = 1$. So we evaluate $v(t)$ at $t = 0$, $t = 1$, and $t = 2$:

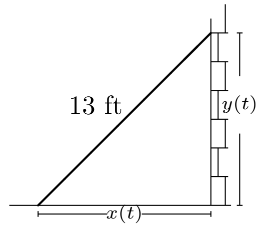
$$\begin{aligned} v(0) &= 5 \\ v(1) &= 1 + \frac{3}{2} - 6 + 5 = \frac{3}{2} \\ v(2) &= 8 + 6 - 12 + 5 = 7. \end{aligned}$$

So the maximum value of $v(t)$ is 7, which is less than 8; and the minimum value of $v(t)$ is $\frac{3}{2}$, which is more than 1. That means that Spaceman Spiff will be able to reach his spaceship safely and escape.[2]

7. A 13 foot ladder is leaning against a vertical wall. The foot of the ladder starts slipping away from the wall at a rate of 0.5 ft/sec. The top of the ladder remains on the wall, but starts sliding down. How fast is the top of the ladder sliding down the wall the instant that the foot of the ladder is 5 ft from the wall? *(10 points)*

   Answer. This problem was in the homework, and in the Chapter 3 test.

   Let $x(t)$ be the distance from the foot of the ladder to the wall at time t, in feet, and $y(t)$ the distance from the top of the ladder to the ground at time t, also in feet. We measure t in seconds:



[1] This is, in fact, a slanderous remark. It is well known that the Ravenous Bugblatter Beast of Traal will only eat spacemen if they are prepared in a nice, light, and fluffy quiche.

[2] Huzzah!

We are told that $\frac{dx}{dt} = \frac{1}{2}$ ft/sec (positive because it is moving away from the wall). We want to know the value of $\frac{dy}{dt}$ when $x = 5$.

An equation that connects $x(t)$ and $y(t)$ is $13^2 = x^2 + y^2$. Differentiating implicitly, we get

$$0 = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}.$$

We want to plug in the values of x and $\frac{dx}{dt}$ and solve for $\frac{dy}{dt}$; but we also need the value of y when $x = 5$. If $13^2 = x^2 + y^2$, then at $x = 5$ we have $169 = 25 + y^2$, so $y^2 = 144$. Therefore, $y = 12$. Plugging in and solving for $\frac{dy}{dt}$, we have

$$\begin{aligned}
0 &= 2x\frac{dx}{dt} + 2y\frac{dy}{dt} \\
0 &= 2(5)\left(\frac{1}{2}\right) + 2(12)\frac{dy}{dt} \\
0 &= 5 + 24\frac{dy}{dt} \\
-5 &= 24\frac{dy}{dt} \\
-\frac{5}{24} &= \frac{dy}{dt}.
\end{aligned}$$

Negative because the ladder is sliding down.

So when the foot of the ladder is 5 feet from the wall, the top of the ladder is sliding down the wall at a rate of $\frac{5}{24}$ feet per second.

8. Use the first and second derivatives of the function

$$f(x) = x^4 + 4x^3$$

to determine all the intervals in which the function is increasing and concave up (intervals in which both things happen).

(10 points)

Answer. We have:

$$\begin{aligned}
f'(x) &= 4x^3 + 12x^2 = 4x^2(x+3) \\
f''(x) &= 12x^2 + 24x = 12x(x+2).
\end{aligned}$$

So the points where the first derivative might change signs are $x = 0$ and $x = -3$. And the points where the second derivative might changes signs are $x = 0$ and $x = -2$. So we need to consider the following intervals: $(-\infty, -3)$, $(-3, -2)$, $(-2, 0)$, and $(0, \infty)$. We do a sign table:

	$(-\infty, -3)$	$(-3, -2)$	$(-2, 0)$	$(0, \infty)$
x^2	$+$	$+$	$+$	$+$
$x+3$	$-$	$+$	$+$	$+$
$f'(x)$	$-$	$+$	$+$	$+$
x	$-$	$-$	$-$	$+$
$x+2$	$-$	$-$	$+$	$+$
$f''(x)$	$+$	$+$	$-$	$+$
$f(x)$				

So $f(x)$ is both increasing and concave up on $(-3,-2)$ and on $(0,\infty)$.

9. Find the average value of the function $f(x) = x^2 + x$ on the interval $[-1,3]$. *(6 points)*

Answer. The average value is

$$\begin{aligned} \overline{f} &= \frac{1}{3-(-1)} \int_{-1}^{3} (x^2+x)\,dx = \frac{1}{4}\left(\frac{1}{3}x^3 + \frac{1}{2}x^2 \Big|_{-1}^{3} \right) \\ &= \frac{1}{4}\left(\left(\frac{27}{3} + \frac{9}{2} \right) - \left(\frac{-1}{3} + \frac{1}{2} \right) \right) = \frac{1}{4}\left(\frac{28}{3} + \frac{8}{2} \right) \\ &= \frac{7}{3} + \frac{2}{2} = \frac{10}{3}. \end{aligned}$$

10. Find the net signed area of the region bounded by the graph of $y = 6\cos x$, the x-axis, and the vertical lines $x = -\frac{\pi}{2}$ and $x = \pi$. *(6 points)*

Answer. This is just the integral of $f(x) = 6\cos x$ on the interval $[-\frac{\pi}{2}, \pi]$. We have:

$$\text{Area} = \int_{-\pi/2}^{\pi} 6\cos x\,dx = 6\sin x \Big|_{-\pi/2}^{\pi} = 6\sin(\pi) - 6\sin\left(-\frac{\pi}{2}\right) = 0 - 6(-1) = 6.$$

11. Use a local linear approximation to estimate the value of

$$\ln(1.05)$$

by selecting a function $f(x)$, a point x where $f(x) = \ln(1.05)$, and a point a "near x" where the value $f(a)$ is easy to compute.

(i) Write out explicitly the function $f(x)$ that you will use to approximate $\ln(1.05)$. *(1 point)*
Answer. The easiest choice is probably $f(x) = \ln(x)$.

(ii) State the value of a (the "easy point") you will use for the approximation. *(1 point)*
Answer. Given $f(x) = \ln(x)$, a good choice for a is $a = 1$.

(iii) Give the exact values of $f(a)$ and $f'(a)$. *(2 points)*
Answer. Then we have $f(a) = \ln(1) = 0$. And $f'(x) = \frac{1}{x}$, so $f'(a) = f'(1) = 1$.

(iv) Using the function $f(x)$ from (i), and the values of a, $f(a)$, and $f'(a)$ from (ii) and (iii), give the value you get as a local linear approximation of $\ln(1.05)$. *(2 points)*
Answer. The general formula is

$$f(x) \approx f(a) + f'(a)(x-a) \text{ for } x \text{ near } a.$$

So:

$$\ln(1.05) \approx f(1) + f'(1)(1.05-1) = 0 + 1(0.05) = 0.05.$$

So $\ln(1.05) \approx 0.05$.

(v) Use $f''(a)$) to state whether your approximation in (iv) is an overestimate or an underestimate. *(2 point)*
Answer. Since $f'(x) = \frac{1}{x}$, then $f''(x) = -\frac{1}{x^2}$, so $f''(1) = -1$. Because $f''(1) < 0$, that means the function is concave down around $a = 1$. That means the tangent lines above the graph, so the approximation is an OVERESTIMATE.