Comments, suggestions, corrections, and further references are most welcomed!

ON THE CAPABILITY OF FINITE GROUPS OF CLASS TWO AND PRIME EXPONENT

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Abstract. We consider the capability of p-groups of class two and odd prime exponent. The question of capability is shown to be equivalent to a statement about vector spaces and linear transformations, and using the equivalence we give proofs of some old results and several new ones. In particular, we establish a number of new necessary and new sufficient conditions for capability, including a sufficient condition based only on the ranks of $G/Z(G)$ and $[G,G]$. Finally, we characterise the capable groups among the 5-generated groups in this class.

1. Introduction.

In his landmark paper [12] on the classification of finite p-groups, P. Hall remarked:

> The question of what conditions a group G must fulfill in order that it may be the central quotient group of another group H, $G \cong H/Z(H)$, is an interesting one. But while it is easy to write down a number of necessary conditions, it is not so easy to be sure that they are sufficient.

Following M. Hall and Senior [11], we make the following definition:

Definition 1.1. A group G is said to be *capable* if and only if there exists a group H such that $G \cong H/Z(H)$.

Capability of groups was first studied by R. Baer in [2], where, as a corollary of deeper investigations, he characterised the capable groups that are direct sums of cyclic groups. Capability of groups has received renewed attention in recent years, thanks to results of Beyl, Felgner, and Schmid [3] characterising the capability of a group in terms of its epicenter; and more recently to work of Graham Ellis [7] that describes the epicenter in terms of the nonabelian tensor square of the group.

We will consider here the special case of nilpotent groups of class two and exponent an odd prime p. This case was studied in [13], and also addressed elsewhere (e.g., Prop. 9 in [7]). As noted in the final paragraphs of [1], currently available techniques seem insufficient for a characterisation of the capable finite p-groups of class 2, but a characterisation of the capable finite groups of class 2 and exponent p seems a more modest and possibly attainable goal. The present work is a contribution towards achieving that goal. We began to study this situation in [18]; here we will introduce what I believe is clearer notation as well as a general setting to frame the discussion. We will also be able to use our methods to extend the necessary

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condition from [13] to include groups that do not satisfy $Z(G) = [G,G]$, and to provide a short new proof of the sufficient condition from [7]. We will also prove a sufficient condition which is closer in flavor to the necessary condition of Heineken and Nikolova.

In the remainder of this section we will give basic definitions and our notational conventions. In Section 2 we will obtain a necessary and sufficient condition for the capability of a given group G of class at most two and exponent p in terms of a "canonical witness." In Section 3 we discuss the general setting in which we will work from the point of view of Linear Algebra, and the specific instance of that general setting that occurs in this work is introduced. We proceed in Section 4 to obtain several easy consequences of this set-up, and their equivalent statements in terms of capability. In Section 5 we use a counting argument to give a sufficient condition for the capability of G that depends only on the ranks of $G/Z(G)$ and $[G,G]$. Next, in Section 6, we prove a slight strengthening of the necessary condition first proven in [13], which also depends only on the ranks of $G/Z(G)$ and $[G,G]$.

In Section 7 we characterise the capable groups among the 5-generated p-groups of prime exponent and class at most two. We also give an alternative geometric proof for a key part of the classification in the 4-generated case, since it highlights the way in which the set-up using linear algebra allows us to invoke other tools (in this case, algebraic geometry) to study our problem. We should mention that the approach using linear algebra and geometry has been used before in the study of groups of class two and exponent p; in particular, the work of Brahana [4, 5] exploits geometry in a very striking fashion to classify certain groups of class two and exponent p in terms of points, lines, planes, and spaces in a projective space over $\mathbb{F}_p$. This classification, found in [4], will also play a role in our classification in the 5-generated case, allowing us to deal with certain groups of order p^8 and p^9.

Finally, in Section 8 we discuss some of the limits of our results so far, and state some questions.

Throughout the paper p will be an odd prime, and $\mathbb{F}_p$ will denote the field with p elements. All groups will be written multiplicatively, and the identity element will be denoted by e; if there is danger of ambiguity or confusion, we will use e_G to denote the identity of the group G. The center of G is denoted by $Z(G)$. Recall that if G is a group, and $x, y \in G$, the commutator of x and y is defined to be $[x,y] = x^{-1}y^{-1}xy$; we use x^y to denote the conjugate $y^{-1}xy$. We write commutators left-normed, so that $[x,y,z] = [[x,y],z]$. Given subsets A and B of G we define $[A,B]$ to be the subgroup of G generated by all elements of the form $[a,b]$ with $a \in A$, $b \in B$. The terms of the lower central series of G are defined recursively by letting $G_1 = G$, and $G_{n+1} = [G_n, G]$. A group is *nilpotent of class at most k* if and only if $G_{k+1} = \{e\}$, if and only if $G_k \subset Z(G)$. We usually drop the "at most" clause, it being understood. The class of all nilpotent groups of class at most k is denoted by $\mathfrak{N}_k$. Though we will sometimes use indices to denote elements of a family of groups, it will be clear from context that we are not refering to the terms of the lower central series in those cases.

The following commutator identities are well known, and may be verified by direct calculation:

Proposition 1.2. *Let G be any group. Then for all $x, y, z \in G$,*

(a) $[xy,z] = [x,z][x,z,y][y,z]$.
(b) $[x,yz] = [x,z][z,[y,x]][x,y]$.

(c) $[x,y,z][y,z,x][z,x,y] \equiv e \pmod{G_4}$.
(d) $[x^r, y^s] \equiv [x,y]^{rs}[x,y,x]^{s\binom{r}{2}}[x,y,y]^{r\binom{s}{2}} \pmod{G_4}$.
(e) $[y^r, x^s] \equiv [x,y]^{-rs}[x,y,x]^{-r\binom{s}{2}}[x,y,y]^{-s\binom{r}{2}} \pmod{G_4}$.

Here, $\binom{n}{2} = \frac{n(n-1)}{2}$ *for all integers* n.

As in [17], our starting tool will be the nilpotent product of groups, specifically the 2-nilpotent and 3-nilpotent product of cyclic groups. We restrict Golovin's original definition [9] to the situation we will consider:

Definition 1.3. Let $A_1, \ldots, A_n$ be nilpotent groups of class at most k. The k-nilpotent product of $A_1, \ldots, A_n$, denoted by $A_1 \amalg^{\mathfrak{N}_k} \cdots \amalg^{\mathfrak{N}_k} A_n$, is defined to be the group $G = F/F_{k+1}$, where F is the free product of the A_i, $F = A_1 * \cdots * A_n$, and F_{k+1} is the $(k+1)$-st term of the lower central series of F.

From the definition it is clear that the k-nilpotent product is the coproduct in the variety $\mathfrak{N}_k$, so it will have the usual universal property. Note that if the A_i lie in $\mathfrak{N}_k$, and G is the $(k+1)$-nilpotent product of the A_i, then $G \in \mathfrak{N}_{k+1}$ and G/G_{k+1} is the k-nilpotent product of the A_i.

When we take the k-nilpotent product of cyclic p-groups, with $p \geq k$, we may write each element uniquely as a product of basic commutators of weight at most k on the generators, as shown in in [23, Theorem 3]; see [10, §12.3] for the definition of basic commutators which we will use. In our applications, where each cyclic group is of order p, the order of each basic commutator is likewise equal to p.

Finally, when we say that a group is *k-generated* we mean that it can be generated by k elements, but may in fact need less. If we want to say that it can be generated by k elements, but not by m elements for some $m < k$, we will say that it is *minimally k-generated*, or *minimally generated by k elements*.

2. A canonical witness.

The idea behind our development is the following: given a group G, we attempt to construct a witness for the capability of G; meaning a group H such that $H/Z(H) \cong G$. The relations among the elements of G force in turn relations among the elements of H. When G is not capable, this will manifest itself as undesired relations among the elements of H, forcing certain elements whose image should not be trivial in G to be central in H.

When G is a group of class two, this can be achieved by starting from the relatively free group of class three in an adequate number of generators. However, any further reductions that can be done in the starting potential witness group H will yield dividends of simplicity later on; this is the main goal of the following result; the argument for condition (ii) appears *en passant* in the proof of [13, Theorem 1].

Theorem 2.1. *Let G be a group, generated by $g_1, \ldots, g_n$. If G is capable, then there exists a group H, such that $H/Z(H) \cong G$, and elements $h_1, \ldots, h_n \in H$ which map onto $g_1, \ldots, g_n$, respectively, under the isomorphism such that:*

(i) *$H = \langle h_1, \ldots, h_n \rangle$, and*
(ii) *The order of h_i is the same as the order of g_i, $i = 1, \ldots, n$.*

Moreover, if G is finite, then H can be chosen to be finite as well.

Proof. If G is capable, then there exists a group K such that $K/Z(K) \cong G$; if G is finite, then by [14, Lemma 2.1] we may choose K to be finite.

Pick $k_1,\ldots,k_n \in K$ mapping to $g_1,\ldots,g_n$, respectively, and let M be the subgroup of K generated by $k_1,\ldots,k_n$. Since $MZ(K) = K$, it follows that $Z(M) = M\cap Z(K)$, hence $M/Z(M)\cong K/Z(K)\cong G$. Thus, replacing K by M if necessary, we may assume that K is generated by $k_1,\ldots,k_n$, mapping onto $g_1,\ldots,g_n$, respectively.

Fix $i_0\in\{1,\ldots,n\}$; we show that we can replace K with a group H with generators $h_1,\ldots,h_n$, such that $H/Z(H)\cong G$, where h_i maps to g_i for each i, the order of h_{i_0} is the same as the order of g_{i_0}, and for all $i\neq i_0$, the order of h_i is the same as the order of k_i. Repeating the construction for $i_0=1,\ldots,n$ will yield the desired group H.

Let $C=\langle x\rangle$ be a cyclic group, with x of the same order as k_{i_0}, and consider $K\times C$. Let m be the order of g_{i_0} (set $m=0$ if g_{i_0} is not torsion), and consider the group $M=(K\times C)/\langle (k_{i_0}^m, x^{-m})\rangle$. Since the intersection of the subgroup generated by $(k_{i_0}^m,x^{-m})$ with the commutator subgroup of $K\times C$ is trivial, it follows that if (k,x^a) maps to the center of M, then $[(k,x^a),K\times C]$ must be trivial, so $k\in Z(K)$. That is, $Z(M)$ is the image of $Z(K)\times C$. Therefore, $M/Z(M)\cong (K\times C)/(Z(K)\times C)\cong K/Z(K)\cong G$. Note that the isomorphism identifies the image of (k_j,x^a) with g_j for all j and all integers a.

For $i\neq i_0$, let h_i be the image of (k_i,e) in M; and let h_{i_0} be the image of (k_{i_0},x^{-1}) in M. Finally, let H be the subgroup of M generated by $h_1,\ldots,h_n$. Then $HZ(M)=M$, so once again we have $H/Z(H)\cong M/Z(M)\cong G$, and the map $H\to H/Z(H)\cong G$ sends h_i to g_i. Moreover, the order of h_{i_0} is equal to the order of g_{i_0}. This finishes the construction. $\square$

This result now allows us to give a very specific "canonical witness" to the capability of G.

Theorem 2.2. *Let G be a finite noncyclic group of class at most two and exponent an odd prime p. Let $g_1,\ldots,g_n$ be elements of G that project onto a basis for G^{ab}, and let F be the 3-nilpotent product of n cyclic groups of order p generated by $x_1,\ldots,x_n$, respectively. Let N be the kernel of the morphism $\psi\colon F\to G$ induced by mapping $x_i\mapsto g_i$, $i=1,\ldots,n$. Then G is capable if and only if*

$$G\cong \left(F/[N,F]\right)\big/ Z\left(F/[N,F]\right).$$

Proof. Sufficiency is immediate. For the necessity, assume that G is capable, and let H be the group guaranteed by Theorem 2.1 such that $G\cong H/Z(H)$. Note that H is of class at most three. Let $\theta\colon H/Z(H)\to G$ be an isomorphism that maps $h_iZ(H)$ to g_i.

Since $h_1,\ldots,h_n$ are of order p, there exists a (unique surjective) morphism $\varphi\colon F\to H$ induced by mapping x_i to h_i, $i=1,\ldots,n$. If $\pi\colon H\to H/Z(H)$ is the canonical projection, then we must have $\theta\pi\varphi=\psi$ by the universal property of the coproduct. Thus, $\varphi(N)=\ker(\pi)=Z(H)$, so $[N,F]\subset\ker(\varphi)$, and φ factors through $F/[N,F]$; surjectivity of φ implies that $\varphi(Z(F/[N,F]))\subset Z(H)$, hence $G\cong H/Z(H)$ is a quotient of $(F/[N,F])\big/Z(F/[N,F])$.

On the other hand, $N[N,F]\subseteq Z(F/[N,F])$, so $G\cong F/N=F/N[N,F]$ has $(F/[N,F])\big/Z(F/[N,F])$ as a quotient.

Thus we have that G has $(F/[N,F])\ /\ Z(F/[N,F])$ as a quotient, which in turn has G as a quotient. Since G is finite, the only possibility is that the central quotient of $F/[N,F]$ is isomorphic to G, as claimed. $\square$

Corollary 2.3. *Let G be a finite noncyclic group of class at most two and exponent an odd prime p. Let $g_1,\ldots,g_n$ be elements of G that project onto a basis for G^{ab}, and let F be the 3-nilpotent product of n cyclic groups of order p generated by $x_1,\ldots,x_n$, respectively. Let $\psi\colon F\to G$ be the map induced by sending x_i to g_i, $i=1,\ldots,n$. Finally, let C be the subgroup of F generated by the commutators $[x_j,x_i]$, $1\leq i<j\leq n$. If X is the subgroup of C such that $\ker(\psi)=X\oplus F_3$, then G is capable if and only if $\bigl\{c\in C\,|\,[c,F]\subset[X,F]\bigr\}=X$.*

Proof. Let $N=\ker(\psi)$. By Theorem 2.2, G is capable if and only if G is isomorphic to the central quotient of $F/[N,F]$. Thus, G is capable if and only if the center of $F/[N,F]$ is $N/[N,F]$, and no larger .

An element $h[N,F]\in F/[N,F]$ lies in $Z(F/[N,F])$ if and only if $[h,F]\subseteq[N,F]$. Since G is of exponent p, $F_3\subseteq N\subseteq F_2$ and so $[N,F]=[X,F]\subseteq F_3$. In particular, we deduce that if $h[N,F]$ is central, then h must lie in F_2. Write $h=cf$, with $c\in C$ and $f\in F_3$. Then $[h,F]=[c,F]$, so $h[N,F]$ is central if and only if $[c,F]\subset[X,F]$.

If $\bigl\{c\in C\ \bigm|\ [c,F]\subset[X,F]\bigr\}=X$, then it follows that $h[N,F]$ is central if and only if $h=cf$ with $c\in X$ and $f\in F_3$, which means that $h[N,F]$ is central if and only if $h\in N$. Hence, the center of $F/[N,F]$ is $N/[N,F]$, and G is capable.

Conversely, assume that G is capable. Then the center of $F/[N,F]$ is equal to $N/[N,F]$. Therefore, $X\subseteq\bigl\{c\in C\ \bigm|\ [c,f]\subset[X,F]\bigr\}\subseteq N\cap C=X$, giving equality and establishing the corollary. $\square$

One advantage of the description just given is the following: both F_2 and F_3 are vector spaces over $\mathbb{F}_p$, and the maps $[-,f]\colon F_2\to F_3$ are linear transformations for each $f\in F$; hence, the condition just described can be restated in terms of vector spaces, subspaces, and linear transformations. While all the work can still be done at the level of groups and commutators, the author, at any rate, found it easier to think in terms of linear algebra. In addition, once the problem has been cast into linear algebra terms, there is a host of tools (such as geometric arguments) that can be brought to bear on the issue.

We will discuss this translation and more results on capability below, after a brief abstract interlude on linear algebra.

3. Some linear algebra.

We set aside groups and capability temporarily to describe the general construction that we will use in our analysis.

Definition 3.1. Let V and W be vector spaces over the same field, and let $\{\ell_i\}_{i\in I}$ be a nonempty family of linear transformations from V to W. Given a subspace X of V, let X^* be the subspace of W defined by:

$$X^*=\mathrm{span}\bigl(\ell_i(X)\,|\,i\in I\bigr).$$

Given a subspace Y of W, let Y^* be the subspace of V defined by:

$$Y^*=\bigcap_{i\in I}\ell_i^{-1}(Y).$$

It will be clear from context whether we are talking about subspaces of V or W.

It is clear that $X \subset X' \Rightarrow X^* \subset X'^*$ for all subspaces X and X' of V, and likewise $Y \subset Y' \Rightarrow Y^* \subset Y'^*$ for all subspaces Y, Y' of W.

Theorem 3.2. *Let V and W be vector spaces over the same field and let $\{\ell_i\}_{i\in I}$ be a nonempty family of linear transformations from V to W. The operator on subspaces of V defined by $X \mapsto X^{**}$ is a closure operator; that is, it is increasing, isotone, and idempotent. Moreover, $(X^{**})^* = (X^*)^{**} = X^*$ for all subspaces X of V.*

Proof. Since $\ell_i(X) \subseteq X^*$ for all i, it follows that $X \subset X^{**}$, so the operator is increasing. If $X \subset X'$, then $X^* \subset X'^*$, hence $X^{**} \subset X'^{**}$, and the operator is isotone. The equality of $(X^{**})^*$ and $(X^*)^{**}$ is immediate. Since $X \subset X^{**}$, we have $X^* \subset (X^{**})^*$. And by construction $\ell_i(X^{**}) \subset X^*$ for each i, so $(X^{**})^* \subset X^*$ giving equality.

Thus, $(X^{**})^{**} = (X^{***})^* = (X^*)^* = X^{**}$, so the operator is idempotent, finishing the proof. □

It may be worth noting that while this closure operator is algebraic (the closure of a subspace X is the union of the closures of all finitely generated subspaces X' contained in X), it is not topological (in general, the closure of the subspace generated by X and X' is not equal to the subspace generated by X^{**} and X'^{**}).

The dual result holds for subspaces of W:

Theorem 3.3. *Let V and W be vector spaces over the same field, and let $\{\ell_i\}_{i\in I}$ be a nonempty family of linear transformations from V to W. The operator on subspaces of W defined by $Y \mapsto Y^{**}$ is an interior operator; that is, it is decreasing, isotone, and idempotent. Moreover, $(Y^{**})^* = (Y^*)^{**} = Y^*$ for all subspaces Y of W.*

Proof. That the operator is isotone follows as it did in the previous theorem. Since $\ell_i(Y^*) \subset Y$ for each i, it follows that $Y^{**} \subset Y$, showing the operator is decreasing. Set $Z = Y^{**}$; by construction, $Y^* \subset \ell_i^{-1}(Z)$ for each i, so $Y^* \subset Z^*$. Therefore, $Z = Y^{**} \subset Z^{**} \subset Z$. Thus $Z = Z^{**}$, proving the operator is idempotent.

Again, the equality of $(Y^{**})^*$ and $(Y^*)^{**}$ is immediate. To finish we only need to show that Y^* is a closed subspace of V. From Theorem 3.2 we know that $Y^* \subset (Y^*)^{**}$; since $Y^{**} \subset Y$, it follows that $(Y^*)^{**} = (Y^{**})^* \subset Y^*$, giving equality. □

As above, the interior operator is algebraic but in general not topological. However, we do have the following result:

Lemma 3.4. *Let V and W be vector spaces over the same field, and let $\{\ell_i\}_{i\in I}$ be a nonempty family of linear transformations from V to W. If A and B are subspaces of V, then $(A+B)^* = A^* + B^*$.*

Proof. Since A and B are contained in $A+B$, we have $A^*, B^* \subseteq (A+B)^*$, and therefore $A^* + B^* \subseteq (A+B)^*$. Conversely, if $\mathbf{w} \in (A+B)^*$, then we can express $\mathbf{w}$ as a linear combination $\mathbf{w} = \ell_{i_1}(a_1+b_1) + \cdots + \ell_{i_k}(a_k+b_k)$, with $a_i \in A$, $b_i \in B$. This gives $\mathbf{w} = \Big(\ell_{i_1}(a_1) + \cdots + \ell_{i_k}(a_k)\Big) + \Big(\ell_{i_1}(b_1) + \cdots + \ell_{i_k}(b_k)\Big) \in A^* + B^*$, proving the equality. □

The lemma implies that $(A \oplus B)^* = A^* + B^*$; however, in general we cannot replace the sum on the right hand side with a direct sum.

Given a family of linear transformations $\{\ell_i \colon V \to W\}_{i\in I}$, we will say a subspace X of V is $\{\ell_i\}_{i\in I}$-*closed* (or simply *closed* if the family is understood from context) if and only if $X = X^{**}$. Likewise, we will say a subspace Y of W is $\{\ell_i\}_{i\in I}$-*open* (or simply *open*) if and only if $Y = Y^{**}$.

It is easy to verify that the closure and interior operators determined by a nonempty family $\{\ell_i\}_{i\in I}$ of linear transformations is the same as the closure operator determined by the subspace of $\mathcal{L}(V,W)$ (the space of all linear transformations from V to W) spanned by the ℓ_i. Likewise, the following observation is straightforward:

Proposition 3.5. *Let V and W be vector spaces, and X be a subspace of V. Let $\{\ell_i\}_{i\in I}$ be a nonempty family of linear transformations from V to W, and let $\psi \in \mathrm{Aut}(V)$. If we use ** to denote the $\{\ell_i\}_{i\in I}$ closure operator, then the $\{\ell_i\psi^{-1}\}_{i\in I}$-closure of $\psi(X)$ is $\psi(X^{**})$. In particular, X is $\{\ell_i\}$-closed if and only if $\psi(X)$ is $\{\ell_i\psi^{-1}\}$-closed. If $\{\ell_i\}$ and $\{\ell_i\psi^{-1}\}$ span the same subspace of $\mathcal{L}(V,W)$, then X is closed if and only if $\psi(X)$ is closed.*

Back to capability. To tie the construction above back to the problem of capability, we introduce specific vector spaces and linear transformations based on Corollary 2.3. We fix an odd prime p throughout.

Definition 3.6. Let $n > 1$. We let $U(n)$ denote a vector space over $\mathbb{F}_p$ of dimension n. We let $V(n)$ denote the vector space $U(n)\wedge U(n)$ of dimension $\binom{n}{2}$. Finally, we let $W(n)$ be the quotient $(V(n)\otimes U(n))/J$, where J is the subspace spanned by all elements of the form

$$(\mathbf{a}\wedge\mathbf{b})\otimes\mathbf{c} + (\mathbf{b}\wedge\mathbf{c})\otimes\mathbf{a} + (\mathbf{c}\wedge\mathbf{a})\otimes\mathbf{b},$$

with $\mathbf{a},\mathbf{b},\mathbf{c} \in U$. The vector space $W(n)$ has dimension $2\binom{n+1}{3}$. If there is no danger of ambiguity and n is understood from context, we will simply write U, V, and W to refer to these vector spaces.

The following notation will be used only in the context where there is a single specified basis for U, to avoid any possibility of ambiguity:

Definition 3.7. Let $n > 1$, and let U, V, and W be as above. If $u_1,\ldots,u_n$ is a given basis for U, and i, j, and k are integers, $1 \leq i,j,k \leq n$, then we let v_{ji} denote the vector $u_j\wedge u_i$ of V, and w_{jik} denote vector of W which is the image of $v_{ji}\otimes u_k$. The "prefered basis" for V (relative to $u_1,\ldots,u_n$) will consist of the vectors v_{ji} with $1 \leq i < j \leq n$. The "prefered basis" for W will consist of the vectors w_{jik} with $1 \leq i < j \leq n$ and $i \leq k \leq n$.

To specify our closure and interior operators on V and W, we define the following family of linear transformations:

Definition 3.8. Let $n > 1$. We embed U into $\mathcal{L}(V,W)$ as follows: given $\mathbf{u}\in U$ and $\mathbf{v}\in V$, we let $\varphi_{\mathbf{u}}(\mathbf{v}) = \overline{\mathbf{v}\otimes\mathbf{u}}$, where $\overline{\mathbf{x}}$ denotes the image in W of a vector $\mathbf{x}\in V\otimes U$. If $u_1,\ldots,u_n$ is a given basis for U and i is an integer, $1\leq i\leq n$, then we will use φ_i to denote the linear transformation φ_{u_i}.

The closure operator we will consider is determined by the family $\{\varphi_{\mathbf{u}} \mid \mathbf{u}\in U\}$. As noted above, if $u_1,\ldots,u_n$ is a basis for U, then this closure operator is also determined by the family $\{\varphi_1,\ldots,\varphi_n\}$.

Going back to the problem of capability, let F be the 3-nilpotent product of cyclic groups of order p generated by $x_1,\ldots,x_n$. We can identify F_2 with $V\oplus W$ by identifying v_{ji} with $[x_j,x_i]$ and w_{jik} with $[x_j,x_i,x_k]$; this also identifies W with F_3.

Let G be a noncyclic group of class at most two and exponent p, and let $g_1,\ldots,g_n$ be elements of G that project onto a basis for G^{ab}. If we let $\psi\colon F\to G$ be the map induced by mapping $x_i\mapsto g_i$ and $N=\ker(\psi)$, then as above we can write $N=X\oplus F_3$, where X is a subgroup of $C=\langle [x_j,x_i] \bigm| 1\leq i<j\leq n\rangle$. Thus, we can identify X with a subspace of V by identifying the latter with the subgroup C; abusing notation somewhat, we call this subspace X as well.

Theorem 3.9. *Let G, F, C, and X be as in the preceding two paragraphs. Then G is capable if and only if X is $\{\varphi_{\mathbf{u}}\,|\,\mathbf{u}\in U\}$-closed.*

Proof. We know that G is capable if and only if $\bigl\{c\in C\,|\,[c,F]\subset[X,F]\bigr\}=X$. Identifying C with V and F_3 with W, note that φ_i is a map from C to F_3, corresponding to $[-,x_i]$. Thus, $X^*\subseteq W$ corresponds to $[X,F]\subseteq F_3$, and X^{**} corresponds to the set $\bigl\{c\in C\,|\,[c,F]\subset[X,F]\bigr\}$. Therefore, G is capable if and only if

$$X=\bigl\{\mathbf{v}\in V \bigm| \varphi_{\mathbf{u}}(\mathbf{v})\in X^* \text{ for all } \mathbf{u}\in U\bigr\}=X^{**},$$

as claimed. $\square$

In other words, the closure operator codifies exactly the condition we want to check to test the capability of G. Thus the question *"What n-generated p-groups of class two and exponent p are capable?"* is equivalent to the question *"What subspaces of $V(n)$ are $\{\varphi_{\mathbf{u}}\,|\,\mathbf{u}\in U\}$-closed?"*

Of course, different subspaces may yield isomorphic groups. In particular, if we let $\mathrm{GL}(n,p)$ act on U, then this action induces an action of $\mathrm{GL}(n,p)$ on $V=U\wedge U$; if X and X' are on the same orbit relative to this action, then the groups G and H that correspond to X and X', respectively, are isomorphic. By Proposition 3.5 the closures of X and X' will also be in the same orbit under the action and G will be capable if and only if H is capable.

Also of interest is the description of the closure of X when G is not capable. It is clear that the quotient of G determined by X^{**} is the largest quotient of G that is capable. That is, X^{**}/X is isomorphic to the *epicenter* of G, the smallest normal subgroup $N\triangleleft G$ such that G/N is capable. In most cases where a subspace X is not closed, therefore, we will attempt to give an explicit description of X^{**} rather than simply prove X is not closed.

The following explicit descriptions of the linear transformations $\varphi_{\mathbf{u}}$, relative to a given basis, will also be useful and are straightforward:

Lemma 3.10. *Fix $n>1$, let $u_1,\ldots,u_n$ be a basis for U, and let v_{ji}, w_{jik} be the corresponding bases for V and W. For all integers i, j, and k, $1\leq i<j\leq n$, $1\leq k\leq n$, the image of v_{ji} under φ_k in terms of the prefered basis of W is:*

$$\varphi_k(v_{ji})=\begin{cases} w_{jik} & \text{if } k\geq i,\\ w_{jki}-w_{ikj} & \text{if } k<i.\end{cases}$$

4. Basic applications.

In this section, we obtain some consequences of our set-up so far. We assume throughout that we have a specified "preferred basis" $\{u_i\}$ for U, from which we obtain the corresponding basis $\{v_{ji} \,|\, 1 \leq i < j \leq n\}$ for V, and likewise the basis $\{w_{jik} \,|\, 1 \leq i < j \leq n, i \leq k < n\}$ for W.

The following observations follow immediately from the definitions:

Lemma 4.1. *Fix $n > 1$, and let k be an integer, $1 \leq k \leq n$.*

(i) *φ_k is one-to-one, and $W = \langle \varphi_1(V), \ldots, \varphi_n(V) \rangle$.*
(ii) *The trivial and total subspaces of V are closed.*
(iii) *The trivial and total subspaces of W are open.*

Definition 4.2. Let i, j, k be integers, $1 \leq i < j \leq n$, $i \leq k \leq n$. We let $\pi_{ji}\colon V \to \langle v_{ji} \rangle$ and $\pi_{jik}\colon W \to \langle w_{jik} \rangle$ be the canonical projections.

Lemma 4.3. *Let $\mathbf{w} \in \varphi_k(V)$. If $\pi_{rst}(\mathbf{w}) \neq \mathbf{0}$, with $1 \leq s < r \leq n$, $s \leq t \leq n$, then $s \leq k \leq t$, and at most one of the inequalities is strict.*

Proof. It is enough to prove the result for $\mathbf{w}$ an element of a basis of $\varphi_k(V)$. Such a basis is given by the vectors w_{jik} with $1 \leq i < j \leq n$, $i \leq k \leq n$, and the vectors $w_{jki} - w_{ikj}$ with $1 \leq i < j \leq n$ and $1 \leq k < i$. Considering these basis vectors, we see that the first class has $r = j$, $s = i$, $t = k$, so $s \leq k = t$. The second class of vectors will yield either $r = j$, $s = k$, $t = i$, with $s = k < t$; or else $r = i$, $s = k$, $t = j$, with $s = k < t$. This proves the lemma. □

Lemma 4.4. *Let i, j be integers, $1 \leq i < j \leq n$, and r an integer such that $1 \leq r \leq n$. For $\mathbf{v} \in V$, $\pi_{jij}(\varphi_r(\mathbf{v})) \neq \mathbf{0}$ if and only if $\pi_{ji}(\mathbf{v}) \neq \mathbf{0}$ and $r = j$. Likewise, $\pi_{jii}(\varphi_r(\mathbf{v})) \neq \mathbf{0}$ if and only if $\pi_{ji}(\mathbf{v}) \neq \mathbf{0}$ and $r = i$.*

Proof. The vectors w_{jij} occurs in the image of a φ_r exactly when $r = j$ and it is applied a vector with nontrivial π_{ji} projection. Thus, if $\pi_{jij}(\mathbf{v}) \neq \mathbf{0}$ then $\pi_{ji}(\mathbf{v}) \neq \mathbf{0}$. The converse is immediate, and the case of π_{jii} is settled in the same manner. □

Lemma 4.5. *Fix i, j, $1 \leq i < j \leq n$. If $\pi_{ji}(X) = \{\mathbf{0}\}$, then $\pi_{ji}(X^{**}) = \{\mathbf{0}\}$.*

Proof. Since $\pi_{ji}(X) = \{\mathbf{0}\}$, it follows that $\pi_{jii}(X^*) = \{\mathbf{0}\}$ by Lemma 4.3. Therefore, if $\mathbf{v} \in V$ has $\pi_{ji}(\mathbf{v}) \neq \mathbf{0}$ then $\varphi_i(\mathbf{v}) \notin X^*$, hence $\mathbf{v} \notin X^{**}$. Thus, $\pi_{ji}(X^{**}) = \{\mathbf{0}\}$, as claimed. □

These lemmas suffice to establish a result of Ellis [7, Prop. 9], which appears as Corollary 4.7 below.

Theorem 4.6. *If X is a coordinate subspace relative to a basis for U (that is, there is a basis $u_1, \ldots, u_n$ such that X is generated by a subset of $\{v_{ji} \,|\, 1 \leq i < j \leq n\}$), then X is closed.*

Proof. Suppose $S \subseteq \{v_{ji} \,|\, 1 \leq i < j \leq n\}$ is such that $X = \langle S \rangle$. By the previous Lemma, we have that $X^{**} \subseteq \langle S \rangle$; therefore, $\langle S \rangle = X \subseteq X^{**} \subseteq \langle S \rangle = X$, and so $X = X^{**}$. □

Corollary 4.7 ([7, Prop. 9])**.** *Let G be a group of class two and exponent p, and let $x_1, \ldots, x_n$ be elements of G that project onto a basis for $G/Z(G)$. If the nontrivial commutators of the form $[x_j, x_i]$, $1 \leq i < j \leq n$, are distinct and form a basis for $[G, G]$, then G is capable.*

Proof. Such a G corresponds to an X that is a coordinate subspace of V, so capability follows from Theorem 4.6. □

The big, the small, and the mixed. The following definition and proposition will be needed below.

Definition 4.8. Let n be an integer greater than 1, and i an integer, $1 \leq i \leq n$. We define $\Pi_i\colon V \to \langle v_{i,1},\ldots,v_{i,i-1},v_{i+1,i},\ldots,v_{n,i}\rangle$ to be the canonical projection.

Proposition 4.9. *Let $n>1$ and i be an integer, $1\leq i\leq n$. Let W_i be the subspace of W spanned by the basis vectors w_{rst}, $1\leq s<r\leq n$, $s\leq t\leq n$, such that exactly one of r, s, and t is equal to i. If X is a subspace of V such that $\Pi_i(X)=\{\mathbf{0}\}$, then $X^*\cap W_i = \varphi_i(X)$ and X is closed.*

Proof. That $\varphi_i(X)$ is contained in W_i follows because $\Pi_i(X)$ is trivial. Since the subspace $\langle \varphi_j(X)\,|\,j\neq i\rangle$ is contained in the subspace spanned by basis vectors w_{rst} in which none of r,s,t are equal to i, we have $X^* = \varphi_i(X)\oplus\langle\varphi_j(X)\,|\,j\neq i\rangle$ and the equality of intersection follows. To show X is closed, let $\mathbf{v}\in X^{**}$. By Lemma 4.5, we know that $\Pi_i(\mathbf{v})=\mathbf{0}$, and so $\varphi_i(\mathbf{v})$ lies in $X^*\cap W_i = \varphi_i(X)$. Since φ_i is one-to-one, we deduce that $\mathbf{v}\in X$. Thus, X is closed. $\square$

Fix a basis $u_1,\ldots,u_n$ for U. Given r, $1\leq r<n$, we can divide these basis vectors into "small" and "large", according to whether their indices are less than or equal to r, or strictly larger than r, respectively. From this, we obtain a similar partition of the corresponding basis vectors v_{ji}, $1\leq i<j\leq n$ of V, and w_{jik}, $1\leq i<j\leq n$, $i\leq k\leq n$ for W. Namely, we write $V=V_s\oplus V_m\oplus V_{\ell}$, $W=W_s\oplus W_{ms}\oplus W_{m\ell}\oplus W_{\ell}$, where:

$$\begin{aligned}
V_s &= \left\langle v_{ji}\ \middle|\ 1\leq i<j\leq r\right\rangle,\\
V_m &= \left\langle v_{ji}\ \middle|\ 1\leq i\leq r<j\leq n\right\rangle,\\
V_{\ell} &= \left\langle v_{ji}\ \middle|\ r<i<j\leq n\right\rangle,\\
W_s &= \left\langle w_{jik}\ \middle|\ 1\leq i<j\leq r, i\leq k\leq r\right\rangle,\\
W_{ms} &= \left\langle w_{jik}\ \middle|\ 1\leq i<j\leq n, i\leq k\leq n,\ \text{either } j\leq r \text{ or } k\leq r\text{, but not both}\right\rangle,\\
W_{m\ell} &= \left\langle w_{jik}\ \middle|\ 1\leq i\leq r<j,k\leq n\right\rangle,\\
W_{\ell} &= \left\langle w_{jik}\ \middle|\ r<i<j\leq n, i\leq k\leq n\right\rangle.
\end{aligned}$$

We refer informally to V_s as the "small part" of V, and its elements as "small vectors;" V_{ℓ} is the "large part" and contains the "large vectors;" and V_m will be called the "mixed part" while its elements will be refered to as "mixed vectors." A similar informal convention will be followed with W, calling W_s the "small part," W_{ℓ} the "large part," W_{ms} the "mixed-small part," and $W_{m\ell}$ the "mixed-large part" of W.

Lemma 4.10. *Notation as in the previous paragraph. If $n>1$ and r is an integer, $1\leq r<n$, then:*

(i) $V_s^*\subseteq W_s\oplus W_{ms}$.
(ii) $V_{\ell}^*\subseteq W_{m\ell}\oplus W_{\ell}$.
(iii) $V_m^* = W_{ms}\oplus W_{m\ell}$.

Proof. Note that the indices involved in the image of $\varphi_k(v_{ji})$ are i, j, and k. Thus, if both i and j are small (resp. large), then all images are either small or mixed-small (resp. mixed large or large); and if i is small and j is large, then all images are mixed. This proves (i) and (ii), and also proves that V_m^* is contained in $W_{ms}\oplus W_{m\ell}$. To finish the proof of (iii), suppose that w_{jik} is one of the generators of $W_{ms}\oplus W_{m\ell}$, as described above. Note that we must have $i\leq r$ in either case. Then $w_{jik}=\varphi_k(v_{ji})$. If $j>r$, then $v_{ji}\in V_m$, so $w_{jik}\in V_m^*$. If, on the other hand, $j\leq r$, then we must have $k>r$ since w_{jik} is either mixed-small or mixed-large. Then we know that $w_{kij}\in V_m^*$ by the immediately preceding argument. Also, $v_{kj}\in V_m$, hence $\varphi_i(v_{kj}) = w_{kij}-w_{jik}\in V_m^*$. Since $w_{kij}\in V_m^*$, we deduce that $w_{jik}\in V_m^*$ as well, and this finishes the proof of (iii). □

In the following theorem, $\mathrm{cl}_s(X_s)$ is meant to stand for the "small closure of X_s"; that is, the $\{\varphi_i\}_{i=1}^r$-closure of X_s; likewise, $\mathrm{cl}_\ell(X_\ell)$ is the "large closure of X_ℓ."

Theorem 4.11. *Let $n>1$, and let r be an integer, $1\leq r<n$, as above. Suppose that X_s is a subspace of V_s, and X_ℓ is a subspace of V_ℓ. Then:*

(i) $(X_s\oplus X_\ell)^* = X_s^*\oplus X_\ell^*$.
(ii) $(X_s\oplus V_m\oplus X_\ell)^* = \langle \varphi_i(X_s)\,|\,1\leq i\leq r\rangle \oplus W_{ms}\oplus W_{m\ell}\oplus\langle\varphi_i(X_\ell)\,|\,r<i\leq n\rangle$.
(iii) *$X_s\oplus X_\ell$ is closed.*
(iv) *If $\mathrm{cl}_s(X_s)$ is the $\{\varphi_i\}_{i=1}^r$-closure of X_s and $\mathrm{cl}_\ell(X_\ell)$ is the $\{\varphi_i\}_{i=r+1}^n$-closure of X_ℓ, then $(X_s\oplus V_m\oplus X_\ell)^{**} = \mathrm{cl}_s(X_s)\oplus V_m\oplus \mathrm{cl}_\ell(X_\ell)$. In particular, the subspace $X_s\oplus V_m\oplus X_\ell$ is closed if and only if X_s is $\{\varphi_i\}_{i=1}^r$-closed and X_ℓ is $\{\varphi_i\}_{i=r+1}^n$-closed.*

Proof. Part (i) follows from Lemma 3.4 and from Lemma 4.10(i) and (ii).

To prove (ii), note that by Lemmas 3.4 and 4.10, we have:

$$\begin{aligned}(X_s\oplus V_m\oplus X_\ell)^* &= X_s^* + V_m^* + X_\ell^*\\ &= \langle\varphi_i(X_s)\,|\,1\leq i\leq n\rangle + W_{ms}+W_{m\ell}+\langle\varphi_i(X_\ell)\,|\,1\leq i\leq n\rangle\\ &= \langle\varphi_i(X_s)\,|\,1\leq i\leq r\rangle + W_{ms}+W_{m\ell}+\langle\varphi_i(X_\ell)\,|\,1\leq i\leq r\rangle.\end{aligned}$$

Now simply observe that the first summand is contained in W_s and the last in W_ℓ to deduce that the sum is direct.

Moving on to (iii), by Lemma 4.5, we know that $(X_s\oplus X_\ell)^{**}\subseteq V_s\oplus V_\ell$. Let $\mathbf{v}_s+\mathbf{v}_\ell$ be an element of $(X_s\oplus X_\ell)^{**}$, with $\mathbf{v}_s$ a small vector, and $\mathbf{v}_\ell$ a large vector. Then for each i, $\varphi_i(\mathbf{v}_s+\mathbf{v}_\ell)\in X_s^*\oplus X_\ell^*$. Thus, we must have $\varphi_i(\mathbf{v}_s)\in X_s^*$ and $\varphi_i(\mathbf{v}_\ell)\in X_\ell^*$ for each i, so $\mathbf{v}_s\in X_s^{**}$ and $\mathbf{v}_\ell\in X_\ell^{**}$. Thus, $(X_s\oplus X_\ell)^{**}\subseteq X_s^{**}\oplus X_\ell^{**}$, and the reverse inclusion follows because the closure operator is isotonic. It is then enough to show that each of X_s and X_ℓ are closed, and since $\Pi_1(X_\ell)=\Pi_n(X_s)=\{\mathbf{0}\}$, this follows from Proposition 4.9.

Finally, for (iv), note that if $j>r$, then $\varphi_j(V_s)\subseteq W_{sm}\subseteq V_m^*$, so $\mathrm{cl}_s(X_s)$ is contained in the closure; similarly, $\mathrm{cl}_\ell(X_\ell)$ is contained in the closure, so we always have $\mathrm{cl}_s(X_s)\oplus V_m\oplus\mathrm{cl}_\ell(X_\ell)\subseteq (X_s\oplus V_m\oplus X_\ell)^{**}$.

Let $\mathbf{v}=\mathbf{v}_s+\mathbf{v}_m+\mathbf{v}_\ell\in(X_s\oplus V_m\oplus X_\ell)^{**}$, with $\mathbf{v}_s\in V_s$, $\mathbf{v}_\ell\in V_\ell$, and $\mathbf{v}_m\in V_m$. Since V_m is contained in the closure, $\mathbf{v}$ is in the closure if and only if $\mathbf{v}_s+\mathbf{v}_\ell$ is in the closure. We further claim that $\mathbf{v_s}+\mathbf{v}_\ell$ is in the closure if and only if each of $\mathbf{v}_s$ and $\mathbf{v}_\ell$ are in the closure. One implication is immediate. For the converse, suppose that $\mathbf{v}_s+\mathbf{v}_\ell$ is in the closure, and $i\leq r$. Then by (ii) we have:

$$\varphi_i(\mathbf{v}_s)+\varphi_i(\mathbf{v}_\ell)\in\langle\varphi_j(X_s)\,|\,j\leq r\rangle\oplus W_{ms}\oplus W_{m\ell}\oplus\langle\varphi_j(X_\ell)\,|\,r<j\leq n\rangle.$$

In particular, $\varphi_i(\mathbf{v}_s) \in \langle \varphi_j(X_s) \,|\, 1 \leq j \leq r\rangle$. Since V_s is contained in $\varphi_j^{-1}(W_{ms})$ for all $j > r$, we conclude that $\mathbf{v}_s$ lies in the closure of $X_s \oplus V_m \oplus X_\ell$, and hence so does $\mathbf{v}_\ell$. This proves the claim.

Finally, observe as above that $\mathbf{v}_s$ lies in the closure if and only if $\varphi_i(\mathbf{v}_s)$ lies in $\langle \varphi_j(X_s) \,|\, 1 \leq j \leq r\rangle$ for $i = 1, \ldots, r$, if and only if $\mathbf{v}_s$ lies in $\mathrm{cl}_s(X_s)$; and similarly that $\mathbf{v}_\ell$ lies in the closure if and only if it lies in $\mathrm{cl}_\ell(X_\ell)$. Thus, the closure of $X_s \oplus V_m \oplus X_\ell$ is equal to $\mathrm{cl}_s(X_s) \oplus V_m \oplus \mathrm{cl}_\ell(X_\ell)$. This proves the theorem. □

The theorem gives the following two interesting corollaries:

Corollary 4.12. *Let G_1 and G_2 be any two nontrivial groups of class at most two and exponent an odd prime p. Then $G = G_1 \amalg^{\mathfrak{N}_2} G_2$ is capable.*

Proof. If G_1 is minimally r-generated, and G_2 is minimally s-generated, then G is minimally $n = r + s$ generated. If we number the generators of G_1 as $g_1, \ldots, g_r$, and those of G_2 as $g_{r+1}, \ldots, g_n$, then the subspace of V corresponding to G will be of the form $X_s \oplus X_\ell$, where $X_s \subseteq V_s$, $X_\ell \subseteq V_\ell$; namely, X_s corresponds to G_1, and X_ℓ corresponds to G_2. By Theorem 4.11(iii), this subspace is always closed. □

Corollary 4.13. *Let G_1 and G_2 be two finite p-groups of class at most two and exponent p. Then $G_1 \oplus G_2$ is capable if and only if each G_i is either nontrivial cyclic or capable.*

Proof. Proceeding as above, note that the subspace of V corresponding to $G_1 \oplus G_2$ is equal to $X_s \oplus V_m \oplus X_\ell$, so by Theorem 4.11(iv), this subspace is closed if and only if X_s is $\{\varphi_i\}_{i=1}^{r}$ closed and X_ℓ is $\{\varphi_i\}_{i=r+1}^{n}$-closed. For noncyclic G_i this is equivalent to being capable, while for cyclic G_i the closure conditions are trivially met. □

In turn, this yields the following important consequences:

Theorem 4.14. *Let G be a p-group of class at most two and exponent p. Then $G \oplus C_p$ is capable if and only if G is cyclic of order p or capable.*

Corollary 4.15. *Let G be a p-group of class exactly two and exponent p. If we write $G = K \oplus C_p^r$, where $r \geq 0$ is an integer and K is a group of class two satisfying $Z(K) = [K, K]$, then G is capable if and only if K is capable.*

Note that any group of class exactly two and exponent p can be written in the form specified by this corollary.

Amalgamated direct products and amalgamated coproducts. We saw in Corollary 4.12 that if we take two nontrivial groups of class two and exponent p, then their coproduct (in this variety) will always be capable, while the capability of a direct sum depends on the factors.

We will now deal with two similar constructions, the direct product with amalgamation and the coproduct with amalgamation. The first construction includes central products (see for example [15, Section 2.2]) but is more general.

Definition 4.16. Let G and K be two groups, and let H be a subgroup of $Z(G)$. Let $\phi\colon H \to Z(K)$ be an embedding. The *amalgamated direct product of G and K (along ϕ)* is the group $G \times_\phi K$ given by

$$G \times_\phi K = \frac{G \times K}{\{(h, \phi(h)^{-1}) \,|\, h \in H\}}.$$

The maps sending $g \mapsto \overline{(g,e)}$ and $k \mapsto \overline{(e,k)}$ embed copies of G and of K into $G \times_\phi K$, respectively, and the intersection of these images is exactly H (identified with $\phi(H)$). When $H = Z(G)$ and ϕ is an isomorphism, the construction is called the *central product* of G and K in [15], where it is denoted by $G \circ K$. All extra-special p-groups other than those of order p^3 may be constructed as central products of smaller extra-special groups.

The following result was inspired by doing an automated brute force search for non-closed subspaces X of dimensions seven and eight when $n = 5$. It was performed with the computer algebra system GAP [8]. I was able to find many examples, and by examining them was led to the result below. The statement of the linear algebra theorem is somewhat complicated, but it leads to a straightforward group-theoretic corollary: if G and K are groups of class two and exponent p, H is a nontrivial subgroup of $[G,G]$, and ϕ embeds H into $[K,K]$, then the amalgamated direct product $G \times_\phi K$ is not capable.

Theorem 4.17. *Let $n > 3$, and let r be an integer, $2 \leq r \leq n-2$. Let X_s and X_ℓ be subspaces of V_s and V_ℓ, respectively, and let H be a nontrivial subspace of V_s such that $H \cap X_s = \{\mathbf{0}\}$. Let $\phi\colon H \to V_\ell$ be an embedding such that $\phi(H) \cap X_\ell = \{\mathbf{0}\}$. Finally, let X be the subspace $X = X_s \oplus X_\ell \oplus V_m \oplus \{h - \phi(h) \,|\, h \in H\}$. Then the closure of X^{**} is the direct sum of the $\{\varphi_i\}_{i=1}^{r}$-closure of $X_s \oplus H$, the $\{\varphi_i\}_{i=r+1}^{n}$-closure of $X_\ell \oplus \phi(H)$, and V_m. In particular, X is not closed.*

Proof. Note that by Lemma 4.10(iii), we have $W_{ms} \oplus W_{m\ell} = V_m^* \subseteq X^*$. Next, note that $X \cap H = X \cap \varphi(H) = \{\mathbf{0}\}$.

We claim that $H^* \subseteq X^*$, and therefore that $H \subseteq H^{**} \subset X^{**}$. Indeed, let $h \in H$, and let k be an integer, $1 \leq k \leq n$. If $k \leq r$, then $\varphi_k(\phi(h)) \in W_{m\ell}$ (since $\phi(h) \in V_\ell$), so $\varphi_k(h) = \varphi_k(h - \phi(h)) + \varphi_k(\phi(h)) \in X^* + W_{m\ell} = X^*$. And if $r < k \leq n$, then $\varphi_k(h) \in W_{ms} \subseteq X^*$. Thus, $\varphi_k(h) \in X^*$ for $k = 1, \ldots, n$, hence $h \in X^{**}$. This proves that $H^* \subseteq X^*$, hence $H \subseteq H^{**} \subseteq X^{**}$.

Thus, the closure of X contains $X_s \oplus H \oplus X_\ell \oplus \phi(H) \oplus V_m$. The description of the closure of X now follows as in the proof of Theorem 4.11(iv). We conclude that X is not closed, because H is nontrivial, $H \cap X = \{\mathbf{0}\}$, yet $H \subseteq X^{**}$. $\square$

Corollary 4.18 (cf. [13, Proposition 1])**.** *Let G_1 and G_2 be two nonabelian groups of class two and exponent p, let H be a subgroup of $[G_1, G_1]$, and let $\phi\colon H \to [G_2, G_2]$ be an embedding. If G is the amalgamated direct product $G = G_1 \times_\phi G_2$, then G is capable if and only if $H = \{e\}$ and both G_1 and G_2 are capable.*

Proof. Let r be the rank of $G_1^{\rm ab}$, s the rank of $G_2^{\rm ab}$, and $n = r + s$. Since G_1 and G_2 are nonabelian, we must have $2 \leq r \leq n-2$. The subspace X corresponding to $G_1 \times G_2$ is of the form $X_s \oplus V_m \oplus X_\ell$, with X_s and X_ℓ determined by G_1 and G_2, respectively. Abusing notation, the subgroup H can be made to correspond to a subspace H of V_s with $H \cap X_s = \mathbf{0}$, and ϕ induces a linear transformation $\phi\colon H \to V_\ell$ which can also be chosen to have $\phi(H) \cap X_\ell = \{\mathbf{0}\}$. The subspace of V corresponding to $G_1 \times_\phi G_2$ is then equal to $X = X_s \oplus X_\ell \oplus V_m \oplus \{h - \varphi(h) \,|\, h \in H\}$. If $H = \{\mathbf{0}\}$, then we are in the situation of Corollary 4.13. And if $H \neq \{\mathbf{0}\}$, then X is not closed by Theorem 4.17. This proves the result. $\square$

The following is of course well-known, and can be proven using other methods:

Corollary 4.19. *Let G be an extra-special p-group. Then G is capable if and only if it is of order p^3 and exponent p.*

Proof. If G is not of exponent p, then it is generated by elements of order p and one element of order p^2 (see for example [15, Theorem 2.2.10]) and therefore is not capable by [17, Theorem 3.12]. So we may assume G is of exponent p. If G is of order p^{2n+1} with $n > 1$, then it is isomorphic to a direct product with amalgamation of the extra-special p-group of order p^3 and exponent p, and the extra-special p-group of order p^{2n-1} and exponent p, identifying their commutator subgroups; as such, it is not capable by Corollary 4.18 above. The extra-special group of order p^3 and exponent p is closed the coproduct of two cyclic groups of order p, and thus is capable by Corollary 4.12. □

We move now to the case of the coproduct with amalgamation.

Definition 4.20. Let G and K be two groups of class at most two and exponent p. Let H be a subgroup of $[G,G]$, and let $\phi\colon H \to [K,K]$ be an embedding. The *amalgamated coproduct of G and K (along ϕ)* is the group $G \amalg_{\phi}^{\mathfrak{N}_2} K$ given by:

$$G \amalg_{\phi}^{\mathfrak{N}_2} K = \frac{G \amalg^{\mathfrak{N}_2} K}{\{h\phi(h)^{-1} \,|\, h \in H\}}.$$

Note that the elements h and $\phi(h)^{-1}$ are central, so the subset given above is in fact a normal subgroup. Again, it is easy to

In general, if $G, K \in \mathfrak{N}_2$, H is an arbitrary subgroup of G, and $\phi\colon H \to K$ an embedding, then the coproduct with amalgamation $G \amalg_{\phi}^{\mathfrak{N}_2} K$ may or may not contain copies of G and K; and even if it does contain copies of G and K, their intersection may be strictly larger than H. There are necessary and sufficient conditions for each of the situations, given in [16, 20, 21]. When G and K are of exponent p and the identified subgroups are contained in the corresponding commutator subgroups, however, $G \amalg_{\phi}^{\mathfrak{N}_2} K$ always contains copies of G and K, and these copies intersect exactly at H.

As before, the statement of the linear algebra result is somewhat complex.

Theorem 4.21. *Let $n > 3$ and let r be an integer, $2 \leq r \leq n-2$. Let X_s and X_ℓ be subspaces of V_s and V_ℓ, respectively, and let H be a subspace of V_s such that $H \cap X_s = \{\mathbf{0}\}$. Let $\phi\colon H \to V_\ell$ be an embedding such that $\phi(H) \cap X_\ell = \{\mathbf{0}\}$. Finally, let X be the subspace of V given by $X = X_s \oplus X_\ell \oplus \{h - \phi(h) \,|\, h \in H\}$. If $\mathrm{cl}_s(X_s)$ is the $\{\varphi_i\}_{i=1}^{r}$-closure of X_s and $\mathrm{cl}_\ell(X_\ell)$ is the $\{\varphi_i\}_{i=r+1}^{n}$-closure of X_ℓ, then the closure of X is given by:*

$$X^{**} = X \oplus \bigl\{h \in H \bigm| h \in \mathrm{cl}_s(X_s) \text{ and } \phi(h) \in \mathrm{cl}_\ell(X_\ell)\bigr\}.$$

In particular, X is closed if and only if

$$\bigl\{h \in H \bigm| h \in \mathrm{cl}_s(X_s) \text{ and } \phi(h) \in \mathrm{cl}_\ell(X_\ell)\bigr\} = \{\mathbf{0}\}.$$

Proof. Note that $X \subset (X_s \oplus H) \oplus (X_\ell \oplus \phi(H))$; the latter subspace is closed by Theorem 4.11(iii), so it contains X^{**}. Thus, to describe the closure of X it is enough to determine exactly which $h \in H$ lie in the closure.

Suppose that $h \in H \cap X^{**}$. Then $\varphi_i(h) \in X^*$ for $i = 1, \ldots, n$; fix $i \leq r$. Then we know that there exist $x_1, \ldots, x_n \in X_s$, $y_1, \ldots, y_n \in X_\ell$, and $h_1, \ldots, h_n \in H$ such that

$$\varphi_i(h) = \varphi_1(x_1 + y_1 + h_1 - \phi(h_1)) + \cdots + \varphi_n(x_n + y_n + h_n - \phi(h_n)).$$

By looking at the W_s, W_{ms}, $W_{m\ell}$, and W_ℓ components, we deduce that:

$$\begin{aligned}
\varphi_i(h) &= \varphi_1(x_1+h_1)+\cdots+\varphi_r(x_r+h_r),\\
\mathbf{0} &= \varphi_{r+1}(x_{r+1}+h_{r+1})+\cdots+\varphi_n(x_n+h_n),\\
\mathbf{0} &= \varphi_1(y_1-\phi(h_1))+\cdots+\varphi_r(y_r-\phi(h_r)),\\
\mathbf{0} &= \varphi_{r+1}(y_{r+1}-\phi(h_{r+1}))+\cdots+\varphi_n(y_n-\phi(h_n)).
\end{aligned}$$

Now, $\varphi_{r+1}(x_{r+1}+h_{r+1})$ is the only term in the expression that lies in W_{ms} and involves generators w_{jik} with one of j or k (in fact, k) equal to $r+1$. Thus, we must have $\varphi_{r+1}(x_{r+1}+h_{r+1})=\mathbf{0}$, which in turn gives $x_{r+1}=h_{r+1}=\phi(h_{r+1})=\mathbf{0}$, since φ_{r+1} and ϕ are embeddings and $H\cap X_s=\mathbf{0}$. Similarly, we deduce that $x_{r+1}=x_{r+2}=\cdots=x_n=h_{r+1}=h_{r+2}=\cdots=h_n=\mathbf{0}$. We are then left with $\varphi_{r+1}(y_{r+1})+\cdots+\varphi_n(y_n)=\mathbf{0}$ as the only equation involving $y_{r+1},\ldots,y_n$, and so we may also assume $y_{r+1}=\cdots=y_n=\mathbf{0}$.

Consider now $\varphi_1(y_1-\phi(h_1))+\cdots+\varphi_r(y_r-\phi(h_r))$. Again, $\varphi_1(y_1-\phi(h_1))$ is the only term in the expression that lies in $W_{m\ell}$ and involves generators w_{jik} with $i=1$. Thus, we must have $\varphi_1(y_1-\phi(h_1))=\mathbf{0}$, and as above we deduce from this that $y_1=\phi(h_1)=\mathbf{0}$ since φ_1 and ϕ are embeddings and $\phi(H)\cap X_\ell=\mathbf{0}$. Similarly, we obtain $y_1=y_2=\cdots=y_r=h_1=\cdots=h_r=\mathbf{0}$. And so we obtain $\varphi_i(h)=\varphi_1(x_1)+\cdots+\varphi_r(x_r)$ for some vectors $x_1,\ldots,x_r\in X_s$. That is, if $h\in H\cap X^{**}$, then h is $\mathrm{cl}_s(X_s)$.

A symmetric argument, considering $\varphi_i(\phi(h))$ with $i>r$ yields that if $\varphi(h)$ lies in X^{**}, then $\varphi(h)$ must lie in $\mathrm{cl}_\ell(X_\ell)$. Since $h-\phi(h)\in X^{**}$ for all $h\in H$, we obtain that a necessary condition for $h\in H$ to lie in the closure is that $h\in\mathrm{cl}_s(X_s)$ and $\phi(h)\in\mathrm{cl}_\ell(X_\ell)$. The theorem will be proven if we can show that this condition is also sufficient.

Suppose that $h\in H\cap\mathrm{cl}_s(X_s)$ is such that $\phi(h)$ lies in $\mathrm{cl}_\ell(X_\ell)$. Then each of $\varphi_1(h),\ldots,\varphi_r(h)$, $\varphi_{r+1}(\phi(h)),\ldots,\varphi_n(\phi(h))$ lie in X^*. Since $\varphi_i(h-\phi(h))\in X^*$ for all i, we deduce that $\varphi_i(h)\in X^*$ for all i, so $h\in X^{**}$. This proves that the condition given is also sufficient, and so proves the theorem. □

Recall that $Z^{**}(G)$, the epicenter of G, is the smallest normal subgroup N of G such that G/N is capable.

Corollary 4.22. *Let G and K be two nonabelian groups of class two and exponent p. Let H be a nontrivial subgroup of $[G,G]$, and let $\phi\colon H\to[K,K]$ be an embedding. Then $G\amalg_\phi^{\mathfrak{N}_2}K$ is capable if and only if*

$$\{h\in H\,|\,h\in Z^{**}(G)\text{ and }\phi(h)\in Z^{**}(K)\}=\{e\}.$$

In particular, if either G or K are capable, then so is $G\amalg_\phi^{\mathfrak{N}_2}K$.

Remark 4.23. It is perhaps interesting to note that when we passed from coproducts and direct products to their amalgamated counterparts, a kind of reversal took place. The coproduct of two nontrivial groups in our class is always capable, while the capability of the direct product depends on the capability of the two factors. However, when we amalgamate nontrivial subgroups of the commutators, then the amalgamated direct product which is *never* capable, while it is in the amalgamated coproduct that capability depends on the capability of the two groups (and the precise choice of H).

5. Dimension Counting.

In this section we will establish a sufficient condition for the capability of a p-group G of exponent p and class at most two that depends only on the ranks of $G/Z(G)$ and $[G,G]$. The idea is the following: given a subspace X of V, we will find a lower bound for the dimension of X^* in terms of n and the dimension of X. If all subspaces X' of V that properly contain X yield subspaces X'^* of dimension strictly larger than $\dim(X^*)$, then it will follow that X must be closed since $X^* = (X^{**})^*$. In order to establish these bounds, we will consider the images $\varphi_1(X)$, $\varphi_2(X),\ldots,\varphi_n(X)$; since each φ_i is one-to-one, the dimension of X^* will depend on how much "overlap" there can be among these subspaces of W.

Lemma 5.1. *Fix $n>1$, and let i and j be integers, $1\leq i<j\leq n$. Then $\varphi_i(V)\cap\varphi_j(V)=\{\mathbf{0}\}$.*

Proof. Let $\varphi_i(\mathbf{v})\in\varphi_j(V)$, and assume that $\pi_{sr}(\mathbf{v})\neq\mathbf{0}$, $1\leq r<s\leq n$. If $r\leq i$, then $\pi_{sri}(\varphi_i(\mathbf{v}))\neq\mathbf{0}$, and since $\varphi_i(\mathbf{v})\in\varphi_j(V)$, Lemma 4.3 implies $r\leq j\leq i$, contradicting the choice of i and j. If $i<r$, then $\pi_{sir}(\varphi_i(\mathbf{v}))\neq\mathbf{0}$. By Lemma 4.3, we must have $i<j=r$. We also have $\pi_{ris}(\varphi_i(\mathbf{v}))\neq\mathbf{0}$, and since $\varphi_i(\mathbf{v})\in\varphi_j(V)$, this time we deduce $i<j=s$. But then we have $j=r=s$, and this is impossible. This contradiction arises from assuming $\pi_{sr}(\mathbf{v})\neq\mathbf{0}$ for some $1\leq r<s\leq n$, hence $\mathbf{v}=\mathbf{0}$. □

Lemma 5.2. *Fix $n>1$ and $r\leq n$. Let $i_1,\ldots,i_r$ be pairwise distinct integers, $1\leq i_1,\ldots,i_r\leq n$. Then $\varphi_{i_1}^{-1}\bigl(\bigl\langle\varphi_{i_2}(V),\ldots,\varphi_{i_r}(V)\bigr\rangle\bigr)$ is of dimension $\binom{r-1}{2}$, with basis given by the vectors v_{ab}, with $a,b\in\{i_2,\ldots,i_r\}$, $b<a$. In particular, the intersection $\varphi_{i_1}(V)\cap\bigl\langle\varphi_{i_2}(V),\ldots,\varphi_{i_r}(V)\bigr\rangle$ has a basis made up of vectors of the form w_{abi_1} with $a,b\in\{i_2,\ldots,i_r\}$, $b<a$ and $b<i_1$; and vectors of the form $w_{ai_1b}-w_{bi_1a}$, with $a,b\in\{i_2,\ldots,i_r\}$, $i_1<b<a$.*

Proof. By Proposition 3.5, it is enough to consider the case where $i_1=1$. Let A denote the pullback described in the statement.

Given $a,b\in\{i_2,\ldots,i_r\}$, $a>b$, we have $v_{ab}\in A$:

$$\varphi_{i_1}(v_{ab})=w_{ai_1b}-w_{bi_1a}=\varphi_b(v_{ai_1})-\varphi_a(v_{bi_1})\in\langle\varphi_{i_2}(V),\ldots,\varphi_{i_r}(V)\rangle.$$

Conversely, let $\mathbf{v}\in A$, and let a,b be integers, $1\leq b<a\leq n$, such that $\pi_{ab}(\mathbf{v})\neq\mathbf{0}$. We can write

$$\varphi_{i_1}(\mathbf{v})=\varphi_{i_2}(\mathbf{v}_2)+\cdots+\varphi_{i_r}(\mathbf{v}_r).$$

Since $i_1=1$, $\pi_{a1b}(\varphi_{i_1}(\mathbf{v}))=-\pi_{b1a}(\varphi_{i_1}(\mathbf{v}))\neq\mathbf{0}$, and therefore we must have $\pi_{a1b}(\varphi_{i_j}(\mathbf{v}_j))\neq\mathbf{0}$ for some $j\geq 2$. This implies $1\leq i_j\leq b$, with at most one inequality strict by Lemma 4.3. Since $1=i_1\neq i_j$, we have $i_j=b$. Considering π_{b1a} instead, we deduce that $a=i_k$ for some $k\geq 2$, so $a,b\in\{i_2,\ldots,i_r\}$. Therefore, $A\subseteq\langle v_{ab}\,|\,a,b\in\{i_2,\ldots,i_r\},a>b\rangle$. This proves equality.

Since the vectors described are linearly independent, they form a basis. Mapping them via φ_{i_1}, which is one-to-one, proves the final clause. □

Corollary 5.3. *Let $n>1$, $r\leq n$, and let $1\leq i_1<i_2<\cdots<i_r\leq n$ be integers. Then*

$$\dim\left(\langle\varphi_{i_1}(V),\ldots,\varphi_{i_r}(V)\rangle\right)=r\binom{n}{2}-\binom{r}{3}.$$

Proof. For simplicity, let $Y = \langle \varphi_{i_1}(V), \ldots, \varphi_{i_r}(V)\rangle$. We have:

$$\begin{aligned}\dim(Y) &= \left(\sum_{k=1}^{r}\dim(\varphi_{i_k}(V))\right) - \left(\sum_{k=2}^{r}\dim\Big(\varphi_{i_k}(V)\cap\big\langle\varphi_{i_1}(V),\ldots,\varphi_{i_{k-1}}(V)\big\rangle\Big)\right)\\ &= r\binom{n}{2} - \left(\sum_{k=2}^{r}\binom{k-1}{2}\right) = r\binom{n}{2} - \binom{r}{3},\end{aligned}$$

as claimed. □

Definition 5.4. Fix $n > 1$. We define $\Phi\colon V^n \to W$ to be

$$\Phi(\mathbf{v}_1, \ldots, \mathbf{v}_n) = \varphi_1(\mathbf{v}_1) + \cdots + \varphi_n(\mathbf{v}_n).$$

If there is danger of ambiguity, we use Φ_n to denote the map associated to the spaces corresponding to the particular choice of n.

Note that if X is a subspace of V, then $\Phi(X^n) = X^*$.

Proposition 5.5. *The kernel of Φ is of dimension $\binom{n}{3}$. A basis for $\ker(\Phi)$ can be determined as follows: each choice of integers a, b, c, $1 \leq a < b < c \leq n$, gives an element $(\mathbf{v}_1, \ldots, \mathbf{v}_n) \in V^n$ of the basis, with:*

$$\mathbf{v}_i = \begin{cases} v_{cb} & \text{if } i = a,\\ -v_{ca} & \text{if } i = b,\\ v_{ba} & \text{if } i = c,\\ \mathbf{0} & \text{otherwise.}\end{cases}$$

Proof. Denote the element corresponding to $a < b < c$ by $\mathbf{v}_{(abc)}$. Note that $\mathbf{v}_{(abc)}$ is in $\ker(\Phi)$:

$$\Phi(\mathbf{v}_{(abc)}) = \varphi_a(v_{cb}) + \varphi_b(-v_{ca}) + \varphi_c(v_{ba}) = w_{cab} - w_{bac} - w_{cab} + w_{bac} = \mathbf{0}.$$

Since Φ is surjective, $\dim(W) = n\dim(V) - \dim(\ker(\Phi))$, hence

$$\dim(\ker(\Phi)) = n\binom{n}{2} - 2\binom{n+1}{3} = \binom{n}{3},$$

so the proposition will be established in full if we prove that the elements $\mathbf{v}_{(abc)}$ of V^n are linearly independent.

Let $\sum \beta_{abc}\mathbf{v}_{abc} = (\mathbf{0}, \ldots, \mathbf{0})$ be a linear combination equal to zero. If we look at the ith coordinate of these n-tuples, we have:

$$\sum_{1\leq r<s<i\leq n}\beta_{rsi}v_{sr} - \sum_{1\leq r<i<s\leq n}\beta_{ris}v_{sr} + \sum_{1\leq i<r<s\leq n}\beta_{irs}v_{sr} = \mathbf{0}.$$

Each basis vector v_{sr} occurs only once. Thus, if $i \in \{a, b, c\}$, then $\beta_{abc} = 0$. This holds for each choice of i, hence $\beta_{abc} = 0$ for all choices of a, b, c. This proves the $\mathbf{v}_{(abc)}$ are linearly independent. □

Theorem 5.6. *Let $(\mathbf{v}_1, \ldots, \mathbf{v}_n) \in \ker(\Phi)$. Write*

$$\mathbf{v}_k = \sum_{1\leq i<j\leq n}\alpha_{ji}^{(k)}v_{ji},$$

(i) *If $i = k$ or $j = k$, then $\alpha_{ji}^{(k)} = 0$; i.e., $\Pi_k(\mathbf{v}_k) = \mathbf{0}$.*
(ii) *If $1 \leq a < b < c \leq n$, then $\alpha_{ba}^{(c)} = \alpha_{cb}^{(a)} = -\alpha_{ca}^{(b)}$.*

(iii) *Fix i,j, $1\leq i<j\leq n$. Then*

$$\begin{aligned}\Pi_i(\mathbf{v}_j) &= \sum_{r=1}^{i-1}\left(-\alpha_{jr}^{(i)}\right)v_{ir} + \sum_{r=i+1}^{j-1}\alpha_{jr}^{(i)}v_{ri} + \sum_{r=j+1}^{n}\left(-\alpha_{rj}^{(i)}\right)v_{ri},\\ \Pi_j(\mathbf{v}_i) &= \sum_{r=1}^{i-1}\left(-\alpha_{ir}^{(j)}\right)v_{jr} + \sum_{r=i+1}^{j-1}\alpha_{ri}^{(j)}v_{jr} + \sum_{r=j+1}^{n}\left(-\alpha_{ri}^{(j)}\right)v_{rj}.\end{aligned}$$

Proof. Part (i) holds for the basis elements described in Proposition 5.5, hence holds for all vectors in the kernel. For part (ii), note that if $1\leq a<b<c\leq n$, then

$$\begin{aligned}\pi_{bac}\big(\varphi_1(\mathbf{v}_1)+\cdots+\varphi_n(\mathbf{v}_n)\big) &= \left(\alpha_{ba}^{(c)}-\alpha_{cb}^{(a)}\right)w_{bac},\\ \pi_{cab}\big(\varphi_1(\mathbf{v}_1)+\cdots+\varphi_n(\mathbf{v}_n)\big) &= \left(\alpha_{ca}^{(b)}+\alpha_{cb}^{(a)}\right)w_{cab}.\end{aligned}$$

Since both are equal to zero, we deduce that $\alpha_{ba}^{(c)}=\alpha_{cb}^{(a)}$ and $\alpha_{ca}^{(b)}=-\alpha_{cb}^{(a)}$. Finally, for (iii), we know that $\Pi_i(\mathbf{v}_i)=\Pi_j(\mathbf{v}_j)=\mathbf{0}$ from (i), so we can write:

$$\begin{aligned}\Pi_i(\mathbf{v}_j) &= \sum_{r=1}^{i-1}\alpha_{ir}^{(j)}v_{ir} + \sum_{r=i+1}^{j-1}\alpha_{ri}^{(j)}v_{ri} + \sum_{r=j+1}^{n}\alpha_{ri}^{(j)}v_{ri},\\ \Pi_j(\mathbf{v}_i) &= \sum_{r=1}^{i-1}\alpha_{jr}^{(i)}v_{jr} + \sum_{r=i+1}^{j-1}\alpha_{jr}^{(i)}v_{jr} + \sum_{r=j+1}^{n}\alpha_{rj}^{(i)}v_{rj},\end{aligned}$$

and applying (ii) gives the desired identities. □

Corollary 5.7. *Let $\mathbf{v}\in\ker(\Phi)$. If $\Pi_j(\mathbf{v}_i)=\mathbf{0}$, then $\Pi_i(\mathbf{v}_j)=\mathbf{0}$. In particular, if $\mathbf{v}_i=\mathbf{0}$, then $\Pi_i(\mathbf{v}_j)=\mathbf{0}$ for all j.*

Proof. The second assertion follows immediately from the first. The first assertion is trivial if $i=j$; for $i\neq j$, then $\alpha_{jr}^{(i)}=0$ for all $r<j$ and $\alpha_{rj}^{(i)}=0$ for $j<r$, so by Theorem 5.6(iii) it follows that $\Pi_i(\mathbf{v}_j)=\mathbf{0}$. □

Corollary 5.8. *Let $\mathbf{v}\in\ker(\Phi)$, $\mathbf{v}\neq(\mathbf{0},\ldots,\mathbf{0})$. If $\mathbf{v}=(\mathbf{v}_1,\ldots,\mathbf{v}_n)$ then the dimension of $\langle\mathbf{v}_1,\ldots,\mathbf{v}_n\rangle$ is at least 3.*

Proof. Write

$$\mathbf{v}=\sum_{1\leq a<b<c\leq n}\beta_{abc}\mathbf{v}_{(abc)}.$$

Fix a,b,c such that $1\leq a<b<c\leq n$, $\beta_{abc}\neq 0$. We claim that $\mathbf{v}_a$, $\mathbf{v}_b$, and $\mathbf{v}_c$ are linearly independent. Indeed, note that $\Pi_a(\mathbf{v}_a)=\Pi_b(\mathbf{v}_b)=\Pi_c(\mathbf{v}_c)=\mathbf{0}$, and $\pi_{cb}(\mathbf{v}_a)\neq\mathbf{0}$. Therefore, if $\alpha_a\mathbf{v}_a+\alpha_b\mathbf{v}_b+\alpha_c\mathbf{v}_c=\mathbf{0}$, then we must have $\alpha_a=0$. A symmetric argument looking at π_{ca} shows that $\alpha_b=0$, and considering π_{ba} shows that $\alpha_c=0$. □

Corollary 5.9 (Prop. 4.6 in [18])**.** *Fix $n>1$, and let X be a subspace of V. If $\dim(X)=1$, then $\dim(X^*)=n$; if $\dim(X)=2$, then $\dim(X^*)=2n$.*

Proof. We prove the contrapositive. Since $\dim(X^*)=n\dim(X)-\dim(X^n\cap\ker(\Phi))$, if $\dim(X^*)<n\dim(X)$, then $X^n\cap\ker(\Phi)\neq\{\mathbf{0}\}$.

Let $\mathbf{v}=(\mathbf{v}_1,\ldots,\mathbf{v}_n)\in X^n\cap\ker(\Phi)$, $\mathbf{v}\neq\mathbf{0}$. Then $\mathbf{v}_i\in X$ for $i=1,\ldots,n$, so by Corollary 5.8, $\dim(X)\geq 3$, as claimed. □

We now proceed along the lines outlined at the beginning of the section. We first formalize the observation made there.

Proposition 5.10. *Let $X < V$. Assume that for all subspaces Y of V, if Y properly contains X then Y^* properly contains X^*. Then $X = X^{**}$.*

Proof. If X^{**} properly contains X, then X^{***} would properly contain X^*. But $X^{***} = X^*$, a contradiction. $\square$

We are therefore searching for a function $f(k,n)$, defined for k with $1 \leq k \leq \binom{n}{2}$, such that for all $X < V(n)$, if $\dim(X) = k$ then $\dim(X^*) \geq nk - f(k,n)$. This is given by:

$$f(k,n) = \max\Big\{\dim(X^n \cap \ker(\Phi_n)) \;\Big|\; X < V, \dim(X) = k\Big\}.$$

Our objective in this section is to find an expression for $f(k,n)$ in terms of k and n; in fact, it turns out that the value is independent of n. The main workhorse in our calculations will Lemma 5.12 below. The idea is to find $\dim(X^n \cap \ker(\Phi_n))$ by examining the "partial intersections"; namely, the intersections of the form

$$\Big\langle (\mathbf{0},\ldots \mathbf{0},\mathbf{v}_i,\mathbf{v}_{i+1},\ldots,\mathbf{v}_n) \;\Big|\; \mathbf{v}_j \in X \Big\rangle \bigcap \ker(\Phi_n),$$

as i ranges from 1 to $n-2$ (when $i = n-1$ or $i = n$, the intersection is trivial by Corollary 5.8). For a fixed i, we can consider the subspace of X consisting of all vectors $\mathbf{v}_i$ which can be "completed" to an element of $\ker(\Phi)$ by taking and n-tuple with $i-1$ copies of $\mathbf{0}$, followed by $\mathbf{v}_i$, followed by some vectors in X; this is the same as considering the pullbacks $X \cap \varphi_i^{-1}\left(\langle \varphi_{i+1}(X),\ldots,\varphi_n(X)\rangle\right)$. It is easy to verify that the sum of the dimensions of these pullbacks is equal to the dimension of $X^n \cap \ker(\Phi_n)$. We will first use the dimension of these pullbacks to establish a lower bound for the dimension of X; then we will turn around and use these calculations to give an upper bound for the dimension of the pullbacks in terms of the dimension of X.

Making the bounds as precise as possible, however, requires one to keep track of a lot of information; this in turn requires the use of multiple indices and subindices in the proof, for which I apologize in advance. To illustrate the ideas and help the reader navigate through the proof, we will first present an illustration. This is not an example in the sense of a specific X, but rather a run-through the main part of the analysis we will perform below, but with specific values for some of the indices and some of the variables to make it more concrete.

Example 5.11. Set $n = 6$, and let X be a subspace of V. We will be interested in bounding above the dimension of Z_i in terms of $\dim(X)$, where

$$Z_i = X \cap \varphi_i^{-1}\Big(\langle \varphi_{i+1}(X),\ldots,\varphi_6(X)\rangle\Big);$$

i.e., Z_i consists of all $\mathbf{v} \in X$ for which there exist $\mathbf{v}_{i+1},\ldots,\mathbf{v}_6$ in X such that

$$(\mathbf{0},\ldots,\mathbf{0},\mathbf{v},\mathbf{v}_{i+1},\ldots,\mathbf{v}_6) \in X^6 \cap \ker(\Phi_6).$$

To do this, we will obtain a lower bound for $\dim(X)$ in terms of $\dim(Z_i)$. To further fix ideas, set $i = 2$. Note that by Theorem 5.6(i) and (ii), we must have $\Pi_1(Z_2) = \Pi_2(Z_2) = \mathbf{0}$. Order all pairs (j,i) lexicographically from right to left, so $(j,i) < (b,a)$ if and only if $i < a$, or $i = a$ and $j < b$. Doing row reduction, we can find a basis $\mathbf{v}_{1,2}$, $\mathbf{v}_{2,2},\ldots,\mathbf{v}_{k,2}$ for Z_2 (the second index refers to the fact that these vectors are in the second component of an element of $\ker(\Phi_6)$), satisfying that the "leading pair" (smallest nonzero component) of each is strictly smaller than that of its successors, and all other vectors have zero component for that pair. For example, suppose that $\dim(Z_2) = 4$, and that the basis has the form:

$$\begin{aligned} \mathbf{v}_{1,2} &= v_{43} + \alpha_1 v_{53} + \alpha_2 v_{64}, & \mathbf{v}_{3,2} &= v_{54} + \gamma v_{64},\\ \mathbf{v}_{2,2} &= v_{63} + \beta v_{64}, & \mathbf{v}_{4,2} &= v_{65}, \end{aligned}$$

for some coefficients $\alpha_1, \alpha_2, \beta, \gamma \in \mathbb{F}_p$. We know there exist vectors $\mathbf{v}_{i,3}$, $\mathbf{v}_{i,4}$, $\mathbf{v}_{i,5}$, $\mathbf{v}_{i,6}$ such that $(\mathbf{0}, \mathbf{v}_{i,2}, \mathbf{v}_{i,3}, \mathbf{v}_{i,4}, \mathbf{v}_{i,5}, \mathbf{v}_{i,6}) \in X^6 \cap \ker(\Phi_6)$ for $i = 1,2,3,4$. Naturally, X contains all twenty vectors, but there will normally be some linear dependencies between them: some may even be equal to $\mathbf{0}$. We want to extract, in some systematic manner, a subset that we can guarantee is linearly independent. First let us consider the information we can obtain about these vectors from our knowledge of the vectors $\mathbf{v}_{i,2}$.

Since $(\mathbf{0}, \mathbf{v}_{i,2}, \mathbf{v}_{i,3}, \mathbf{v}_{i,4}, \mathbf{v}_{i,5}, \mathbf{v}_{i,6})$ lies in $\ker(\Phi)$, we can use Theorem 5.6(iii) to describe the Π_i-image of each vector $\mathbf{v}_{i,j}$, where $i \leq 2$ and $j > 2$. The Π_1-image must be trivial, and for the Π_2 image we obtain the following:

$$\begin{aligned} \Pi_2(\mathbf{v}_{1,3}) &= v_{42} + \alpha_1 v_{52}, & \Pi_2(\mathbf{v}_{2,3}) &= v_{62},\\ \Pi_2(\mathbf{v}_{1,4}) &= -v_{32} + \alpha_2 v_{62}, & \Pi_2(\mathbf{v}_{2,4}) &= \beta v_{62},\\ \Pi_2(\mathbf{v}_{1,5}) &= -\alpha_1 v_{32}, & \Pi_2(\mathbf{v}_{2,5}) &= \mathbf{0},\\ \Pi_2(\mathbf{v}_{1,6}) &= -\alpha_2 v_{42}. & \Pi_2(\mathbf{v}_{2,6}) &= -v_{32} - \beta v_{42}. \end{aligned}$$

$$\begin{aligned} \Pi_2(\mathbf{v}_{3,3}) &= \mathbf{0}, & \Pi_2(\mathbf{v}_{4,3}) &= \mathbf{0},\\ \Pi_2(\mathbf{v}_{3,4}) &= v_{52} + \gamma v_{62}, & \Pi_2(\mathbf{v}_{4,4}) &= \mathbf{0},\\ \Pi_2(\mathbf{v}_{3,5}) &= -v_{42}, & \Pi_2(\mathbf{v}_{4,5}) &= v_{62},\\ \Pi_2(\mathbf{v}_{3,6}) &= -\gamma v_{42}. & \Pi_2(\mathbf{v}_{4,6}) &= -v_{52}. \end{aligned}$$

One way to obtain these without too much confusion is as follows: to find $\Pi_2(\mathbf{v}_{j,k})$, go through the expression for $\mathbf{v}_{j,2}$ replacing all indices k by 2, remembering that $v_{ab} = -v_{ba}$. Any v_{ba} in which neither a nor b are equal to k are simply removed.

To extract systematically a set of linearly independent vectors, we proceed in the following manner: consider all the pairs which are leading components of the basis vectors $\mathbf{v}_{i,2}$; in this case, $(4,3)$, $(6,3)$, $(5,4)$, and $(6,5)$. The individual indices that occur are 3, 4, 5, and 6. For each of them, we identify the smallest pair in which it occurs. Thus, 3 first occurs in pair number one, as does 4. The index 5 first occurs in pair number three, and 6 first occurs in pair number two.

Since the first pair in which 3 appears is the *first* pair (corresponding to the first basis vectors $\mathbf{v}_{1,2}$, $(4,3)$, where it is paired with 4, we will select the vector $\mathbf{v}_{1,4}$; this vector has first nontrivial component $(3,2)$. The next index is 4, again in the *first* pair, paired with 3; so this time we select $\mathbf{v}_{1,3}$. This has nontrivial $(4,2)$ copmonent, and trivial (j,i) component for all $(j,i) < (4,2)$.

The next index is 5, which first occurs in the third pair (corresponding to $\mathbf{v}_{3,2}$) paired with 4. So we select $\mathbf{v}_{3,4}$, a vector with trivial (j,i) component for all $(j,i) < (5,2)$, and nontrivial $(5,2)$ component. Next we go to the index 6, that first occurs in second pair together with 3; so we select the vector $\mathbf{v}_{2,3}$, a vector with nontrivial $(6,2)$ component, and trivial (j,i) component for all $(j,i) < (6,2)$.

In summary, we want to consider our original basis vectors $\mathbf{v}_{1,2}$, $\mathbf{v}_{2,2}$, $\mathbf{v}_{2,3}$, and $\mathbf{v}_{2,4}$, plus the vectors we have selected based on the location of the indices, to wit the vectors $\mathbf{v}_{1,4}, \mathbf{v}_{1,3}, \mathbf{v}_{3,4}, \mathbf{v}_{2,3}$ corresponding, respectively, to the indices 3, 4, 5, and 6. The choices we have made ensure that the Π_2-images of these latter four vectors are linearly independent, and so the vectors themselves must be linearly independent. Since $\Pi_2(Z_2) = \mathbf{0}$, the full collection of eight vectors is linearly independent, and so we can conclude that X must have dimension at least 8.

What is more, note that none of the four vectors $\mathbf{v}_{1,4}$, $\mathbf{v}_{1,3}$, $\mathbf{v}_{3,4}$, and $\mathbf{v}_{2,3}$ will occur in a similar analysis involving Z_3 (or more generally Z_i with $i > 2$): when performing a similar analysis, all vectors will have trivial Π_i-image when $i < 3$, and these vectors have nontrivial Π_2-image. Note as well that the number of indices, in this case 4, must satisfy $\dim(Z_2) \leq \binom{4}{2}$, since we need to be able to obtain at least $\dim(Z_2)$ pairs out of the indices that occur.

Thus we have seen that if $\dim(Z_2) = 4$, then $\dim(X) \geq 8$. If we move on to Z_3, we will obtain new vectors that must lie in X; while the vectors in the basis for Z_2 may again occur in that analysis, the vectors $\mathbf{v}_{1,4}$, $\mathbf{v}_{1,3}$, $\mathbf{v}_{3,4}$, and $\mathbf{v}_{2,3}$ will not, and so by keeping track of them we can give an even better lower bound for $\dim(X)$. $\Box$

What ensures that this process will work the way we want is how we choose the vectors of the basis and the vectors that "correspond" to each index. The former count towards the value of $\dim(X^n \cap \ker(\Phi_n))$, while the latter may be removed from consideration when we move on to Z_{i+1}. This is all done in generality in the proof of the following promised lemma:

Lemma 5.12. *Fix $n > 1$, and let X be a subspace of V. For each i, $1 \leq i \leq n$, let*

$$Z_i = X \cap \varphi_i^{-1}\Big(\langle \varphi_{i+1}(X), \ldots, \varphi_n(X)\rangle\Big);$$

i.e., Z_i consists of all $\mathbf{v} \in X$ for which there exist $\mathbf{v}_{i+1}, \ldots, \mathbf{v}_n$ in X such that

$$(\mathbf{0}, \ldots, \mathbf{0}, \mathbf{v}, \mathbf{v}_{i+1}, \ldots, \mathbf{v}_n) \in X^n \cap \ker(\Phi).$$

If $\dim\big(X \cap \langle v_{sr} \,|\, i \leq r < s \leq n\rangle\big) = d_i$ and $\dim(Z_i) = r_i$, then $r_i \leq \binom{d_i - r_i}{2}$. Morevoer, if s_i is the smallest positive integer such that $r_i \leq \binom{s_i}{2}$, then we must have $d_{i+1} \leq d_i - s_i$.

Proof. Fix i_0, $1 \leq i_0 \leq n$. For simplicity, write $r = r_{i_0}$. By Theorem 5.6, if $\mathbf{v} \in Z_{i_0}$ then $\Pi_i(\mathbf{v}) = \mathbf{0}$ for all $i \leq i_0$.

Let $\mathbf{v}_{1i_0}, \ldots, \mathbf{v}_{ri_0}$ be a basis for Z_{i_0}. We will modify it as follows:

Order all pairs (j,i), $i_0 < i < j \leq n$ by letting $(j,i) < (b,a)$ if and only if $i < a$ or $i = a$ and $j < b$ (lexicographically from right to left). Let (j_1, i_1) be the smallest pair for which $\pi_{j_1 i_1}(\mathbf{v}_{ki_0}) \neq \mathbf{0}$ for some k, $1 \leq k \leq r$. Reordering if necessary we may assume $k = 1$. Replacing $\mathbf{v}_{1i_0}$ with a scalar multiple of itself and adding adequate multiples to the remaining $\mathbf{v}_{ki_0}$ if necessary we may also assume that

$$\pi_{j_1 i_1}(\mathbf{v}_{ki_0}) = \begin{cases} v_{j_1 i_1} & \text{if } k = 1;\\ \mathbf{0} & \text{if } k \neq 1.\end{cases}$$

Let (j_2, i_2) be the smallest pair for which $\pi_{j_2 i_2}(\mathbf{v}_{ki_0}) \neq \mathbf{0}$ for some k, $2 \leq k \leq r$. Again we may assume $k = 2$, and that

$$\pi_{j_2 i_2}(\mathbf{v}_{ki_0}) = \begin{cases} v_{j_2 i_2} & \text{if } k = 2;\\ \mathbf{0} & \text{if } k \neq 2.\end{cases}$$

Proceeding in the same way for $k = 3, \ldots, r$, we obtain an ordered list of pairs $(j_1, i_1) < (j_2, i_2) < \ldots < (j_r, i_r)$ and a basis $\mathbf{v}_{1i_0}, \ldots, \mathbf{v}_{ri_0}$ such that

$$\pi_{j_\ell i_\ell}(\mathbf{v}_{ki_0}) = \begin{cases} v_{j_\ell i_\ell} & \text{if } \ell = k, \\ \mathbf{0} & \text{if } \ell \neq k; \end{cases}$$

and such that $\pi_{ba}(\mathbf{v}_{ki_0}) = \mathbf{0}$ for all $(b, a) < (j_k, i_k)$. Write $\mathbf{v}_{ki_0} = \sum_{i_0 < i < j \leq n} \alpha_{ji}^{(k,i_0)} v_{ji}$. From the above we have:

$$\alpha_{ji}^{(k,i_0)} = \begin{cases} 1 & \text{if } (j,i) = (j_k, i_k), \\ 0 & \text{if } (j,i) < (j_k, i_k). \end{cases}$$

For $k = 1, \ldots, r$ and $i = i_0 + 1, \ldots, n$, let $\mathbf{v}_{ki}$ be vectors in X such that

$$(\mathbf{0}, \ldots, \mathbf{0}, \mathbf{v}_{ki_0}, \mathbf{v}_{ki_0+1}, \ldots, \mathbf{v}_{kn}) \in \ker(\Phi) \cap X^n.$$

By Theorem 5.6(iii) we have

$$\Pi_{i_0}(\mathbf{v}_{kj}) = \sum_{m=i_0+1}^{j-1} \alpha_{jm}^{(k,i_0)} v_{mi_0} - \sum_{m=j+1}^{n} \alpha_{mj}^{(k,i_0)} v_{mi_0}.$$

For simplicity, set $\alpha_{ji}^{(k,i_0)} = -\alpha_{ij}^{(k,i_0)}$, and $\alpha_{jj}^{(k,i_0)} = 0$; then we can rewrite the above expression as:

$$\Pi_{i_0}(\mathbf{v}_{kj}) = \sum_{m=i_0+1}^{n} \alpha_{jm}^{(k,i_0)} v_{mi_0}. \tag{5.13}$$

Let s be the cardinality of the set $\{i_1, j_1, \ldots, i_r, j_r\}$; that is, s is the number of distinct indices that occur in the list $(j_1, i_1), \ldots, (j_r, i_r)$. Note that $r \leq \binom{s}{2}$. Let $a_1 < a_2 < \cdots < a_s$ be the list of these distinct indices. For each ℓ with $1 \leq \ell \leq s$, let $(j_{k(\ell)}, i_{k(\ell)})$ be the smallest pair among $(j_1, i_1), \ldots, (j_r, i_r)$ that has $a_\ell \in \{i_{k(\ell)}, j_{k(\ell)}\}$. If $a_\ell = i_{k(\ell)}$, let $b_\ell = j_{k(\ell)}$; if $a_\ell = j_{k(\ell)}$, let $b_\ell = i_{k(\ell)}$. Consider the following list of vectors from X:

$$\mathbf{v}_{1i_0}, \mathbf{v}_{2i_0}, \ldots, \mathbf{v}_{ri_0}, \mathbf{v}_{k(1)b_1}, \mathbf{v}_{k(2)b_2}, \ldots, \mathbf{v}_{k(s)b_s}.$$

Note that all of these vectors lie in $X \cap \langle v_{ji} \mid i_0 \leq i < j \leq n \rangle$. We will show that these vectors are linearly independent. Since $\mathbf{v}_{1i_0}, \ldots, \mathbf{v}_{ri_0}$ are linearly independent and $\Pi_{i_0}(\mathbf{v}_{ki_0}) = \mathbf{0}$ for $k = 1, \ldots, r$, it suffices to show that $\Pi_{i_0}(\mathbf{v}_{k(1)b_1}), \ldots, \Pi_{i_0}(\mathbf{v}_{k(s)b_s})$ are linearly independent.

First, from (5.13) we have $\pi_{a_\ell i_0}\left(\mathbf{v}_{k(m)b_m}\right) = \alpha_{b_m a_\ell}^{(k(m),i_0)}$. We claim that if $\ell < m$, then $\alpha_{b_m a_\ell}^{(k(m),i_0)} = 0$. By construction, this claim will follow if we can show that either $a_\ell = b_m$, or else the pair made up of b_m and a_ℓ is strictly smaller than the pair made up of a_m and b_m (which is equal to $(j_{k(m)}, i_{k(m)})$); the claim will then follow because $\alpha_{ba}^{(k,i_0)} = 0$ whenever $(b, a) < (j_k, i_k)$. Indeed, we know that $a_\ell < a_m$. If $a_m = i_{k(m)}$ and $b_m = j_{k(m)}$, then replacing a_m in the pair (b_m, a_m) with something smaller (namely a_ℓ) gives a smaller pair: $(b_m, a_\ell) < (b_m, a_m)$. If, on the other hand, we have $a_m = j_{k(m)}$ and $b_m = i_{k(m)}$, then if $a_\ell > b_m$ we have $(a_\ell, b_m) < (a_m, b_m)$, and if $a_\ell < b_m$ then we also have $(b_m, a_\ell) < (a_m, b_m)$. The only remaining possibility is $a_\ell = b_m$, which is of course no trouble.

Thus, we conclude that $\alpha_{b_m a_\ell}^{k(m),i_0} = 0$ whenever $\ell < m$. To see that the vectors $\Pi_2(\mathbf{v}_{k(1)b_1}),\ldots,\Pi_2(\mathbf{v}_{k(s)b_s})$ are linearly independent, note that

$$\pi_{a_\ell i_0}\left(\mathbf{v}_{k(m)b_m}\right) = \alpha_{b_m a_\ell}^{(k(m),i_0)} = \begin{cases} \mathbf{0} & \text{if } \ell < m,\\ v_{b_\ell a_\ell} & \text{if } m = \ell.\end{cases}$$

Therefore, if $\beta_1\Pi_{i_0}(\mathbf{v}_{k(1)b_1}) + \cdots + \beta_s\Pi_{i_0}(\mathbf{v}_{k(s)b_s}) = \mathbf{0}$, then $\beta_1 = 0$ since the only vector with nontrivial (a_1,i_0)-component is $\Pi_{i_0}(\mathbf{v}_{k(1)b_1})$. Hence $\beta_2 = 0$, because the only remaining vector with nontrivial (a_2,i_0)-component is $\Pi_{i_0}(\mathbf{v}_{k(2)b_2})$; and continuing this way we conclude $\beta_j = 0$ for all j. So the vectors are indeed linearly independent. Thus we have established that

$$\mathbf{v}_{1i_0},\mathbf{v}_{2i_0},\ldots,\mathbf{v}_{ri_0},\mathbf{v}_{k(1)b_1},\mathbf{v}_{k(2)b_2},\ldots,\mathbf{v}_{k(s)b_s}$$

is a collection of linearly independent vectors in $X\cap\langle v_{sr}\,|\, i_0\leq r<s\leq n\rangle$.

Thus we conclude that $d_{i_0}\geq r+s$. Since $r\leq\binom{s}{2}$, it follows that

$$r\leq\binom{s}{2}\leq\binom{d_{i_0}-r}{2},$$

as claimed.

To complete the proof, it only remains to establish the upper bound on d_{i_0+1}. We have $d_{i_0} = d_{i_0+1} + \dim\bigl(\langle v_{ji}\,|\, i_0\leq i<j\leq n\rangle \cap \{\mathbf{v}\in X \mid \Pi_{i_0}(\mathbf{v})\neq\mathbf{0}\}\bigr)$. Since the vectors $\mathbf{v}_{k(1)b_1},\ldots,\mathbf{v}_{k(s)b_s}$ are linearly independent, have nontrivial Π_{i_0} projection, and lie in $X\cap\bigl\langle v_{ji}\ \bigm|\ i_0\leq i<j\leq n\bigr\rangle$, we have $d_{i_0}\geq d_{i_0+1}+s$. Moreover, since $r\leq\binom{s}{2}$, we also have $s_{i_0}\leq s$; therefore, $d_{i_0+1}\leq d_{i_0}-s\leq d_{i_0}-s_{i_0}$, as desired. $\square$

Note that Z_{n-1} and Z_n are always trivial.

Definition 5.14. Let d be a nonnegative integer. We define $r(d)$ to be the largest integer such that $r(d)\leq d$ and $r(d)\leq\binom{d-r(d)}{2}$.

Theorem 5.15. *Fix $n>1$ and let $X<V$. Fix i_0, $1\leq i_0\leq n-2$, and let*

$$Z_{i_0} = X\cap\varphi_{i_0}^{-1}\left(\Bigl\langle\varphi_{i_0+1}(X),\ldots,\varphi_n(X)\Bigr\rangle\right).$$

If $\dim(X\cap\langle v_{ji}\,|\, i_0\leq i<j\leq n\rangle) = d$, then $\dim(Z_{i_0})\leq r(d)$. Equivalently,

$$\dim(Z_{i_0})\leq d-\left\lceil\frac{\sqrt{8d+1}-1}{2}\right\rceil \tag{5.16}$$

where $\lceil x\rceil$ is the smallest integer greater than or equal to x.

Proof. Let $\dim(Z_{i_0}) = r$. By Lemma 5.12, $r\leq\binom{d-r}{2}$, so $r\leq r(d)$, as claimed. From $r(d)\leq\binom{d-r(d)}{2}$ we easily obtain (5.16). $\square$

We have two other ways of describing the function $r(d)$, which will prove useful below:

Corollary 5.17. *Let d be a positive integer. Then $r(d)$ is the number of nontriangular numbers strictly less than d. Equivalently, if we write $d = \binom{t}{2}+s$, with $0<s\leq t$, then $r(d) = \binom{t-1}{2}+(s-1)$.*

Proof. Since $r(d)\leq\binom{d-r(d)}{2}\leq\binom{(d+1)-r(d)}{2}$, it follows that $r(d+1)\geq r(d)$. We also have

$$r(d)+2>r(d)+1>\binom{d-(r(d)+1)}{2} = \binom{(d+1)-(r(d)+2)}{2},$$

so $r(d+1) < r(d)+2$. If $r(d) < \binom{d-r(d)}{2}$, then

$$r(d)+1 \leq \binom{d-r(d)}{2} = \binom{(d+1)-(r(d)+1)}{2},$$

so $r(d+1) \geq r(d)+1$ and in this case we have $r(d+1) = r(d)+1$. If $r(d) = \binom{d-r(d)}{2}$, then $r(d)+1 > \binom{(d+1)-(r(d)+1)}{2}$, hence $r(d+1) < r(d)+1$ and we conclude that $r(d+1) = r(d)$. In summary, we have:

$$r(d+1) = \begin{cases} r(d)+1 & \text{if } r(d) < \binom{d-r(d)}{2},\\ r(d) & \text{if } r(d) = \binom{d-r(d)}{2}. \end{cases}$$

We claim that $r(d) = \binom{d-r(d)}{2}$ if and only if d is a triangular number: when $d = \binom{t+1}{2}$ for some $t \geq 0$, we have

$$\binom{t}{2} = \binom{\binom{t+1}{2} - \binom{t}{2}}{2} = \binom{d-\binom{t}{2}}{2},$$

so $r(d) = \binom{t}{2} = \binom{d-r(d)}{2}$. Conversely, if $r(d) = \binom{d-r(d)}{2}$, then solving for d we obtain $d = \binom{d-r(d)+1}{2}$, proving that d is a triangular number. Therefore, we have:

$$r(d+1) = \begin{cases} r(d)+1 & \text{if } d \text{ is not a triangular number},\\ r(d) & \text{if } d \text{ is a triangular number}. \end{cases}$$

Since $r(1) = 0$, we conclude that $r(d)$ is the number of nontriangular numbers strictly smaller than d, as claimed. To establish the formula, note that the value of r at $\binom{t}{2}$ is $\binom{t-1}{2}$, and therefore $r\left(\binom{t}{2}+s\right) = \binom{t-1}{2} + (s-1)$ for $0 < s < t$, since there are exactly $s-1$ more nontriangular numbers strictly less than $\binom{t}{2}+s$ than there are strictly less than $\binom{t}{2}$. And $\binom{t}{2}+t = \binom{t+1}{2}$, so we also get equality when $s = t$. $\square$

Remark 5.18. These alternate descriptions can also be obtained by examining sequence A083920 in [22]; for example, compare the closed formula there with (5.16). I first realized these alternate descriptions hold by calculating the first few values of $r(d)$ directly, and then consulting [22].

We can now obtain an upper bound for $\sum \dim(Z_k)$ in terms of $\dim(X)$, which in turn gives a lower bound for $\dim(X^*)$ in terms of $\dim(X)$.

Definition 5.19. For $n > 0$ and integer m, $0 \leq m \leq \binom{n}{2}$, we let $f(m,n)$ denote the largest possible value of $\sum \dim(Z_k)$ for a subspace X of V with $\dim(X) = m$; equivalently,

$$f(m,n) = \max\Big\{\dim\big(X^n \cap \ker(\Phi_n)\big) \;\Big|\; X < V(n),\ \dim(X) = m\Big\}.$$

Remark 5.20. As we will see below, the value of $f(m,n)$ does not depend on n; meaning that if $m \leq \binom{n}{2}$ and $n \leq N$, then $f(m,n) = f(m,N)$. It is easy to verify that $f(m,n) \leq f(m,N)$: if X is a subspace of $V(n)$ of dimension m, we can also consider it as a subspace of $V(N)$. If the dimension of X^* with respect to $\{\varphi_i\}_{i=1}^n$ is $nm-r$, then the dimension of X^* with respect to $\{\varphi_i\}_{i=1}^N$ is $Nm-r$; so we have

$$\dim(X^n \cap \ker(\Phi_n)) = \dim(X^N \cap \ker(\Phi_N)).$$

Intuitively, the reason the reverse inequality also holds is that the largest value of $f(m,n)$ occurs when the vectors in Z_i use fewer indices rather than more. Because more indices means a larger value of s in the proof of Lemma 5.12, which means more vectors are "taken out of circulation" for Z_{i+1}, which gives a smaller possible value for $X \cap \langle v_{rs} \,|\, i < r < s < n\rangle$. So the "best" strategy for larger intersection with $\ker(\Phi_n)$ is to keep X confined to as small a number of indices as possible. The proof below will formalize this intuition, and show that indeed the value of f depends only on m.

Theorem 5.21. *Let $m > 0$, and write $m = \binom{T}{2} + s$, $0 \leq s \leq T$. If $m \leq \binom{n}{2}$, then*

$$f(m,n) = \binom{T}{3} + \binom{s}{2}.$$

Remark 5.22. Although there is some ambiguity in the expression for m, since $\binom{T}{2} + T = \binom{T+1}{2}$, note that the values $\binom{T}{3} + \binom{T}{2}$ and $\binom{T+1}{3} + \binom{0}{2}$ are equal, so the given value of $f(m)$ is well-defined.

Proof. By replacing $\binom{T+1}{2}$ with $\binom{T}{2} + T$ if necessary, we may assume $s > 0$. Note that we must have $T < n$ in this situation. First we show that $f(m,n) \geq \binom{T}{3} + \binom{s}{2}$.

Let X be the m-dimensional coordinate subspace of $V(n)$ generated by all v_{ji} with $1 \leq i < j \leq T$, and the vectors $v_{T+1,1}, \ldots, v_{T+1,s}$. Then X^* is the coordinate subspace of $W(n)$ generated by all vectors of the form w_{jik} with $1 \leq i < j \leq T$, $i \leq k \leq n$; plus the vectors of the form $w_{T+1,i,k}$ with $1 \leq i \leq s$, $i \leq k \leq n$. There are $2\binom{T+1}{2} + (n-T)\binom{n}{2}$ vectors of the first kind, and

$$n + (n-1) + (n-2) + \cdots + n - (s-1) = sn - \binom{s}{2}$$

of the second kind. Thus $\dim(X^*) = 2\binom{T+1}{2} + (n-T)\binom{T}{2} + sn - \binom{s}{2}$; and we have:

$$\begin{aligned}
n\dim(X) - \dim(X^*) &= T\binom{T}{2} - 2\binom{T+1}{3} + \binom{s}{2}\\
&= (T-2)\binom{T}{2} - 2\binom{T}{3} + \binom{s}{2}\\
&= \binom{T}{3} + \binom{s}{2}.
\end{aligned}$$

Therefore, $f(m,n) \geq \binom{T}{3} + \binom{s}{2}$.

For the reverse inequality, we will apply induction. Assume the for any X' space of $V(n)$ with $\dim(X') < m$. Write $m = \binom{T}{2} + s$ with $0 < s \leq T$, and $T < n$, and let X be a subspace of V of dimension m. We want to show that $\sum \dim(Z_i)$ is bounded above by $\binom{T}{3} + \binom{s}{2}$. If all Z_i are trivial, this follows. Otherwise, assume i is the smallest index with nontrivial Z_i, and that $\dim(Z_i) = k > 0$. Then $k \leq r(m)$, and if ℓ is the smallest positive integer such that $k \leq \binom{\ell}{2}$ then

$$\dim\bigl(X \cap \langle v_{sr} \,|\, i < r < s \leq n\rangle\bigr) \leq m - \ell.$$

So the sum of the dimensions of the Z_j with $j > i$ is at most $f(m-\ell, n)$; that is, the sum over all k is bounded:

$$\sum \dim(Z_k) \leq k + f(m-\ell, n).$$

We want to show that $k + f(m-\ell,n) \leq \binom{T}{3} + \binom{s}{2}$ for all k and ℓ that satisfy the relevant conditions. It is easy to show that for $m = 1, 2, 3, 4$, and 5, all values of the form $k + f(m-\ell, n)$, $k \leq r(m)$ and ℓ as above are less than or equal to $\binom{T}{3} + \binom{s}{2}$.

If $\ell = T = m - r(m)$, then since $k \leq r(m)$ we have

$$\begin{aligned}
k + f(m-\ell, n) &\leq r(m) + f(r(m), n)\\
&= \binom{T-1}{2} + (s-1) + f\left(\binom{T-1}{2} + (s-1), n\right)\\
&= \binom{T-1}{2} + (s-1) + \binom{T-1}{3} + \binom{s-1}{2}\\
&= \binom{T}{3} + \binom{s}{2};
\end{aligned}$$

If $\ell < T$, since $k \leq \binom{\ell}{2}$, it is enough to to show that for $1 < \ell < T$,

$$\binom{\ell}{2} + f(m-\ell, n) \leq \binom{T}{3} + \binom{s}{2}.$$

If $2 \leq \ell \leq s$, then:

$$\begin{aligned}
\binom{\ell}{2} + f(m-\ell, n) &= \binom{\ell}{2} + f\left(\binom{T}{2} + (s-\ell), n\right)\\
&= \binom{\ell}{2} + \binom{T}{3} + \binom{s-\ell}{2}\\
&\leq \binom{T}{3} + \binom{s}{2}.
\end{aligned}$$

The last inequality follows since $\binom{\ell}{2} + \binom{s-\ell}{2}$ is the number of two element subsets of $\{1, \ldots, s\}$, where either both elements are less than or equal to ℓ, or both strictly larger than ℓ.

If $s < \ell < T$, then write $\ell = s + a$, $a > 0$. We then have

$$m - \ell = \binom{T}{2} + s - (s+a) = \binom{T-1}{2} + (T-1-a),$$

so

$$\binom{\ell}{2} + f(m-\ell, n) = \binom{\ell}{2} + \binom{T-1}{3} + \binom{T-1-a}{2}.$$

Since $\ell + 1 - T \leq 0$ and $a > 0$, we must have

$$6a(s + a + 1 - T) \leq 0.$$

Rewriting and introducing suitable terms we have:

$$6as + 3a^2 - 3a - 3T^2 + 9T - 6 + 3T^2 - 9T - 6aT + 9a + 3a^2 + 6 \leq 0$$

In turn, this can be rewritten as

$$6as + 3a^2 - 3a - 3(T-1)(T-2) + 3(T-a-1)(T-a-2) \leq 0.$$

This gives:

$$3(s^2 + 2as + a^2 - s - a) - 3(T-1)(T-2) + 3(T-a-1)(T-a-2) \leq 3(s^2 - s),$$

and so

$$3((s+a)^2 - (s+a)) - 3(T-1)(T-2) + 3(T-a-1)(T-a-2) \leq 3(s^2 - s).$$

Substituting ℓ for $s + a$ and adding $T(T-1)(T-2)$ to both sides we have

$$3(\ell^2 - \ell) + (T-3)(T-2)(T-1) + 3(T-a-1)(T-a-2) \leq T(T-1)(T-2) + 3(s^2 - s);$$

dividing through by 6 yields the desired inequality:

$$\binom{\ell}{2} + f(m-\ell,n) \leq \binom{\ell}{2} + \binom{T-1}{3} + \binom{T-1-a}{2} \leq \binom{T}{3} + \binom{s}{2}.$$

We therefore conclude that $f(m,n) \leq \binom{T}{3}+\binom{s}{2}$, which completes the proof. Note that indeed, the value of n is not relevant to the value of $f(m,n)$, so long as n is large enough to satisfy $m \leq \binom{n}{2}$. $\square$

Since the value of $f(m,n)$ does not depend on n, we will drop the second argument and simly call this function $f(m)$.

Theorem 5.23. *Fix $n > 1$ and let X be a subspace of V. Write $\dim(X) = \binom{T}{2}+s$, $0 \leq s \leq T$. Then*

$$n\dim(X) - \binom{T}{3} - \binom{s}{2} \leq \dim(X^*) \leq \min\left\{ n\dim(X),\ 2\binom{n+1}{3}\right\}.$$

Proof. The lower bound follows from $\dim(X^*) \geq n\dim(X) - f(\dim(X))$, and the upper bound is immediate. $\square$

Corollary 5.24. *Fix $n > 1$ and let X be a subspace of V with $\dim(X) = m$. If $\dim(X^*) = nm - k$ and $n + k > f(m+1)$, then X is closed.*

Proof. Suppose X is as in the statement, and let Y be any subspace of V of dimension $m+1$. From the definition of f we know that

$$\dim(Y^*) \geq n(m+1) - f(m+1),$$

so $\dim(Y^*) - \dim(X^*) \geq n + k - f(m+1) > 0$. Therefore every Y strictly larger than X must have $\dim(X^*) < \dim(Y^*)$, which shows that X is closed by Proposition 5.10. $\square$

Corollary 5.25. *Fix $n > 1$ and let X be a subspace of V with $\dim(X) = m$. Write $m = \binom{T}{2} + s$, $0 \leq s < T$. If $\binom{T}{3} + \binom{s+1}{2} < n$, then X is closed.*

Proof. This follows from the previous corollary and the formula for $f(m+1)$ in Theorem 5.21. $\square$

For reference, Table 1 contains the values of $f(m)$, $3 \leq m \leq 50$. Note that $f(1) = f(2) = 0$ by Corollary 5.9. The sequence of values of $f(m)$ appears as sequence A111138 in [22].

Translating back into group theory, we obtain the following:

Theorem 5.26. *Let G be a group of class at most two and exponent p, where p is an odd prime. Let $\mathrm{rank}(G^{\mathrm{ab}}) = n$, and let $\mathrm{rank}([G,G]) = m$. If $f\left(\binom{n}{2} - m + 1\right) < n$, where $f(k)$ is the function in Theorem 5.21, then G is capable.*

Proof. The subspace X of $V(n)$ corresponding to G has dimension $\binom{n}{2} - m$; so the result follows directly from Corollary 5.25. $\square$

m	$f(m)$	m	$f(m)$	m	$f(m)$
3	1	19	26	35	77
4	1	20	30	36	84
5	2	21	35	37	84
6	4	22	35	38	85
7	4	23	36	39	87
8	5	24	38	40	90
9	7	25	41	41	94
10	10	26	45	42	99
11	10	27	50	43	105
12	11	28	56	44	112
13	13	29	56	45	120
14	16	30	57	46	120
15	20	31	59	47	121
16	20	32	62	48	123
17	21	33	66	49	126
18	23	34	71	50	130

Table 1. Explicit values of $f(m)$, $3 \leq m \leq 50$

6. A Necessary Condition.

In this section, we use our set-up to give a proof of a slight strengthening of the necessary condition proven by Heineken and Nikolova in [13]. The proof is essentially that given in [13] "translated" into our notation and set-up. We do gain two improvements on their result: a necessary condition for equality to hold, and a weakening of the hypothesis by dropping an assumption. In [13] the authors assume throughout that the capable group G they investigate satisfies the condition $Z(G) = [G,G]$, and so their result is restricted to that situation. We will be able to obtain their result with this assumption dropped.

The object of this section is to prove that if G is capable, of class at most two and exponent p, and $[G,G]$ is of order p^k, then $G/Z(G)$ is of order at most $p^{2k+\binom{k}{2}}$.

It is interesting to note that while the results from the previous sections, leading to sufficient conditions, have focused on the closure operator on the subspaces of V, the proof here will proceed by placing considerable emphasis on the interior operator on the subspaces of W. I do not know if this is simple happenstance, or if we can indeed expect that considerations of the interior operator on W will generally point towards necessary conditions while the closure operator on V will give sufficient ones.

In addition to the linear transformations $\varphi_{\mathbf{u}}$, an important role in the proof is played by elements $g \in Z(G)$ which have nontrivial image in G^{ab}. In order to account for these elements in our setting, we will use another family of linear transformations which we introduce now:

Definition 6.1. Let $n > 1$. We embed U into $\mathcal{L}(U,V)$ as follows: given $\mathbf{u} \in U$, we define $\psi_{\mathbf{u}}(\mathbf{a}) = \mathbf{a} \wedge \mathbf{u}$ for all $\mathbf{a} \in U$. If $u_1, \ldots, u_n$ is a given basis for U and i is an integer, $1 \leq i \leq n$, then we let ψ_i denote the transformation ψ_{u_i}. Note that for any $\mathbf{a}, \mathbf{b} \in U$, $\psi_{\mathbf{a}}(\mathbf{b}) = -\psi_{\mathbf{b}}(\mathbf{a})$.

Fix an isomorphism between G^{ab} and U. Let $g \in G$ be an element whose image in G^{ab} is nontrivial, and let $\mathbf{u}_g$ be the corresponding element of U. Then $g \in Z(G)$ if and only if $\psi_{\mathbf{u}_g}(U)$ is contained in X. Note also that for any $\mathbf{u} \in U$, $\psi_{\mathbf{u}}(U) = \langle \mathbf{u} \rangle^* = \mathrm{span}(\psi_1(\mathbf{u}), \ldots, \psi_n(\mathbf{u}))$. This is how we will use the maps above to address the central elements of G that are not in $[G, G]$

An explicit description of the maps ψ in terms of a basis for U is easy:

Lemma 6.2. *Fix $n > 1$, let $u_1, \ldots, u_n$ be a basis for U, and let v_{ji} be the corresponding basis for V. For all integers i and j, $1 \leq i, j \leq n$, the image of u_j under ψ_i in terms of the basis v_{ji} is given by:*

$$\psi_i(u_j) = \begin{cases} v_{ji} & \text{if } i < j, \\ \mathbf{0} & \text{if } i = j, \\ -v_{ij} & \text{if } i > j. \end{cases}$$

Let Y be a subspace of W. If we let $X = Y^*$ then X is closed by Theorem 3.3; moreover, any closed subspace X of V can be realized this way, by letting $Y = X^*$. Given such an X and Y, we define two subsets of U as follows:

$$\begin{aligned} Z &= \left\{ \mathbf{u} \in U \;\middle|\; \psi_{\mathbf{u}}(U) \subseteq X \right\}, \\ C &= \left\{ \mathbf{u} \in U \;\middle|\; \varphi_{\mathbf{u}}(V) \subseteq Y \right\}. \end{aligned}$$

Let F be the 3-nilpotent product of n cyclic groups of order p, and let U, V, W correspond to F^{ab}, $\langle [x_j, x_i] \,|\, 1 \leq i < j \leq n \rangle$, and F_3 respectively, as in Section 3, Let N correspond to X, and $G = F/(X \oplus F_3)$, $H = F/Y$. Then G is capable (since X is closed), H is a witness for the capability of G, and it is not hard to see that Z will correspond to the image of $Z_2(H)$ in H^{ab} (this is the same as the image of $Z(G)$ in G^{ab}, i.e., those elements that are central in G but do not come from commutators), while C will correspond to the image of the centralizer $C([H, H])$ in H^{ab}. These two sets (in fact, subspaces as we will prove below) play a key role in our analysis.

Lemma 6.3. *Let Y be a subspace of W, and let $X = Y^*$. If*

$$Z = \left\{ \mathbf{u} \in U \;\middle|\; \psi_{\mathbf{u}}(U) \subseteq X \right\} \qquad \text{and} \qquad C = \left\{ \mathbf{u} \in U \;\middle|\; \varphi_{\mathbf{u}}(V) \subseteq Y \right\},$$

then both Z and C are subspaces of U, and $Z \subseteq C$.

Proof. The map $\mathbf{u} \mapsto \psi_{\mathbf{u}}$ is a linear embedding from U to $\mathcal{L}(U, V)$. The canonical projection $V \to V/X$ induces a map $U \to \mathcal{L}(U, V/X)$. The kernel of this map is Z, so Z is a subspace.

Similarly, the kernel of the composite map $U \to \mathcal{L}(V, W) \to \mathcal{L}(V, W/Y)$, given by composing the embedding $\mathbf{u} \mapsto \varphi_{\mathbf{u}}$ with the map induced by the canonical projection $W \to W/Y$ has kernel C, so C is a subspace.

To prove that $Z \subseteq C$, let $\mathbf{z} \in Z$. If $\mathbf{z} = \mathbf{0}$, then trivially $\mathbf{z} \in C$. If $\mathbf{z} \neq \mathbf{0}$, then complete it to a basis $\mathbf{z} = u_1, \ldots, u_n$ of U, and let v_{ji}, w_{jik} be the corresponding prefered bases for V and W. Since $\mathbf{z} = u_1 \in Z$, it follows that $v_{j1} \in X$ for $j = 2, \ldots, n$. We want to show that $\varphi_1(v_{ji}) \in Y$ for all i, j, $1 \leq i < j \leq n$. If $i = 1$, then $v_{ji} \in X$, so $\varphi_{\mathbf{u}}(v_{ji}) \in Y$ for all $\mathbf{u} \in U$ and there is nothing to do. If $i > 1$, then $\varphi_1(v_{ji}) = w_{j1i} - w_{i1j} = \varphi_i(v_{j1}) - \varphi_j(v_{i1})$. Since $v_{j1}, v_{i1} \in X$, we have that both $\varphi_i(v_{j1})$ and $\varphi_j(v_{i1})$ lie in Y, hence $\varphi_1(v_{ji}) \in Y$. This proves that $\mathbf{z} \in C$, as claimed. $\square$

We continue by stating some results on the interactions between the maps $\psi_{\mathbf{u}}$ and the maps $\varphi_{\mathbf{u}'}$.

Lemma 6.4. *For* $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}\in U$, $\varphi_{\mathbf{a}}(\psi_{\mathbf{b}}(\mathbf{c}))+\varphi_{\mathbf{c}}(\psi_{\mathbf{a}}(\mathbf{b}))+\varphi_{\mathbf{b}}(\psi_{\mathbf{c}}(\mathbf{a}))=\mathbf{0}$.

Proof. This is simply the Jacobi identity. Evaluating the left hand side, we obtain

$$\overline{(\mathbf{c}\wedge\mathbf{b})\otimes\mathbf{a}}+\overline{(\mathbf{b}\wedge\mathbf{a})\otimes\mathbf{c}}+\overline{(\mathbf{a}\wedge\mathbf{c})\otimes\mathbf{b}},$$

where $\overline{(r\wedge s)\otimes t}$ represents the image of $r\wedge s\otimes t$ in $(V\otimes U)/J$ (see Definition 3.6). But since this element is one of the generators of the subspace J, the left hand side is trivial in W, as claimed. $\square$

Lemma 6.5. *Let Y be a subspace of W, and let $X=Y^*$. Let*

$$C=\bigl\{\mathbf{u}\in U\;\bigm|\;\varphi_{\mathbf{u}}(V)\subseteq Y\bigr\}.$$

If $\mathbf{c}\in C$, then for all $\mathbf{a},\mathbf{b}\in U$,

$$\varphi_{\mathbf{b}}(\psi_{\mathbf{c}}(\mathbf{a}))\equiv\varphi_{\mathbf{a}}(\psi_{\mathbf{c}}(\mathbf{b}))\pmod{Y}.$$

Proof. From Lemma 6.4, we know that $\varphi_{\mathbf{a}}(\psi_{\mathbf{b}}(\mathbf{c}))+\varphi_{\mathbf{b}}(\psi_{\mathbf{c}}(\mathbf{a}))=-\varphi_{\mathbf{c}}(\psi_{\mathbf{a}}(\mathbf{b}))$. Since $\mathbf{c}\in C$, we must have $\varphi_{\mathbf{c}}(\psi_{\mathbf{a}}(\mathbf{b}))\in Y$. Therefore,

$$\begin{aligned}\varphi_{\mathbf{b}}(\psi_{\mathbf{c}}(\mathbf{a}))&\equiv-\varphi_{\mathbf{a}}(\psi_{\mathbf{b}}(\mathbf{c}))\pmod{Y}\\&\equiv\varphi_{\mathbf{a}}(-\psi_{\mathbf{b}}(\mathbf{c}))\pmod{Y}\\&\equiv\varphi_{\mathbf{a}}(\psi_{\mathbf{c}}(\mathbf{b}))\pmod{Y}.\end{aligned}$$

This proves the lemma. $\square$

Lemma 6.6. *Let Y be a subspace of W, $X=Y^*$, and let*

$$C=\bigl\{\mathbf{u}\in U\;\bigm|\;\varphi_{\mathbf{u}}(V)\subseteq Y\bigr\}.$$

If $\mathrm{codim}_W(Y)=1$, then $\mathrm{codim}_V(X)=\mathrm{codim}_U(C)$.

Proof. The map $C\to\mathcal{L}(V,W/Y)$ factors through $\mathcal{L}(V/X,W/Y)$, and the kernel of the induced map $U\to\mathcal{L}(V/X,W/Y)$ is C. Therefore,

$$\begin{aligned}\mathrm{codim}_U(C)=\dim(U/C)&\leq\dim(\mathcal{L}(V/X,W/Y))=\dim(V/X)\dim(W/Y)\\&=\mathrm{codim}_U(X)\mathrm{codim}_W(Y)=\mathrm{codim}_U(X),\end{aligned}$$

proving one inequality.

To prove the reverse inequality, let $\mathbf{w}\in W\backslash Y$, $\mathrm{codim}_V(X)=k$, and pick elements $\mathbf{v}_1,\ldots,\mathbf{v}_k$ of V whose images in the quotient V/X form a basis. Since $\mathbf{v}_1\notin X$, there exists $\mathbf{u}_1\in U$ such that $\varphi_{\mathbf{u}_1}(\mathbf{v}_1)\notin Y$. Note that $\mathbf{u}_1\notin C$. Adjusting $\mathbf{v}_1$ by a scalar if necessary, and adding multiples of $\mathbf{v}_1$ to $\mathbf{v}_2,\ldots,\mathbf{v}_k$ if necessary, we may assume that

$$\varphi_{\mathbf{u}_1}(\mathbf{v}_1)\equiv\mathbf{w}\pmod{Y}\qquad\text{and}\qquad\varphi_{\mathbf{u}_1}(\mathbf{v}_i)\equiv0\pmod{Y}\quad\text{for }i>i.$$

Since $\mathbf{v}_2\notin X$, there exists $\mathbf{u}_2\in U$ such that $\psi_{\mathbf{u}_2}(\mathbf{v}_2)\notin Y$. Multiplying $\mathbf{v}_2$ by a scalar and adding multiples of $\mathbf{v}_2$ to $\mathbf{v}_3,\ldots,\mathbf{v}_k$ if necessary, we may assume that

$$\varphi_{\mathbf{u}_2}(\mathbf{v}_2)\equiv\mathbf{w}\pmod{Y}\qquad\text{and}\qquad\varphi_{\mathbf{u}_2}(\mathbf{v}_i)\equiv0\pmod{Y}\quad\text{for }i>2.$$

Proceeding in the same manner, we obtain elements $\mathbf{u}_1,\ldots,\mathbf{u}_k\in U$ such that $\varphi_{\mathbf{u}_i}(\mathbf{v}_i)\equiv\mathbf{w}\pmod{Y}$ for $i=1,\ldots,k$, and $\varphi_{\mathbf{u}_i}(\mathbf{v}_j)\in Y$ for $j>i$. Let $\overline{\varphi_{\mathbf{u}_1}},\ldots,\overline{\varphi_{\mathbf{u}_k}}$ be the images of $\mathbf{u}_1,\ldots,\mathbf{u}_k$ in $\mathcal{L}(V,W/Y)$. These linear transformations are linearly independent, because if $\alpha_1\overline{\varphi_{\mathbf{u}_1}}+\cdots+\alpha_k\overline{\varphi_{\mathbf{u}_k}}$ is the zero transformation, then evaluating at $\mathbf{v}_k$ we deduce that $\alpha_k=0$; then evaluating at $\mathbf{v}_{k-1}$ we obtain $\alpha_{k-1}=0$; etc. Since the images of $\mathbf{u}_1,\ldots,\mathbf{u}_k$ are linearly independent under a map with kernel C, it follows that their images in U/C are also linearly independent, proving that $\mathrm{codim}_U(C)=\dim(U/C)\geq k$. This proves the reverse inequality, and we are done. □

From the proof above we also deduce the following technical corollary; we will use it in argument below:

Corollary 6.7. *Let Y be a subspace of W, and let $X=Y^*$. Let*

$$C=\bigl\{\mathbf{u}\in U \bigm| \varphi_{\mathbf{u}}(V)\subseteq Y\bigr\},$$

and let $\mathbf{w}\in W\backslash Y$. If $\mathrm{codim}_W(Y)=1$ and $\mathrm{codim}_V(X)=k$, then there exist $\mathbf{v}_1,\ldots,\mathbf{v}_k\in V$ and $\mathbf{u}_1,\ldots,\mathbf{u}_k\in U$ such that:

(i) *The images of $\mathbf{v}_1,\ldots,\mathbf{v}_k$ in V/X form a basis for V/X.*
(ii) *The images of $\mathbf{u}_1,\ldots,\mathbf{u}_k$ in U/C form a basis for U/C. In particular, $U=\langle\mathbf{u}_1,\ldots,\mathbf{u}_k\rangle\oplus C$.*
(iii) *$\varphi_{\mathbf{u}_i}(\mathbf{v}_i)\equiv\mathbf{w} \pmod{Y}$ for $i=1,\ldots,k$.*
(iv) *$\varphi_{\mathbf{u}_i}(\mathbf{v}_j)\equiv\mathbf{0} \pmod{Y}$ for all i,j, $1\leq i<j\leq k$.*

Lemma 6.8. *Let Y be a subspace of W, and let $X=Y^*$. Let*

$$Z=\bigl\{\mathbf{u}\in U \bigm| \psi_{\mathbf{u}}(U)\subseteq X\bigr\}\quad\textit{and}\quad C=\bigl\{\mathbf{u}\in U \bigm| \varphi_{\mathbf{u}}(V)\subseteq Y\bigr\}.$$

Let $\mathbf{u}_1,\ldots,\mathbf{u}_k\in U$ be elements such that $U=\langle\mathbf{u}_1,\ldots,\mathbf{u}_k\rangle+C$. If $\mathbf{c}\in C$ is such that $\psi_{\mathbf{c}}(\mathbf{u}_i)\in X$ for $i=1,\ldots,k$, then $\mathbf{c}\in Z$.

Proof. To prove that $\mathbf{v}\in V$ lies in X, it is enough to show that $\varphi_{\mathbf{u}_i}(\mathbf{v})\in Y$ for $i=1,\ldots,k$; this follows since U is spanned by the vectors $\mathbf{u}_1,\ldots,\mathbf{u}_k$ and C, the latter always mapping into Y, and $X=\bigl\{\mathbf{v}\in V \bigm| \varphi_{\mathbf{u}}(\mathbf{v})\in Y \text{ for all } \mathbf{u}\in U\bigr\}$. Thus, to prove that $\mathbf{c}\in Z$, it is enough to show that for every $\mathbf{u}\in U$ and $i=1,\ldots,k$, $\varphi_{\mathbf{u}_k}(\psi_{\mathbf{c}}(\mathbf{u}))\in Y$. Since $\mathbf{c}\in C$, we know from Lemma 6.5 that

$$\varphi_{\mathbf{u}_k}(\psi_{\mathbf{c}}(\mathbf{u}))\equiv\varphi_{\mathbf{u}}(\psi_{\mathbf{c}}(\mathbf{u}_k)) \pmod{Y}.$$

By assumption, $\psi_{\mathbf{c}}(\mathbf{u}_k)\in X$, and therefore $\varphi_{\mathbf{u}}(\psi_{\mathbf{c}}(\mathbf{u}_k))\in Y$. Thus, $\varphi_{\mathbf{u}_k}(\psi_{\mathbf{c}}(\mathbf{u}))$ lies in Y, and we are done. □

The following counting argument will be needed a few times, and will be the key tool used to establish the upper bounds.

Lemma 6.9. *Let A and B be vector spaces over the same field, $\dim(A)=n$ and $\dim(B)=1$. Let $\mathbf{a}_1,\ldots,\mathbf{a}_n$ be a basis for A, and let $\mathbf{b}\in B$ be a nonzero vector. Let $f_1,\ldots,f_n\in\mathcal{L}(A,B)$ be linear transformations such that*

$$f_i(\mathbf{a}_i)=\mathbf{b}\quad\textit{for } i=1,\ldots,n;\qquad\textit{and}\qquad f_i(\mathbf{a}_j)=\mathbf{0}\quad\textit{if } 1\leq i<j\leq n.$$

Then the dimension of the subspace

$$\Bigl\{\bigl(\mathbf{v}_1,\ldots,\mathbf{v}_n\bigr)\in A^n \Bigm| f_i(\mathbf{v}_j)=f_j(\mathbf{v}_i),\quad 1\leq i,j\leq n\Bigr\}$$

is $n+\binom{n}{2}$.

Proof. Express $\mathbf{v}_i$ in terms of the basis for A, $\mathbf{v}_i = \alpha_{i1}\mathbf{a}_1 + \cdots + \alpha_{in}\mathbf{a}_n$. We have n degrees of freedom for choosing $\mathbf{v}_1$. Once $\mathbf{v}_1$ is fixed, we must have

$$\alpha_{21}\mathbf{b} = f_1(\mathbf{v}_2) = f_2(\mathbf{v}_1),$$

which fixes the value of α_{21}, leaving $n-1$ degrees of freedom for choosing $\mathbf{v}_2$. Once both $\mathbf{v}_1$ and $\mathbf{v}_2$ are fixed, $\mathbf{v}_3$ must satisfy

$$\begin{aligned} \alpha_{31}\mathbf{b} &= f_1(\mathbf{v}_3) = f_3(\mathbf{v}_1),\\ \alpha_{31}f_2(\mathbf{a}_1) + \alpha_{32}\mathbf{b} &= f_2(\mathbf{v}_3) = f_3(\mathbf{v}_2). \end{aligned}$$

The first equation completely determines α_{31}, which together with the second equation completely determines α_{32}, leaving $n-2$ degrees of freedom for choosing $\mathbf{v}_3$.

Continuing in this manner we have $n-3$ degrees of freedom for $\mathbf{v}_4$, $n-4$ for $\mathbf{v}_5$, and so on, until we have one degree of freedom left for $\mathbf{v}_n$. In total, we have

$$n + (n-1) + (n-2) + \cdots + 2 + 1 = n + \binom{n}{2}$$

degrees of freedom in choosing the n-tuple; this proves that the subspace in question has dimension $n + \binom{n}{2}$, as claimed. $\square$

Our proof that we can bound $\dim(U/Z)$ in terms of $\dim(V/X)$ will proceed by induction on $\mathrm{codim}_W(Y)$. The basis of the induction is contained in the following lemma:

Lemma 6.10 (cf. [13, Lemma 1])**.** *Let Y be a subspace of W of codimension one. Let $X = Y^*$, and let $Z = \{\mathbf{u} \in U \mid \psi_{\mathbf{u}}(U) \subseteq X\}$. If $\mathrm{codim}_V(X) = k$, then $\dim(U/Z) \leq 2k + \binom{k}{2}$.*

Proof. As before, let $C = \{\mathbf{u} \in U \mid \varphi_{\mathbf{u}}(V) \subseteq Y\}$. From Lemma 6.6 we know that $\dim(U/C) = k$, so we only need to prove that $\dim(C/Z) \leq k + \binom{k}{2}$. Fix $\mathbf{w} \in W\backslash Y$, and let $\mathbf{v}_1, \ldots, \mathbf{v}_k$ and $\mathbf{u}_1, \ldots, \mathbf{u}_k$ be the vectors given by Corollary 6.7.

Consider the linear map $C \mapsto (V/X)^k$ defined by:

$$\mathbf{c} \mapsto \left(\overline{\psi_{\mathbf{c}}(\mathbf{u}_1)}, \ldots, \overline{\psi_{\mathbf{c}}(\mathbf{u}_k)}\right),$$

where $\overline{\mathbf{v}}$ is the image of $\mathbf{v} \in V$ under the canonical projection $V \to V/X$. By Lemma 6.8, the kernel of the map is Z, so we obtain an embedding of C/Z into $(V/X)^k$. Note also that the image of $\mathbf{c} \in C$ is a vector $(\overline{\psi_{\mathbf{c}}(\mathbf{u}_1)}, \ldots, \overline{\psi_{\mathbf{c}}(\mathbf{u}_k)})$ that satisfies the congruence $\varphi_{\mathbf{u}_j}(\psi_{\mathbf{c}}(\mathbf{u}_i)) \equiv \varphi_{\mathbf{u}_i}(\psi_{\mathbf{c}}(\mathbf{u}_j)) \pmod{Y}$ by Lemma 6.5. This is well defined, since elements of X always map into Y via any $\varphi_{\mathbf{u}}$.

By Lemma 6.9, the image of $\mathbf{c}$ lies in a subspace of dimension $k + \binom{k}{2}$; therefore, C/Z is of dimension at most $k + \binom{k}{2}$, By Lemma 6.6:

$$\dim(U/Z) = \dim(U/C) + \dim(C/Z) \leq k + k + \binom{k}{2} = 2k + \binom{k}{2},$$

proving the lemma. $\square$

One final observation is needed:

Lemma 6.11. *Let Y be a subspace of W, and let Y' be a subspace of W with the same interior as Y; that is, such that $Y'^{**} = Y^{**}$. Then $Y'^{*} = Y^{*}$.*

Proof. This follows from Theorem 3.3, since $Y'^{*} = Y'^{***} = Y^{***} = Y$. $\square$

We can now prove the main result of this section:

Theorem 6.12 (cf. [13, Theorem 1])**.** *Let Y be a subspace of W, let $X = Y^*$, and let $Z = \bigl\{\mathbf{u}\in U \bigm| \psi_{\mathbf{u}}(U)\subseteq X\bigr\}$. If $\mathrm{codim}_V(X) = k$, then $\dim(U/Z)\leq 2k+\binom{k}{2}$, and equality holds only if there exists a subspace Y' of W such that $\mathrm{codim}_W(Y')\leq 1$ and $Y'^{**}=Y^{**}$.*

Proof. We proceed by induction on $r = \mathrm{codim}_W(Y)$. If $\mathrm{codim}_W(Y) = 0$, then $X = V$, $Z = U$, and the result holds trivially. If $\mathrm{codim}_W(Y) = 1$, then the result is Lemma 6.10; the final clause also holds, since the consequent is trivially true with $Y' = Y$. Assume then that $\mathrm{codim}_W(Y)\geq 2$. As before, let

$$C = \bigl\{\mathbf{u}\in U \bigm| \varphi_{\mathbf{u}}(V)\subseteq Y\bigr\}.$$

Suppose inductively that the result holds for any subspace Y' of W such that $\mathrm{codim}_W(Y') < \mathrm{codim}_W(Y) = r$. If there exists Y' with $\mathrm{codim}_W(Y') < \mathrm{codim}_W(Y)$ and $Y'^{**} = Y^{**}$, then we can replace Y with Y'. Note that since $Y'^{*} = Y^{*}$, the subspaces Z of U and X of V are not affected by change, so the result holds by induction. We may therefore assume that if Y_2 is any subspace of W that properly contains Y, then Y^* is properly contained in Y_2^*. To prove the result for Y, we will need to establish that the strict inequality holds in this situation.

Among all Y_2 such that $Y\subseteq Y_2$, and $\dim(Y_2) = \dim(Y)+1$, we pick one for which $X_2 = Y_2^*$ is of minimal dimension. Note that X is properly contained in X_2; if we let $\omega = \dim(X_2/X)$, then $0<\omega<k$ and $\mathrm{codim}_V(X_2) = k-\omega$. Let $Z_2 = \bigl\{\mathbf{u}\in U \bigm| \psi_{\mathbf{u}}(U)\subseteq X_2\bigr\}$; again we have that $Z\subseteq Z_2$. By the induction hypothesis, we know that $\dim(U/Z_2)\leq 2(k-\omega)+\binom{k-\omega}{2}$. We now want to estimate the dimension of Z_2/Z. We will do this in two steps, first by giving an upper bound for the dimension of $Z_2/(Z_2\cap C)$, and then giving an upper bound for the dimension of $(Z_2\cap C)/Z$.

Let Y_3 be any subspace of W that contains Y, $\dim(Y_3) = \dim(Y)+1$, and $Y_3\neq Y_2$. This is possible because $\dim(Y)<\dim(W)-1$, so there are at least $p+1$ subspaces of dimension one more than $\dim(Y)$. Let $X_3 = Y_3^*$; by choice of Y_2, $\dim(X_3/X)\geq\omega$, and so $\dim(V/X_3)\leq k-\omega$.

If $\mathbf{v}\in X_3$ and $\mathbf{u}\in Z_2$, then $\varphi_{\mathbf{u}}(\mathbf{v})\in Y_2\cap Y_3 = Y$; so the map $Z_2\mapsto\mathcal{L}(V, Y_2/Y)$ defined by $\mathbf{u}\mapsto\varphi_{\mathbf{u}}$ factors through $\mathcal{L}(V/X_3, Y_2/Y)$. The kernel of this map is $Z_2\cap C$, and therefore

$$\dim(Z_2/(Z_2\cap C))\leq\dim(\mathcal{L}(V/X_3, Y_2/Y)) = \dim(V/X_3)\leq k-\omega.$$

Finally, we want an upper bound for $\dim((Z_2\cap C)/Z)$. This is the difficult part of the induction.

By Corollary 6.7, we can select elements $\mathbf{x}_1,\ldots,\mathbf{x}_\omega$ in X_2, and $\mathbf{u}_1,\ldots,\mathbf{u}_\omega$ in U, such that the $\mathbf{x}_i$ projecto onto a basis for X_2/X, and

$$\begin{aligned}
\varphi_{\mathbf{u}_i}(\mathbf{x}_i)&\equiv\mathbf{w}\pmod{Y}\qquad 1\leq i\leq\omega,\\
\varphi_{\mathbf{u}_i}(\mathbf{x}_j)&\equiv\mathbf{0}\pmod{Y}\qquad 1\leq i<j\leq\omega.
\end{aligned}$$

Since the images of $\mathbf{u}_1,\ldots,\mathbf{u}_\omega$ are linearly independent when we project to U/C, if we project to $U/(Z_2\cap C)$ the images are also linearly independent. Extend this list to $\mathbf{u}_1,\ldots,\mathbf{u}_\omega,\mathbf{u}_{\omega+1},\ldots,\mathbf{u}_s$ such that the projections form a basis for $U/(Z_2\cap C)$.

Fix $j>\omega$. Adding suitable multiples of $\mathbf{u}_\omega$ to $\mathbf{u}_j$, we may assume that $\varphi_{\mathbf{u}_j}(\mathbf{x}_\omega)$ is in Y. Then, adding multiples of $\mathbf{u}_{\omega-1}$ to $\mathbf{u}_j$, which will not change the value of $\varphi_{\mathbf{u}_j}(\mathbf{x}_\omega)$ modulo Y, we may also assume that $\varphi_{\mathbf{u}_j}(\mathbf{x}_{\omega-1})\in Y$. Continuing in this manner, adding multiples of $\mathbf{u}_{\omega-2},\ldots,\mathbf{u}_1$, we may assume that $\varphi_{\mathbf{u}_j}(\mathbf{x}_i)\in Y$ for $i=1,\ldots,\omega$; repeating this for each $j>\omega$, we obtain:

$$\varphi_{\mathbf{u}_j}(\mathbf{x}_i)\in Y\quad\text{for all } i,j,\ 1\leq i\leq\omega<j\leq s.$$

Since $X_2=\langle\mathbf{x}_1,\ldots,\mathbf{x}_\omega\rangle+X$, we thus have that for $j>\omega$, $\varphi_{\mathbf{u}_j}(X_2)\subseteq Y$.

Given $\mathbf{u}\in Z_2\cap C$, consider

$$\left(\overline{\psi_{\mathbf{u}}(\mathbf{u}_1)},\ldots,\overline{\psi_{\mathbf{u}}(\mathbf{u}_\omega)},\overline{\psi_{\mathbf{u}}(\mathbf{u}_{\omega+1})},\ldots,\overline{\psi_{\mathbf{u}}(\mathbf{u}_s)}\right)\in(X_2/X)^s.$$

Since $\mathbf{u}\in C$, if $1\leq i<j\leq s$ then by Lemma 6.5 the coordinates satisfy

$$\varphi_{\mathbf{u}_j}(\psi_{\mathbf{u}}(\mathbf{u}_i))\equiv\varphi_{\mathbf{u}_i}(\psi_{\mathbf{u}}(\mathbf{u}_j))\pmod{Y};$$

the values are well defined modulo X, since $X=Y^*$. In addition, $\mathbf{u}\in Z_2$, and so $\psi_{\mathbf{u}}(\mathbf{u}_i)\in X_2$; therefore, if $j>\omega$, then $\varphi_{\mathbf{u}_j}(\psi_{\mathbf{u}}(\mathbf{u}_i))\in\varphi_{\mathbf{u}_j}(X_2)\subseteq Y$. In particular, if $j>\omega$, then $\varphi_{\mathbf{u}_j}(\psi_{\mathbf{u}}(\mathbf{u}_i))\equiv 0\pmod{Y}$ for all i, and therefore

$$\varphi_{\mathbf{u}_1}(\psi_{\mathbf{u}}(\mathbf{u}_j))\equiv\cdots\equiv\varphi_{\mathbf{u}_s}(\psi_{\mathbf{u}}(\mathbf{u}_j))\equiv 0\pmod{Y}.$$

Since $U=\langle\mathbf{u}_1,\ldots,\mathbf{u}_s\rangle+C$, it follows that if $j>\omega$ then $\varphi_{\mathbf{a}}(\psi_{\mathbf{u}}(\mathbf{u}_j))\in Y$ for all $\mathbf{a}\in U$. Therefore $\psi_{\mathbf{u}}(\mathbf{u}_j)\in X$ for all $j>\omega$. Thus, we conclude that for all $\mathbf{u}\in Z_2\cap C$ and all $j>\omega$, $\psi_{\mathbf{u}}(\mathbf{u}_j)\in X$. Therefore, in the s-tuple

$$\left(\overline{\psi_{\mathbf{u}}(\mathbf{u}_1)},\ldots,\overline{\psi_{\mathbf{u}}(\mathbf{u}_\omega)},\overline{\psi_{\mathbf{u}}(\mathbf{u}_{\omega+1})},\ldots,\overline{\psi_{\mathbf{u}}(\mathbf{u}_s)}\right)\in(X_2/X)^s$$

only the first ω components may be nontrivial.

Consider then the linear map $Z_2\cap C\longmapsto(X_2/X)^s$ given by

$$\mathbf{u}\mapsto\left(\overline{\psi_{\mathbf{u}}(\mathbf{u}_1)},\ldots,\overline{\psi_{\mathbf{u}}(\mathbf{u}_\omega)},\overline{\psi_{\mathbf{u}}(\mathbf{u}_{\omega+1})},\ldots,\overline{\psi_{\mathbf{u}}(\mathbf{u}_s)}\right).$$

We claim that the kernel of this map is Z.

Certainly, Z is contained in the kernel. Conversely, let $\mathbf{u}\in Z_2\cap C$ be such that $\psi_{\mathbf{u}}(\mathbf{u}_j)\in X$ for $j=1,\ldots,s$. Since $U=\langle\mathbf{u}_1,\ldots,\mathbf{u}_s\rangle+(Z_2\cap C)$, to prove that $\mathbf{u}\in Z$ it is enough to show that $\psi_{\mathbf{u}}(Z_2\cap C)\subseteq X$. In turn, to establish this it is enough to show that if $\mathbf{z}\in Z_2\cap C$, then for all $\mathbf{a}\in U$, $\varphi_{\mathbf{a}}(\psi_{\mathbf{u}}(\mathbf{z}))\in Y$. Since $\mathbf{u}\in Z_2\cap C\subseteq C$, we know that $\varphi_{\mathbf{a}}(\psi_{\mathbf{u}}(\mathbf{z}))\equiv\varphi_{\mathbf{z}}(\psi_{\mathbf{u}}(\mathbf{a}))\pmod{Y}$; and since $\mathbf{z}\in Z_2\cap C\subseteq C$, we also have that $\varphi_{\mathbf{z}}(\psi_{\mathbf{u}}(\mathbf{a}))\in Y$. Therefore, $\varphi_{\mathbf{a}}(\psi_{\mathbf{u}}(\varphi_{\mathbf{z}}))\in Y$ for all $\mathbf{a}\in U$ and we conclude that $\psi_{\mathbf{u}}(\mathbf{z})\in X$ as desired. Therefore $\mathbf{u}\in Z$, proving the claim. Putting this claim together with the observation that only the first ω components can be nontrivial in any case, we conclude that we have an embedding

$$\begin{array}{rcl}(Z_2\cap C)/Z & \hookrightarrow & (X_2/X)^\omega\\ \mathbf{u} & \longmapsto & \left(\overline{\psi_{\mathbf{u}}(\mathbf{u}_1)},\ldots,\overline{\psi_{\mathbf{u}}(\mathbf{u}_\omega)}\right),\end{array}$$

and that the ω-tuples in the image satisfy

$$\varphi_{\mathbf{u}_j}(\psi_{\mathbf{u}}(\mathbf{u}_i))\equiv\varphi_{\mathbf{u}_i}(\psi_{\mathbf{u}}(\mathbf{u}_j))\pmod{Y}\qquad\text{for all } i,j,\ 1\leq i<j\leq\omega.$$

Applying Lemma 6.9, we have that $(Z_2\cap C)/Z$ embeds into a subspace of dimension $\omega+\binom{\omega}{2}$, and therefore $\dim((Z_2\cap C)/Z)\leq\omega+\binom{\omega}{2}$.

Thus we have shown that:

$$\begin{aligned}
\dim(U/Z) &= \dim(U/Z_2) + \dim(Z_2/(Z_2\cap C)) + \dim((Z_2\cap C)/Z),\\
\dim(U/Z_2) &\leq 2(k-\omega) + \binom{k-\omega}{2},\\
\dim(Z_2/(Z_2\cap C)) &\leq k-\omega,\\
\dim((Z_2\cap C)/Z) &\leq \omega + \binom{\omega}{2}.
\end{aligned}$$

Putting it all together, we have (cf. Theorem 1 in [13]):

$$\begin{aligned}
\dim(U/Z) &\leq 3k-2\omega+\binom{k-\omega}{2}+\binom{\omega}{2}\\
&= 3k-2\omega+\frac{k^2-k}{2}+\omega(\omega-k)\\
&= 2k+\binom{k}{2}+(k-\omega)(1-\omega)-\omega\\
&= 2k+\binom{k}{2}-\Big((\omega-1)(k-\omega-1)+1\Big)\\
&< 2k+\binom{k}{2}.
\end{aligned}$$

The last inequality holds since $0<\omega<k$, and therefore both $\omega-1$ and $k-\omega-1$ are nonnegative.

This strict inequality finishes the inductive step, and proves the theorem. $\square$

Translating back to groups we obtain the promised improvement on the necessary condition of Heineken and Nikolova:

Theorem 6.13. *Let G be a p-group of class at most 2 and exponent p. If G is capable, and $[G,G]$ is of rank k, then $G/Z(G)$ is of rank at most $2k+\binom{k}{2}$. Moreover, equality holds only if there exists a witness H to capability such that H_3 is cyclic (possibly trivial).*

Proof. If we fix an isomorphism $G^{\mathrm{ab}}\cong U$ and let X be the corresponding subspace of V, then $[G,G]\cong V/X$ and $G/Z(G)\cong U/Z$, where $Z=\big\{\mathbf{u}\in U \;\big|\; \psi_{\mathbf{u}}(U)\subseteq X\big\}$. Thus, the inequality by Theorem 6.12. For the "moreover" clause, note that if we let H be F/M, where F is the 3-nilpotent product of n groups of order p (with n the rank of G^{ab}), and M is the subspace of H_3 corresponding to any Y with $Y^*=X$, then H will be a witness fot the capability of X (as we are assuming that X is closed). By picking the Y' of codimension at most 1 guaranteed by the theorem, we obtain a witness with the desired property. $\square$

7. The 5-generated case.

In this section we combine our results so far to characterise the capable groups among the 5-generated groups of class at most two and exponent p.

One way to interpret Corollary 5.25 is that if G is of class exactly two and exponent p, and the commutator subgroup of G is "large enough," then G will be capable. On the other hand, Theorem 6.13 says that if G is capable, of class exactly two and exponent p, then the commutator subgroup of G cannot be "too small". Put together, the results seem to indicate that a group of class exactly two and prime exponent will be capable if and only if it is "nonabelian enough." The characterisation below seems to reinforce this intuition.

The 4-generated case. It is of course well known that a nontrivial cyclic group cannot be capable. It has also also been long known that an extra-special p-group is capable if and only if it is of order p^3 and exponent p. The following result shows that, at least for 4-generated groups in the class we are considering, these are the *only* exceptions to capability.

Theorem 7.1. *Let p be a prime, and let G be a 4-generated group of class at most 2 and exponent p. Then G is one and only one of the following:*

(i) *Cyclic and nontrivial;*
(ii) *Extra-special of order p^5 and exponent p;*
(iii) *Capable.*

Proof. Following the notation of Theorem 5.26, let n be the rank of G^{ab}, and let m be the rank of $[G, G]$.

The case of $p = 2$ is trivial, since G is abelian in this case. Assume then $p > 2$. The three categories are of course disjoint, so we only need to show that any such G is one of the three. If G is trivial, then it is capable. If G is minimally 1-generated, then it is nontrivial cyclic.

For G minimally 2-generated, Theorem 5.26 shows that G is capable: we have $n = 2$ and $m = 0$ or 1, and in either case $f(\binom{2}{2} - m + 1) < 2$. For G minimally 3-generated, again Theorem 5.26 settles the problem: here we have $m = 0$, 1, 2, or 3, and $f(\binom{3}{2} - m + 1) < 3$ in all cases.

Consider then the case of G minimally 4-generated; m must satisfy $0 \leq m \leq 6$. If $m \geq 2$, then $f(\binom{4}{2} - m + 1) < 4$, so G will be capable. If $m = 0$, then $G \cong C_p^4$, which is capable. Thus, the only case not covered is when $m = 1$, i.e., the commutator subgroup is cyclic.

If $Z(G) \neq [G, G]$, then we apply Theorem 4.14 and the $n = 3$ case to deduce that G is capable. Finally, if $Z(G) = [G, G]$ then we apply Theorem 6.13: the group cannot be capable, since $4 > 2(1) + \binom{1}{2}$. Alternatively, G is of order p^5, exponent p, and extra-special, and so we apply Corollary 4.19. □

The minimally 5-generated case. We next consider the case of $n = 5$. Here, Theorem 5.26 settles the cases $m \geq 4$; and the case $m = 0$ is of course trivial. We can finish the characterisation applying some easy group theory, and finally by applying non-trivial work of Brahana [4] to obtain a very satisfying result similar to Theorem 7.1.

If $m = 1$ then our group G has cyclic commutator subgroup. We cannot then have $Z(G) = [G, G]$ since G^{ab} is of order p^5, and so the group G will be either of the form $E \oplus C_p$, where E is extra-special of order p^5 and exponent p (hence G is not capable), or else of the form $K \oplus C_p^3$ where K is the nonabelian (extra-special) group of order p^3 and exponent p (and so G will be capable).

To discuss the cases of $m = 2$ and $m = 3$, recall that if V is a vector space and k is an integer, $0 \leq k \leq \dim(V)$, then the Grassmannian $Gr(k, V)$ is the set of all k-dimensional subspaces of V. This set has a rich geometric structure, though we will only touch on it briefly.

To solve the cases of $m = 2$ and $m = 3$, by Proposition 3.5 we only need to consider one representative from each orbit of the action of $\mathrm{GL}(5, p)$ in $Gr(7, V)$ (for the case $m = 3$) and $Gr(8, V)$ (for the case $m = 2$). In [4], Brahana shows that there are 6 orbits in $Gr(2, V)$ and 22 orbits in $Gr(3, V)$. By taking the orthogonal complement of each subspace (relative to our prefered basis v_{ji}, $1 \leq i < j \leq n$, with $\langle v_{ji}, v_{rs}\rangle = 1$ if $(j, i) = (r, s)$ and 0 otherwise) we obtain a well-known duality that shows that the number of orbits in $Gr(k, V)$ is the same as the number of orbits in $Gr(\binom{n}{2} - k, V)$ (see for example the paragraphs leading to [6, Theorem 1]; the argument there is for $n = 4$ and $k = 6$, but it trivially generalizes); thus, we can take the lists from [4] and by taking orthogonal complements, obtain a complete list of orbit representatives for the cases we are interested in. It is then an easy matter to check which ones correspond to closed subspaces and which do not.

There are six orbits of 8-dimensional subspaces under the action of $\mathrm{GL}(5, p)$: we give representatives of the orbits as orthogonal complements to the representatives found under the heading *"the lines of S"* in [4, p. 547]:

1. The coordinate subspace $X_1 = \langle v_{41}, v_{51}, v_{32}, v_{42}, v_{52}, v_{43}, v_{53}, v_{54}\rangle$; this is closed by Theorem 4.6. Alternatively, note that u_5 is central in the corresponding G, so we can apply Corollary 4.15 to reduce to the $n = 4$, $\dim(X) = 4$ case.
2. The coordinate subspace $X_2 = \langle v_{31}, v_{41}, v_{51}, v_{32}, v_{42}, v_{52}, v_{53}, v_{54}\rangle$; again, this is closed either by appplying Theorem 4.6 or Corollary 4.15.
3. The subspace $X_3 = \langle v_{21} - v_{43}, v_{31}, v_{41}, v_{51}, v_{42}, v_{52}, v_{53}, v_{54}\rangle$. Again, note that $\psi_5(U)$ is contained in X_3, so by Corollary 4.15 we conclude that X_3 is closed.
4. The subspace $X_4 = \langle v_{21} - v_{43}, rv_{31} - v_{42}, v_{41}, v_{51}, v_{32}, v_{52}, v_{53}, v_{54}\rangle$, with r not a square in $\mathbb{F}_p$. Since $\psi_5(U) \subseteq X_4$, we conclude as before that X_4 is closed.
5. The subspace $X_5 = \langle v_{21} - v_{43}, v_{31}, v_{41}, v_{32}, v_{42}, v_{52}, v_{53}, v_{54}\rangle$. In this case, X_5 is *not* closed: it corresponds to the amalgamated direct product of two groups: a 2-nilpotent product of two cyclic groups of order p, generated by g_3 and g_4; and the 2-nilpotent product of a cyclic group of order p generated by g_1 and the direct sum of two cyclic groups of order p, generated by g_2 and g_5. We amalgamate along the subgroup generated by $[g_4, g_3]$, identifying it with $[g_2, g_1]$. Theorem 4.17 shows X_5 is therefore not closed.
6. The subspace $X_6 = \langle v_{21} - v_{43}, v_{31} - v_{52}, v_{41}, v_{51}, v_{32}, v_{42}, v_{53}, v_{54}\rangle$. This subspace is closed, as can be verified with a simple computation in GAP. Alternatively, if X_6 were not closed then the closure would contain either v_{21} or v_{31}, but it is not hard to verify that neither w_{213} nor w_{312} lie in X_6^*.

Moving on to the 7-dimensional spaces, we obtain representatives of the orbits as orthogonal complements of the twenty-two *planes of S* listed in [4, pp. 547–548]. We present them in the same order as Brahana. The first six orbits correspond to groups G with $Z(G) \neq [G, G]$; this allows us reduce the problem to a subspace with $n = 4$ and codimension 3, all of which are necessarily closed as already noted. In all six cases, u_5 corresponds to a central element:

1. The subspace $X_1 = \langle v_{41}, v_{51}, v_{42}, v_{52}, v_{43}, v_{53}, v_{54}\rangle$.
2. The subspace $X_2 = \langle v_{51}, v_{32}, v_{42}, v_{52}, v_{43}, v_{53}, v_{54}\rangle$.
3. The subspace $X_3 = \langle v_{41}, v_{51}, v_{32}, v_{42}, v_{52}, v_{53}, v_{54}\rangle$.
4. The subspace $X_4 = \langle v_{21} - v_{43}, v_{51}, v_{32}, v_{42}, v_{52}, v_{53}, v_{54}\rangle$.

5. The subspace $X_5 = \langle v_{21} - v_{43}, v_{41}, v_{51}, v_{32}, v_{52}, v_{53}, v_{54}\rangle$.
6. The subspace $X_6 = \langle v_{21} - v_{43}, rv_{31} - v_{42}, v_{51}, v_{32}, v_{52}, v_{53}, v_{54}\rangle$, with r not a square in $\mathbb{F}_p$.

The next fifteen orbits correspond to subspaces that are closed; this is easy to determine using GAP, and not hard to verify by hand as well (either by applying one of our theorems, or by explicit computation). We list them without comment and leave routine (though often tedious) verification that they are indeed closed to the interested reader:

7. The subspace $X_7 = \langle v_{21} - v_{43} + rv_{53}, v_{31} - v_{52}, v_{41}, v_{51}, v_{32}, v_{42} - v_{53}, v_{54}\rangle$, where $x^3 + rx - 1$ is irreducible over $\mathbb{F}_p$.
8. The subspace $X_8 = \langle v_{21} - v_{43}, v_{31} - v_{52}, v_{41} + rv_{32}, v_{51}, v_{42}, v_{53}, v_{54}\rangle$, with r not a square in $\mathbb{F}_p$.
9. The subspace $X_9 = \langle v_{21} - v_{43}, v_{31} - v_{52}, v_{41}, v_{51}, v_{32} - v_{54}, v_{42}, v_{53}\rangle$.
10. The subspace $X_{10} = \langle v_{21} - v_{43}, v_{31} - v_{52}, v_{41}, v_{51}, v_{42}, v_{53}, v_{54}\rangle$.
11. The subspace $X_{11} = \langle v_{21} - v_{43}, v_{31} - v_{52}, v_{41}, v_{51}, v_{32}, v_{53}, v_{54}\rangle$.
12. The subspace $X_{12} = \langle v_{21} - v_{43}, v_{31} - v_{52}, v_{51}, v_{32}, v_{42}, v_{53}, v_{54}\rangle$.
13. The subspace $X_{13} = \langle v_{21} - v_{43}, v_{31} - v_{52} - rv_{42}, v_{41}, v_{51}, v_{32}, v_{53}, v_{54}\rangle$, with r not a square in $\mathbb{F}_p$.
14. The subspace $X_{14} = \langle v_{21} - v_{43}, v_{31} - v_{42}, v_{41} - rv_{51}, v_{32}, v_{42}, v_{53}, v_{54}\rangle$, $r \neq 0$.
15. The subspace $X_{15} = \langle v_{21} - v_{43}, v_{31} - v_{52}, v_{41}, v_{51}, v_{32}, v_{42}, v_{53}\rangle$.
16. The subspace $X_{16} = \langle v_{31} - v_{52}, v_{41}, v_{51}, v_{32}, v_{42}, v_{53}, v_{54}\rangle$.
17. The subspace $X_{17} = \langle v_{21} - v_{41} - v_{43}, v_{31} - v_{52}, v_{51}, v_{32}, v_{42}, v_{53}, v_{54}\rangle$.
18. The subspace $X_{18} = \langle v_{21} - v_{31} - v_{43} + v_{52}, v_{41}, v_{51}, v_{32}, v_{42}, v_{53}, v_{54}\rangle$.
19. The subspace $X_{19} = \langle v_{31}, v_{41}, v_{51}, v_{32}, v_{42}, v_{52}, v_{43}\rangle$.
20. The subspace $X_{20} = \langle v_{21} - v_{43}, v_{31}, v_{41}, v_{51}, v_{42}, v_{52}, v_{54}\rangle$.
21. The subspace $X_{21} = \langle v_{21} - v_{43}, v_{31}, v_{41}, v_{51}, v_{32}, v_{42}, v_{54}\rangle$.

The twenty-second and final orbit corresponds to an amalgamated direct product of the 2-nilpotent product of two cyclic groups of order p, generated by g_1 and g_2, with the 2-nilpotent product of three cyclic groups of order p, generated by g_3, g_4, and g_5, amalgamating by identifying the commutator $[g_2, g_1]$ with $[g_4, g_3]$. Thus, by Theorem 4.17 it gives the only nonclosed subspace of dimension 7 when $n = 5$ (up to the action of $GL(5, p)$):

22. The subspace $X_{22} = \langle v_{21} - v_{43}, v_{31}, v_{41}, v_{51}, v_{32}, v_{42}, v_{52}\rangle$.

We then obtain:

Theorem 7.2. *Let G be a minimally 5-generated p-group of class at most two and exponent p. Then G is one and only one of the following:*

(i) *Isomorphic to a direct product $E \times C_p$, where E is the extra-special p-group of order p^5 and exponent p;*

(ii) *Isomorphic to the amalgamated direct product*
$$\Bigl(\langle x_1\rangle \amalg^{\mathfrak{N}_2} \langle x_2\rangle\Bigr) \times_\phi \Bigl(\langle x_3\rangle \amalg^{\mathfrak{N}_2} \langle x_4\rangle \amalg^{\mathfrak{N}_2} \langle x_5\rangle\Bigr),$$
with each x_i of order p, and $\phi([x_2, x_1]) = [x_4, x_3]$;

(iii) *Isomorphic to the amalgamated direct product*
$$\Bigl(\langle x_1\rangle \amalg^{\mathfrak{N}_2} \langle x_2\rangle\Bigr) \times_\phi \Bigl(\bigl(\langle x_3\rangle \amalg^{\mathfrak{N}_2} \langle x_4\rangle \amalg^{\mathfrak{N}_2} \langle x_5\rangle\bigr)/\langle [x_5, x_4]\rangle\Bigr),$$
with each x_i of order p and $\phi([x_2, x_1]) = [x_4, x_3]$;

(iv) *Capable.*

If we recall that the extraspecial group of order p^5 and exponent p is obtained by taking the central product of two nonabelian groups of order p^3 and exponent p (more precisely, a central product) , we combine Theorems 7.1 and 7.2 into a single statement:

Theorem 7.3. *Let G be a 5-generated group of class at most 2 and exponent p. Then G is one and only one of the following:*

(i) *Nontrivial cyclic;*
(ii) *Isomorphic to an amalgamated direct product $G_1 \times_\phi G_2$ of two nonabelian groups, amalgamating a nontrivial cyclic subgroup of the commutator subgroups.*
(iii) *Capable.*

An alternative geometrical proof. The only part of the proof of Theorem 7.1 that does not follow by applying Theorem 5.26 is the case of $n = 4$ and $\dim(X) = 5$. I would like to present an alternative proof for this case. The reason for doing so is that a key step in the proof is geometric rather than algebraic. This highlights what I believe to be one of the potential strengths of the approach through linear algebra, namely that by casting the problem in terms of linear algebra we have an array of tools that can be brought to bear on the problem, most particularly geometric tools whose application may not be so easy to discern when the problem is presented in terms of commutators. This can also be seen in [4], though it will not be apparent in our presentation above. The geometric part of the argument is due to David McKinnon.

Fix $n = 4$. Given a vector $\mathbf{u} \in U$, $\mathbf{u} \neq \mathbf{0}$, we obtain a subspace $\psi_{\mathbf{u}}(U)$ of V; it is easy to verify that this subspace is 3-dimensional. Moreover, any nontrivial scalar multiple of $\mathbf{u}$ will yield the same subspace. Thus we obtain a map from the one dimensional subspaces of V (which form projective 3-space over $\mathbb{F}_p$) to $Gr(3, V)$; that is, a map $\Psi\colon \mathbb{P}^3 \to Gr(3, V)$. Explicitly, given $[\alpha_1 \colon \alpha_2 \colon \alpha_3 \colon \alpha_4] \in \mathbb{P}^3$, we associate to it the subspace $U \wedge (\alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3 + \alpha_4 u_4)$.

Turning now to $\ker(\Phi)$, where Φ is the map from Definition 5.4, it is easy to verify that if $\mathbf{p} \in \mathbb{P}^3$, $\mathbf{v} \in V$ is an arbitrary vector, and $X = \langle \Psi(\mathbf{p}), \mathbf{v}\rangle$, then $X^4 \cap \ker\Phi$ is trivial if and only if $\mathbf{v} \in \Psi(\mathbf{p})$.

Let $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4)$ be a nontrivial element of $\ker(\Phi)$. The subspace of V spanned by $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ is exactly 3-dimensional. This gives a mapping from the one-dimensional subspaces of $\ker(\Phi)$ to the 3-dimensional subspaces of V,

$$\Upsilon\colon Gr(1, \ker(\Phi)) \to Gr(3, V).$$

We can identify $Gr(1, \ker(\Phi))$ with $\mathbb{P}^3$ (or to be more precise, with $\mathbb{P}^{\binom{4}{3}-1}$): we have a bijection between a basis for $\ker(\Phi)$ and the choice of triples from $\{1, 2, 3, 4\}$, so a point $[\alpha_1 \colon \alpha_2 \colon \alpha_3 \colon \alpha_4]$ can be identified, for example, with the element $\alpha_1\mathbf{v}_{(234)} + \alpha_2\mathbf{v}_{(134)} + \alpha_3\mathbf{v}_{(124)} + \alpha_4\mathbf{v}_{(123)}$ (using the notation from the proof of Proposition 5.5). Thus we have two maps with domain $\mathbb{P}^3$ and codomain $Gr(3, V)$.

Consider a 5-dimensional subspace X of V. From Theorem 5.23, we know that $\dim(X^*) = 18$, $\dim(X^*) = 19$, or $\dim(X^*) = 20$. Since the only subspace of V that properly contains X is V itself, we deduce that X is closed if and only if $X^4 \cap \ker(\Phi_4)$ is nontrivial; that is, a 5-dimensional subspace of V is closed if and only if there exists $\mathbf{q} \in \mathbb{P}^3$ such that $\Upsilon(\mathbf{q}) \subseteq X$. As noted above, if X contains

$\Psi(\mathbf{p})$ for some $\mathbf{p} \in \mathbb{P}^3$, then X will be closed. The result we want is the converse: that if X is closed, then there exists $\mathbf{p} \in \mathbb{P}^3$ such that $\Psi(\mathbf{p}) \subseteq X$. This result can be established by considering the maps Ψ, Υ, and using a little algebraic geometry.

Suppose first we are working over the algebraic closure $\overline{\mathbb{F}_p}$ of $\mathbb{F}_p$ (so we can do algebraic geometry). The maps $\Psi\colon \mathbb{P}^3 \to Gr(3,V)$ and $\Upsilon\colon \mathbb{P}^3 \to Gr(3,V)$ are both regular maps, since they are defined everywhere and are locally (relative to the Zariski topology) determined by rational functions on the coordinates. We define two subsets of the algebraic variety $Gr(4,V) \times \mathbb{P}^3$, namely:

$$A = \bigl\{(X,\mathbf{p}) \bigm|, \Psi(\mathbf{p}) \subseteq X\bigr\}, \qquad \text{and} \qquad B = \bigl\{(X,\mathbf{q}) \bigm| \Upsilon(\mathbf{q}) \subseteq X\bigr\}.$$

Since both Ψ and Υ are regular, both A and B are closed subvarieties of $Gr(4,V) \times \mathbb{P}^3$. If we now consider the projections,

$$\begin{aligned} p_1\colon Gr(4,V) \times \mathbb{P}^3 &\to Gr(4,V)\\ p_2\colon Gr(4,V) \times \mathbb{P}^3 &\to \mathbb{P}^3, \end{aligned}$$

we obtain maps from each of A and B into $Gr(4,V)$ and $\mathbb{P}^3$, respectively. The maps to $\mathbb{P}^3$ are surjections, and the fibers all have dimension 2 because the fiber over $\mathbf{p}$ (resp. over $\mathbf{q}$) is the set of all 4-dimensional subspaces of V that contain the 3-dimensional space $\Psi(\mathbf{p})$ (resp. $\Upsilon(\mathbf{q})$); this set is isomorphic to the set of lines in the quotient space $V/\Psi(\mathbf{p})$ (resp. $V/\Upsilon(\mathbf{q})$), which in turn is isomorphic to the projective plane $\mathbb{P}^2$, hence 2-dimensional.

The maps are also smooth, so we have smooth maps of fiber dimension 2 over a smooth 3-dimensional variety; this means that both A and B are of dimension $3+2=5$.

Consider now the projections to $Gr(4,V)$. We know that $p_1(A)$ and $p_1(B)$ are irreducible subvarieties of $Gr(4,V)$ of dimension at most 5, and that $p_1(A)$ is contained in $p_1(B)$ (to see this last assertion, note that if $(X,\mathbf{p}) \in A$, then $X^4 \cap \ker(\Phi)$ is nontrivial, so there exists $\mathbf{q}$ such that $(X,\mathbf{q}) \in B$). If we can show that $p_1(A)$ is of dimension exactly 5, then the irreducibility of B will imply that $p_1(A) = p_1(B)$. To show that $p_1(A)$ is of dimension exactly 5 it is enough to show that it is generically finite; for this it is, in turn, enough to show there is at least one $X \in Gr(4,V)$ such that $p_1^{-1}(X)$ is nonempty and finite. But in fact $p_1^{-1}(X)$ has at most one element, for if $\mathbf{p} \neq \mathbf{q}$, then $\langle \Psi(\mathbf{p}), \Psi(\mathbf{q})\rangle$ contains $U \wedge U'$ with U' of dimension 2 (spanned by the lines corresponding to $\mathbf{p}$ and $\mathbf{q}$); and this subspace is of dimension 5. So the conclusion that $p_1(A) = p_1(B)$ holds over $\overline{\mathbb{F}_p}$.

Thus, if $\mathbf{q} \in \mathbb{P}^3$ and $X \in Gr(4,V)$ are such that $(X,\mathbf{q}) \in B$, then there exists $\mathbf{p} \in \mathbb{P}^3$ such that $(X,\mathbf{p}) \in A$. We want to show that if $\mathbf{q}$ and X are defined over $\mathbb{F}_p$, then $\mathbf{p}$ is also defined over $\mathbb{F}_p$. If we apply a Galois automorphism to the varieties over $\overline{\mathbb{F}_p}$, both X and $\mathbf{q}$ are fixed, and every conjugate of $\mathbf{p}$ will also satisfy the conclusion; however, we know that if $(X,\mathbf{p}),(X,\mathbf{p}') \in A$, then $\mathbf{p} = \mathbf{p}'$, by the argument above, so we conclude that $\mathbf{p}$ is fixed by all Galois automorphisms of $\overline{\mathbb{F}_p}$, proving it is indeed defined over $\mathbb{F}_p$.

This proves what we want: if X' is a 5-dimensional subspace of V, and if there exists $(X,\mathbf{p}) \in A$ such that $X \subseteq X'$, then X' is closed. And if X' is closed, then there exists $(X,\mathbf{q}) \in B$ with $X \subseteq X'$, and this implies the existence of $\mathbf{p} \in \mathbb{P}^3$ with $(X,\mathbf{p}) \in A$. Thus, X' is closed if and only if it contains $\Psi(\mathbf{p})$ for some $\mathbf{p} \in \mathbb{P}^3$. In terms of the groups, it says that a group G of class two, exponent p, with G^{ab} of rank 4 and $[G,G]$ of order p is capable if and only if $[G,G] \neq Z(G)$. That is, a

5-dimensional subspace of V is closed if and only if the corresponding group is not extra-special.

Remark 7.4. The proof that there exist $\mathbf{p} \in \mathbb{P}^3$ such that $\Psi(\mathbf{p}) \subseteq X$ if and only if there exists $\mathbf{q} \in \mathbb{P}^3$ such that $\Upsilon(\mathbf{q}) \subseteq X$ can be done purely at an algebraic level; see for example [19]. However, I find the geometric argument more satisfying.

8. Final remarks and questions.

The gap between our necessary and sufficient condition, unfortunately, grows with n. Thus, when $n = 4$ the necessary condition allows us to discard the case $\dim(X) = 5$ (when X does not contain $\Psi(\mathbf{u})$ for some nontrivial $\mathbf{u} \in U$), while the sufficient condition handles the remaining cases with $\dim(X) \leq 4$. When we move to $n = 5$, however, Theorem 6.13 deals only with $\dim(X) = 9$ (where we are reduced to the case $n = 4$ as above), while Corollary 5.25 dispatches $\dim(X) \leq 6$, leaving us to deal with the cases of dimension 7 and 8. Our success above was achieved thanks to the careful geometric analysis of Brahana. With $n = 6$, Theorem 6.13 would handle $\dim(X) = 14$ and 15 (we can either reduce to a smaller n, or else the subspace is not closed), and Corollary 5.25 deals with $\dim(X) \leq 7$, leaving now six potential dimensions open. As n increases, the gap between our numerical necessary and sufficient conditions continues to widen, making them less and less useful.

Heineken proved that the necessary condition is sharp, in that there are examples of capable groups in which the inequality from Theorem 6.13 is an equality. We might likewise wonder if we can sharpen the sufficient condition. There is some hope this might be possible, since for example Corollary 5.24 considers all subspaces of dimension strictly larger than X, while Proposition 5.10 only requires us to look at those subspaces that properly contain X. So we ask:

Question 8.1. *Is the sufficient condition in Corollary 5.25 sharp? That is, is it true that for all $n > 1$, if m is the smallest integer such that $0 < m < \binom{n}{2}$ and $f(m+1) \geq n$, then there exists $X < V$ such that $\dim(X) = m$ and $X \neq X^{**}$?*

Note that if we can find a non-closed subspace $X < V(n)$ with $\dim(X) = k$, then we can find non-closed subspaces $X' < V(n)$ with $\dim(X') = r$ for any r satisfying $k \leq r < \binom{n}{2}$: enlarge X by adding vectors from X^{**} not in X until we obtain a subspace of codimension one in its closure; and then continue by adding vectors that do not lie in X^{**} until we obtain a subspace of codimension 1 in $V(n)$. So it is enough to ask about the smallest value of m with $\dim(X) = m$ and $X \neq X^{**}$.

For $m \leq 5$, the answer to Question 8.1 is affirmative. Consider then $n = 6$; by taking an amalgamated central product of the 2-nilpotent product of two cyclic groups of order p and the 2-nilpotent product of 4 cyclic groups of order p we can find a non-closed subspace of dimension 9; the least m, however, for which $f(m+1) \geq 6$ is $m = 8$. So we ask:

Question 8.2. *Is there a subspace X of $V(6)$ with $\dim(X) = 8$ and $X \neq X^{**}$?*

I do not know the answer to this question yet; I have done a brute force search using GAP and have found no examples yet. However, though the search has considered over one hundred million subspaces, the total number of eight dimensional subspaces of the fifteen dimensional space $V(6)$ is approximately 9.3×10^{26} if we work over $\mathbb{F}_3$, so the negative results in this search are hardly significant.

In general, given n, taking an amalgamated central product of two relatively free groups, one of rank 2 and one of rank $n-2$, and identifying a subgroup of order p from each, yields a non-closed subspace of dimension $2n-3$ (we need $2(n-2)$ relations to state the generators from one relatively free group commute with those of the other, and one relation to identify one nontrivial commutator from each factor with each other). This is the smallest nonclosed subspace we can obtain with amalgamated direct products, but it is not necessarily the smallest non-closed. For example, with $n=8$, the amalgamated direct product yields a non-closed X of dimension 13; but if we take the amalgamated coproduct of two extra-special groups of order p^5 and exponent p, identifying the commutator subgroups, we obtain a non-closed X of dimension 11 (we will need 5 relations to describe each of the extra-special groups, plus one relation to identify the two commutator subgroups). This eleven dimensional subspace still falls two short of the 9-dimensional example we would need for $n=8$ if Corollary 5.25 is indeed sharp.

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