Math 566 - Homework 6

Due Wednesday March 6, 2024

1. Let $R = \mathbb{Z}_6$, and $S = \{2, 4\}$. Prove that S is a multiplicative subset of R, and that $S^{-1}R \cong \mathbb{Z}_3$.

2. Let $S = \{\pm 1001^k \mid k \text{ a positive integer}\}$. Let $\varphi\colon \mathbb{Z} \to S^{-1}\mathbb{Z}$ be the canonical map, $\varphi(a) = \frac{1001a}{1001}$. Describe the prime factorization of all $a \in \mathbb{Z}$ such that $\varphi(a)$ is a unit in $S^{-1}\mathbb{Z}$.

3. Let P be a nonzero prime ideal of $\mathbb{Z}$, and let $\mathbb{Z}_P$ be the localization of $\mathbb{Z}$ at P; that is, $\mathbb{Z}_P = (\mathbb{Z} - P)^{-1}\mathbb{Z}$. Show that we can identify $\mathbb{Z}_P$ with the subring of $\mathbb{Q}$ consisting of the rationals that can be written as $\frac{a}{b}$ with $a, b \in \mathbb{Z}$ and $b \notin P$.

4. Show that if we view $\mathbb{Z}_P$ as a subring of $\mathbb{Q}$ as in Problem 3, then
$$\bigcap_P \mathbb{Z}_P = \mathbb{Z},$$
where the intersection runs over all nonzero prime ideals of $\mathbb{Z}$.

5. *Fractions of quotients.* Let R be a commutative ring, I be an ideal of R, and let $\pi\colon R \to R/I$ be the canonical projection onto the quotient.
   (i) Show that if S is a multiplicative subset of R, then $\pi S = \{\pi(s) \mid s \in S\}$ is a multiplicative subset of R/I.
   (ii) Show that $\theta\colon S^{-1}R \to (\pi S)^{-1}(R/I)$ given by $\theta(\frac{r}{s}) = \frac{\pi(r)}{\pi(s)}$ is a well-defined surjective ring homomorphism.
   (iii) Recall that $S^{-1}I \triangleleft S^{-1}R$. Prove that $(S^{-1}R)/S^{-1}I \cong (\pi S)^{-1}(R/I)$.

6. *Fractions of fractions.* Let R be a commutative ring, and S a multiplicative subset of R. Let T be the a multiplicative subset of $S^{-1}R$, and let
$$S_* = \left\{ r \in R \;\middle|\; \frac{r}{s} \in T \text{ for some } s \in S \right\}.$$
   (i) Show that S_* is a multiplicative subset of R.
   (ii) Prove that if $t \in S_*$ and $s \in S$, then $sr \in S_*$.
   (iii) Define $f\colon T^{-1}(S^{-1}R) \to S_*^{-1}R$ by $f(\frac{a/t}{b/u}) = \frac{au}{bt}$. Show that this is a well-defined ring homomorphism.
   (iv) Prove that $T^{-1}(S^{-1}R) \cong S_*^{-1}R$.

   Note: This means that one can realize a ring of quotients of a ring of quotients of R as a ring of quotients of R; this is analogous to the fact that a quotient of a quotient of R can be realized as a quotient of R (the Third Isomorphism Theorem).