Math 566 - Homework 2

Due Wednesday January 31, 2024

1. Let $(R, +, \cdot)$ be a ring, and let $(R^{\mathrm{op}}, +, \circ)$ be the opposite ring, as in Homework 1, Problem 1. Let I be a subset of R. Show that I is a left (resp. right) ideal of $(R, +, \cdot)$ if and only if I is a right (resp. left) ideal of $(R^{\mathrm{op}}, +, \circ)$

2. Let R be a ring, and let X be a set. Let R^X be the set of all functions $f\colon X \to R$. Define addition and multiplication in R^X by

$$(f+g)(x) = f(x) + g(x), \qquad (fg)(x) = f(x)g(x)$$

where the operations on the right hand side are the operations of R.

   (i) Prove that R^X with these operations is a ring.
   (ii) Prove that R^X is commutative if and only if R is commutative or X is empty.
   (iii) Prove that R^X has a unity if and only if R has a unity or X is empty.

3. Let R and S be rings with unity, and let $f\colon R \to S$ be a ring homomorphism; recall that we do not require ring homomorphisms to be unital unless we specify that they are.

   (i) Show that if $1_S \in \mathrm{Im}(f)$, then $f(1_R) = 1_S$.
   (ii) Prove that if there exists $u \in R$ such that $f(u)$ is a unit in S, then $f(1_R) = 1_S$.

4. Let p be a prime number.

   (i) Prove that if $1 \leq k \leq p-1$, then $\binom{p}{k}$ is a multiple of p.
   (ii) The Freshman's Dream. Let R be a commutative ring with identity such that $\mathrm{char}(R) = p$. Prove that for all $a, b \in R$ and positive integers n, $(a+b)^{p^n} = a^{p^n} + b^{p^n}$.

5. Let R be a ring. An element $r \in R$ is *nilpotent* if and only if there exists a positive integer n such that $r^n = 0$.

   (i) Show that if R is commutative, then the set of all nilpotent elements of R is an ideal of R.
   (ii) Give an example of a ring R and elements a and b of R such that each of a and b are nilpotent, but neither ab nor $a+b$ are nilpotent. Hint: Try 2×2 matrices.

6. Given a function $f\colon \mathbb{R} \to \mathbb{R}$, the *support of* f is the set

$$\mathrm{supp}(f) = \{r \in R \mid f(r) \neq 0\}.$$

We say f *has compact support* if and only if there exists $N > 0$ such that $\mathrm{supp}(f) \subseteq [-N, N]$.

Let R be the ring of all functions $f\colon \mathbb{R} \to \mathbb{R}$ with pointwise addition and multiplication.

   (i) Let S be the set of all elements of R that are continuous and have compact support. Prove that S is a subring of R.
   (ii) Prove that S does not have an identity, but nonetheless $S^2 = S$.
   (iii) Prove that S is not an ideal of R.