Math 566 - Homework 10

Due Wednesday April 24, 2024

1. Let K be an extension of F, and let $u \in K$. Show that if u is the root of a monic polynomial $x^n + a_{n-1}x^{n-1} + \cdots + a_0 \in K[x]$, and each a_i is algebraic over F, then u is algebraic over F.

2. Let K be an extension of F, and let L and M be intermediate extensions (so $F \subseteq L \subseteq K$ and $F \subseteq M \subseteq K$).

   (i) Prove that $[LM : M] \leq [L : L \cap M]$.

   (ii) Conclude that $[LM : M] \leq [L : F]$.

3. Let K be an extension of F, and let $u, v \in K$ be algebraic over F with $[F(u) : F] = n$ and $[F(v) : F] = m$.

   (i) Prove that $[F(u, v) : F] \leq nm$.

   (ii) Show that if $\gcd(m, n) = 1$, then $[F(u, v) : F] = nm$.

4. Let K be a finite dimensional extension of F and let L and M be intermediate extensions.

   (i) Show that if $[LM : F] = [L : F][M : F]$, then $L \cap M = F$.

   (ii) Show that if $[L : F] = 2$ or $[M : F] = 2$, and $L \cap M = F$, then we will have $[LM : F] = [L : F][M : F]$.

   (iii) Use a real and a nonreal cube root of 2 to give an example of a finite dimensional extension K of $\mathbb{Q}$, and intermediate fields L and M, such that $L \cap M = \mathbb{Q}$ and $[L : \mathbb{Q}] = [M : \mathbb{Q}] = 3$, but $[LM : \mathbb{Q}] < 9$.

5. Prove that $\mathbb{Q}(\sqrt{2})$ is not isomorphic to $\mathbb{Q}(\sqrt{3})$. NOTE: We know there is no isomorphism $\mathbb{Q}(\sqrt{2}) \to \mathbb{Q}(\sqrt{3})$ that sends $\sqrt{2}$ to $\sqrt{3}$; but this, in and of itself, does not preclude the possibility of an isomorphism where $\sqrt{2}$ is mapped to some other element of $\mathbb{Q}(\sqrt{3})$.

6. Let K be an extension of F, where $\mathrm{char}(F) \neq 2$. Prove that $[K : F] = 2$ if and only if $K = F(\sqrt{d})$ for some $d \in F$ that is not a square in F.

7. Let K be an extension of F where $\mathrm{char}(F) \neq 2$. Prove that if $[K : F] = 2$, then K is Galois over F.

8. Let K be a finite dimensional Galois extension of F, and let L and M be intermediate fields. Use the Fundamental Theorem of Galois Theory to prove the following:

   (i) $\mathrm{Aut}_{LM}(K) = \mathrm{Aut}_L(K) \cap \mathrm{Aut}_M(K)$.

   (ii) $\mathrm{Aut}_{L \cap M}(K) = \langle \mathrm{Aut}_L(K), \mathrm{Aut}_M(K) \rangle$.

   You may invoke the theorem even though we have not finished proving it in class. HINT: You should be able to prove this using only the correspondence between intermediate fields and subgroups.