Math 566 - Homework 1
Due Wednesday January 24, 2024

1. Let $(R, +, \cdot)$ be a ring, and define the *opposite ring* $(R^{\mathrm{op}}, +, \circ)$ as follows: the underlying set of R^{op} is R, and addition in R^{op} is the same as addition on R. Multiplication on R^{op}, which we will denote by $\circ$, is defined by $a \circ b = b \cdot a$, where $\cdot$ is the multiplication in R.
   (i) Show that $(R^{\mathrm{op}}, +, \circ)$ is a ring.
   (ii) Show that R has an identity if and only if R^{op} has an identity.
   (iii) Show that R is a division ring if and only if R^{op} is a division ring.
2. Let $(R, +, \cdot)$ be a set, together with two binary operations, and assume that the set and operations satisfy all the axioms of a ring, *except perhaps* for commutativity of addition. That is, $(R, +)$ is a (not necessarily commutative) group, $\cdot$ is associative, and $\cdot$ distributes on both sides over $+$.
   (i) Prove that if R has a multiplicative identity, that is, an element $1_R \in R$ such that for all $a \in R$ we have $a \cdot 1_R = 1_R \cdot a = a$, then $x + y = y + x$ for all $x, y \in R$; that is, commutativity of $+$ is a consequence of the other axioms of a ring, together with the existence of a unity.
   (ii) Give an example to show that commutativity of $+$ does not follow from the other axioms if R does not have a multiplicative identity, by exhibiting an example of a set R, and binary operations $+$ and $\cdot$ such that $(R, +)$ is a *nonabelian* group, and $\cdot$ is an associative operation that distributes over $+$ on both sides.
3. **Cayley's Theorem for Rings.** Let $(R, +, \cdot)$ be a ring; for each $r \in R$, let $\lambda_r \colon R \to R$ be the function given by
$$\lambda_r(a) = ra$$
   (i) Show that for each $r \in R$, λ_r is an element of $\mathrm{End}(R, +)$, the endomorphism group of the abelian group $(R, +)$.
   (ii) Define $\psi \colon R \to \mathrm{End}(R, +)$ by $\psi(r) = \lambda_r$. Prove that this map is a ring homomorphism (where $\mathrm{End}(R, +)$ is a ring with pointwise addition and composition of functions). Prove that if R has a unity, then ψ is one-to-one.
   (iii) Use the Dorroh embedding to show that if R is a ring, with or without unity, then there exists an abelian group A an a one-to-one ring homomorphism $\varphi \colon R \to \mathrm{End}(A, +)$. That is: every ring is [isomorphic to] a subring of the endomorphism ring of an abelian group.
4. A *Boolean ring* is a ring $(R, +, \cdot)$ such that $a^2 = a$ for all $a \in R$. Prove that every Boolean ring is commutative and $a = -a$ for all $a \in R$. *Hint:* Square $(a + a)$ and $(a - b)$. (An element a of a ring such that $a^2 = a$ is called an *idempotent*.)
5. Let X be a set, and let $\mathcal{P}(X)$ be the power set of X (the set of all subsets of X). Define operations $\oplus$ and $\odot$ on $\mathcal{P}(X)$ by:
$$A \oplus B = (A - B) \cup (B - A) \qquad \text{(symmetric difference)}$$
$$A \odot B = A \cap B \qquad \text{(intersection)}$$
   Show that $(\mathcal{P}(X), \oplus, \odot)$ is a Boolean ring with unity.
6. Give an example of a ring R and a subring S such that R has a unity, S has a unity, but $1_S \neq 1_R$.