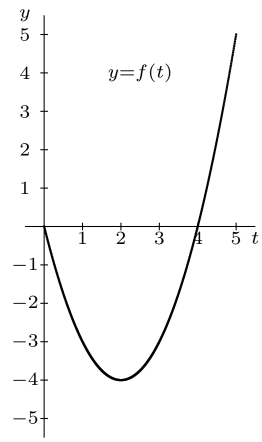
Math 270–005: Calculus I
Prof. Arturo Magidin
Homework 5
SOLUTIONS

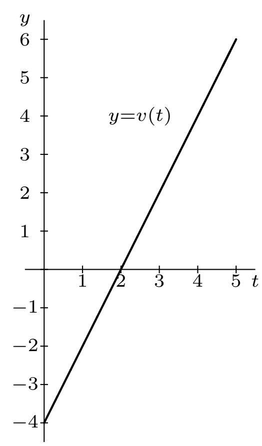
§3.6

15. We have $f(t) = t^2 - 4t$, $0 \leq t \leq 5$. The variable t is measured in seconds, the position $f(t)$ in feet.

   (a) A graph of the function $f(t)$:



   (b) The velocity function is the derivative of the position. So in this case, $v(t) = f'(t) = 2t - 4$. Here's a graph on $[0, 5]$:

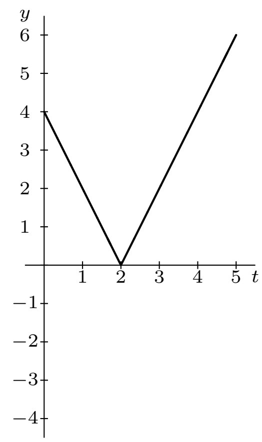


   The object is moving left when $v(t) < 0$, which happens for $0 \leq t < 2$. It is moving right when the velocity is positive, which is $2 < t \leq 5$. And it is stationary when $v(t) = 0$, which happens when $t = 2$.

   (c) The acceleration is the derivative of the velocity, so $a(t) = v'(t) = 2$. Thus, at $t = 1$, the velocity is $v(1) = 2(1) - 4 = -2$ ft/sec, and the acceleration is 2 ft/sec^2.

   (d) When is the velocity zero? When $v(t) = 2t - 4 = 0$. This happens when $t = 2$. The acceleration at that time is 2 ft/sec^2 because the acceleration is *always* 2 ft/sec^2 for this motion.

   (e) The speed is the absolute value of the velocity. If we graph it, we get:



So we see that the speed is increasing for $2 \leq t \leq 5$. Alternatively: the velocity increases when velocity and acceleration have the same sign. Since the acceleration is always positive, this happens on $[2, 5]$, where the velocity is not negative.

23. Now the stone is trown vertically from a cliff. The initial velocity is 32 ft/s, and the initial height is 48 ft. The position function is given by

$$s(t) = -16t^2 + 32t + 48.$$

(a) The velocity function is $v(t) = -32t + 32$.

(b) The highest point is achieved when the velocity is 0, which happens when $t = 1$: one second after throwing the stone.

(c) The height at the highest point is then $s(1) = -16 + 32 + 48 = 64$ ft above the ground.

(d) The stone strikes the ground when $s(t) = 0$. We have

$$\begin{aligned} -16t^2 + 32t + 48 &= 0 \\ -16(t^2 - 2t - 3) &= 0 \\ -16(t-3)(t+1) &= 0. \end{aligned}$$

Since $t = -1$ is impossible, it strikes the ground when $t = 3$: three seconds after we throw it.

(e) The velocity when it strikes the ground is $v(3) = -32(3) + 32 = -32(2) = -64$ ft/sec. Negative because it is moving down.

(f) When is the speed increasing? Since the acceleration is $a(t) = v'(t) = -32$, which is always negative, the speed is increasing when the velocity is negative; this occures on $[1, 3]$: from the moment it starts falling until it hits the ground.

§3.7

27. If $y = (3x^2 + 7x)^{10}$, then the outside function is $f(u) = u^{10}$, the inside function is $u = g(x) = 3x^2 + 7x$. We have:

$$y' = 10(3x^2 + 7x)^9(3x^2 + 7x)' = 10(3x^2 + 7x)^9(6x + 7).$$

35. If $y = \tan e^x$, then the "outside function" is $f(u) = \tan u$, the "inside function" is $u = g(x) = e^x$. We have:

$$y' = \sec^2(e^x)(e^x)' = e^x \sec^2(e^x).$$

§3.8

13. The curve is $x^4 + y^4 = 2$, the point is $(1, -1)$. Note that the point is indeed on the curve.

    (a) First we find the derivative using implicit differentiation:

$$\begin{aligned} x^4 + y^4 &= 2 \\ \frac{d}{dx}(x^4 + y^4) &= \frac{d}{dx}(2) \\ 4x^3 + 4y^3y' &= 0 \\ 4y^3y' &= -4x^3 \\ y' &= -\frac{4x^3}{4y^3} \\ y' &= -\frac{x^3}{y^3}. \end{aligned}$$

    (b) Next we find the curve (the slope of the tangent line to the curve) at $(1, -1)$:

$$y' \Big|_{(1,-1)} = -\frac{(1)^3}{(-1)^3} = -\frac{1}{-1} = 1.$$

19. Here we have the curve $\cos y = x$, at the point $(0, \frac{\pi}{2})$. Note that the point does indeed satisfy the equation.

    (a) The derivative:

$$\begin{aligned} \cos y &= x \\ \frac{d}{dx}(\cos y) &= \frac{d}{dx}(x) \\ (-\sin y)y' &= 1 \\ y' &= \frac{1}{-\sin y} \\ y' &= -\csc(y). \end{aligned}$$

    (b) At the point $(0, \frac{\pi}{2})$, we obtain $y' = -\csc(\frac{\pi}{2}) = -1$.

37. Now we have:

$$\begin{aligned} 6x^3 + 7y^3 &= 13xy \\ \frac{d}{dx}(6x^3 + 7y^3) &= \frac{d}{dx}(13xy) \\ 18x^2 + 21y^2y' &= 13((x)'y + x(y)') \\ 18x^2 + 21y^2y' &= 13(y + xy') \\ 21y^2y' - 13xy' &= 13y - 18x^2 \\ (21y^2 - 13x)y' &= 13y - 18x^2 \\ y' &= \frac{13y - 18x^2}{21y^2 - 13x}. \end{aligned}$$

46. We have the curve $x^3 + y^3 = 2xy$ (the folium of Descartes), and the point $(1, 1)$.

(a) The point lies in the curve, since $(1)^3 + (1)^2 = 2 = 2(1)(1)$.

(b) To find the equation of the tangent line at the point, first we find $y'|_{(1,1)}$:

$$\begin{aligned}
x^3 + y^3 &= 2xy \\
\frac{d}{dx}(x^3 + y^3) &= \frac{d}{dx}(2xy) \\
3x^2 + 3y^2y' &= 2(y + xy') \\
3(1)^2 + 3(1)^2y' &= 2(1 + (1)y') \\
3 + 3y' &= 2 + 2y' \\
y' &= -1.
\end{aligned}$$

So the slope is -1, and the equation of the tangent line is:

$$\begin{aligned}
y - 1 &= (-1)(x - 1) \\
y &= -(x - 1) + 1 \\
y &= 1 - x + 1 \\
y &= 2 - x.
\end{aligned}$$

51. To find the second derivative $\frac{d^2y}{dx^2}$, we have:

$$\begin{aligned}
x + y^2 &= 1 \\
\frac{d}{dx}(x + y^2) &= \frac{d}{dx}(1) \\
1 + 2yy' &= 0 \\
2yy' &= -1 \\
y' &= -\frac{1}{2y}. \\
\frac{d}{dx}y' &= \frac{d}{dx}\left(-\frac{1}{2y}\right) \\
y'' &= -\frac{1}{2}\left(\frac{d}{dx}y^{-1}\right) \\
y'' &= -\frac{1}{2}\left(-y^{-2}y'\right) \\
y'' &= \frac{1}{2y^2}y' \\
y'' &= \frac{1}{2y^2}\left(-\frac{1}{2y}\right) \\
y'' &= -\frac{1}{4y^3}.
\end{aligned}$$